Higher-order networks

An introduction to simplicial complexes
Lesson III

LTCC Course
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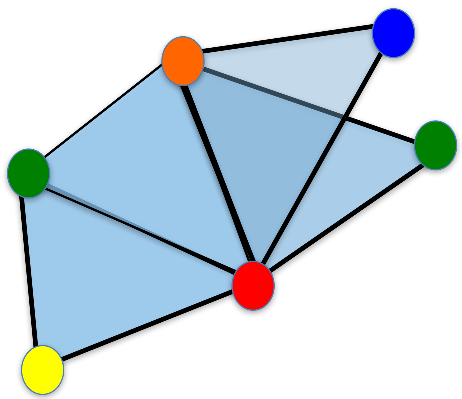
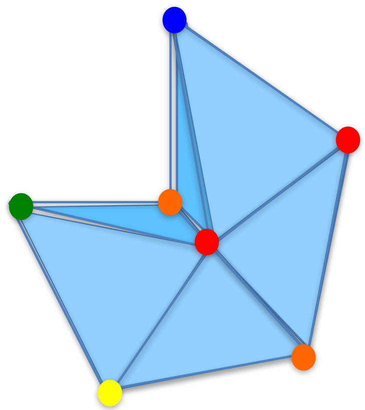
Higher-order networks

Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



d=2 simplicial complex

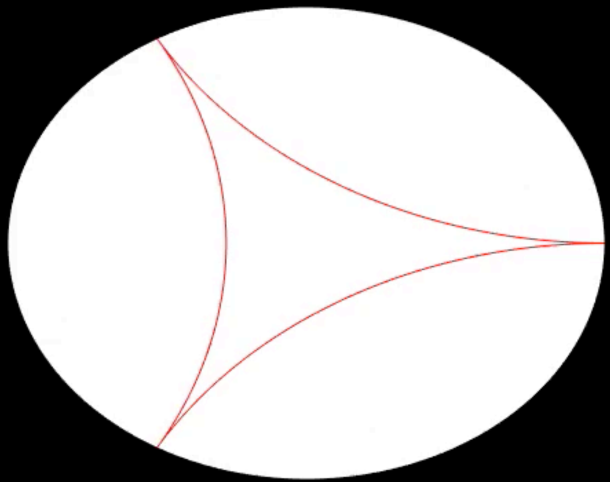


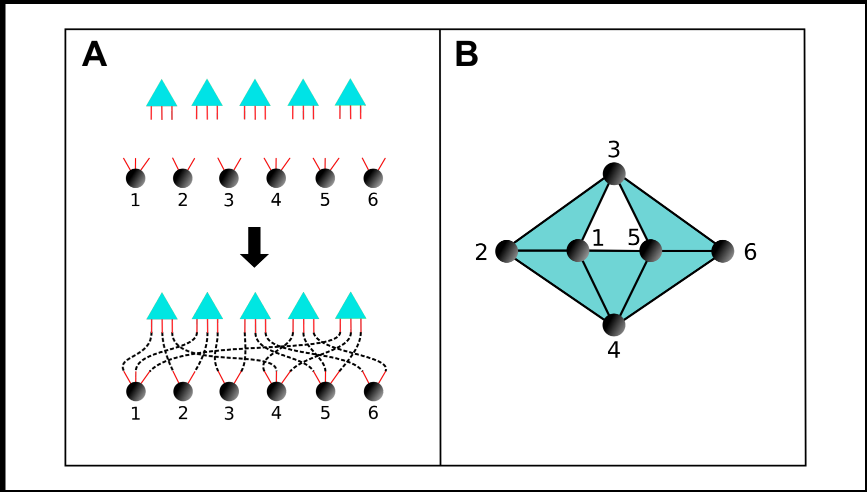
d=3 simplicial complex

Simplicial complex models

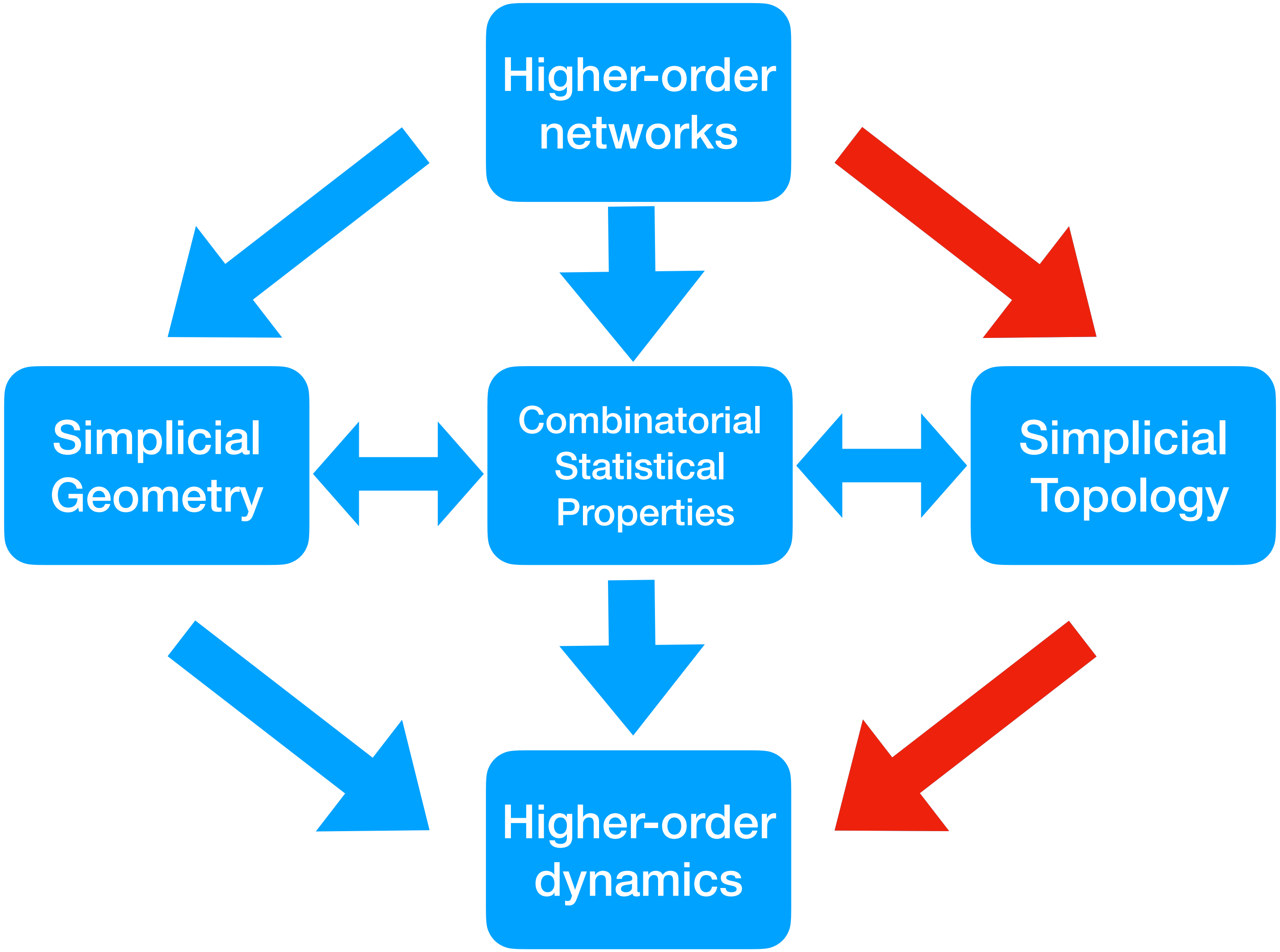
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





Higher-order structure and dynamics



Lesson III:
Topology and higher-order dynamics

- **Spectral properties of the Laplacians**

- **Diffusion**
  - **Heat diffusion on graphs**
  - **Higher-order diffusion of topological signals**

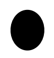
- **Topological Kuramoto model**
  - **The Kuramoto model on graphs**
  - **The Topological Kuramoto model**

- **Global synchronisation of topological signals**
  - **Global synchronisation ongraphs**
  - **Global synchronisation on simplicial and cell complexes**

Summary of
Algebraic Topology

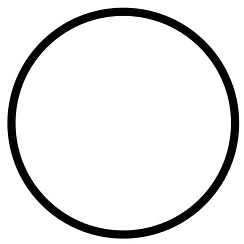
Betti numbers

Point



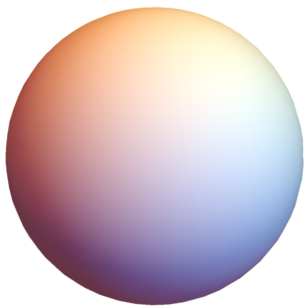
$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 0$

Circle



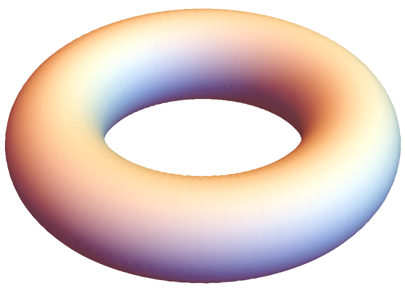
$\beta_0 = 1$
$\beta_1 = 1$
$\beta_2 = 0$

Sphere



$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 1$

Torus

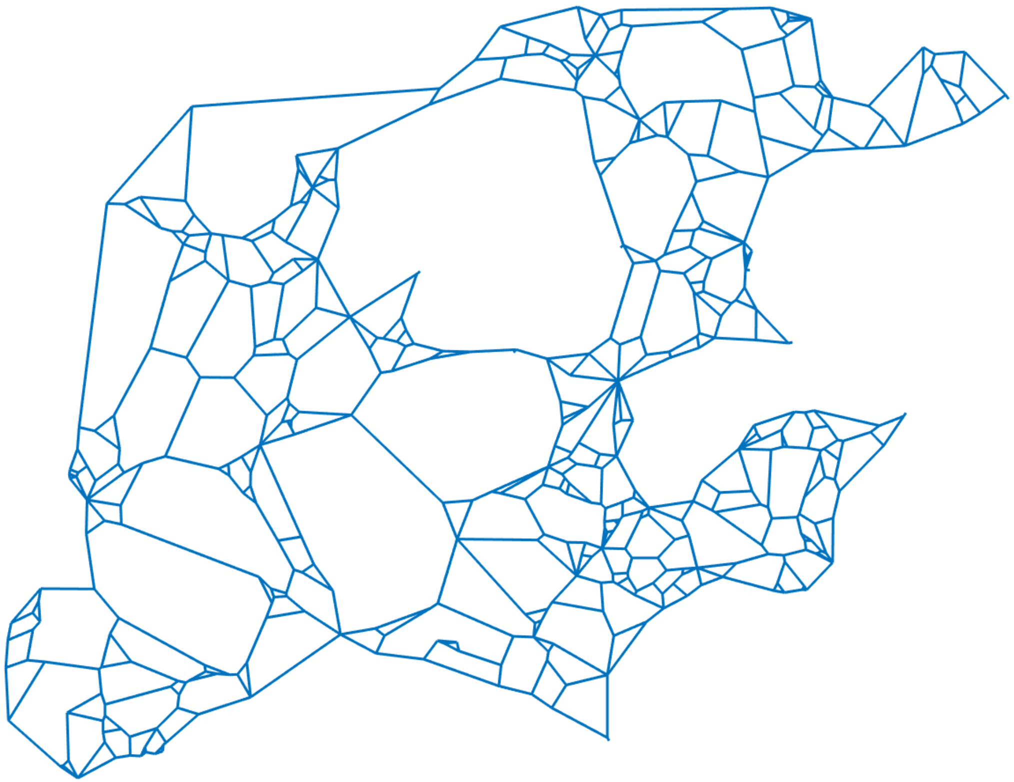


$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$

Euler characteristic

$$\chi = \sum_n (-1)^n \beta_n$$

Betti number 1

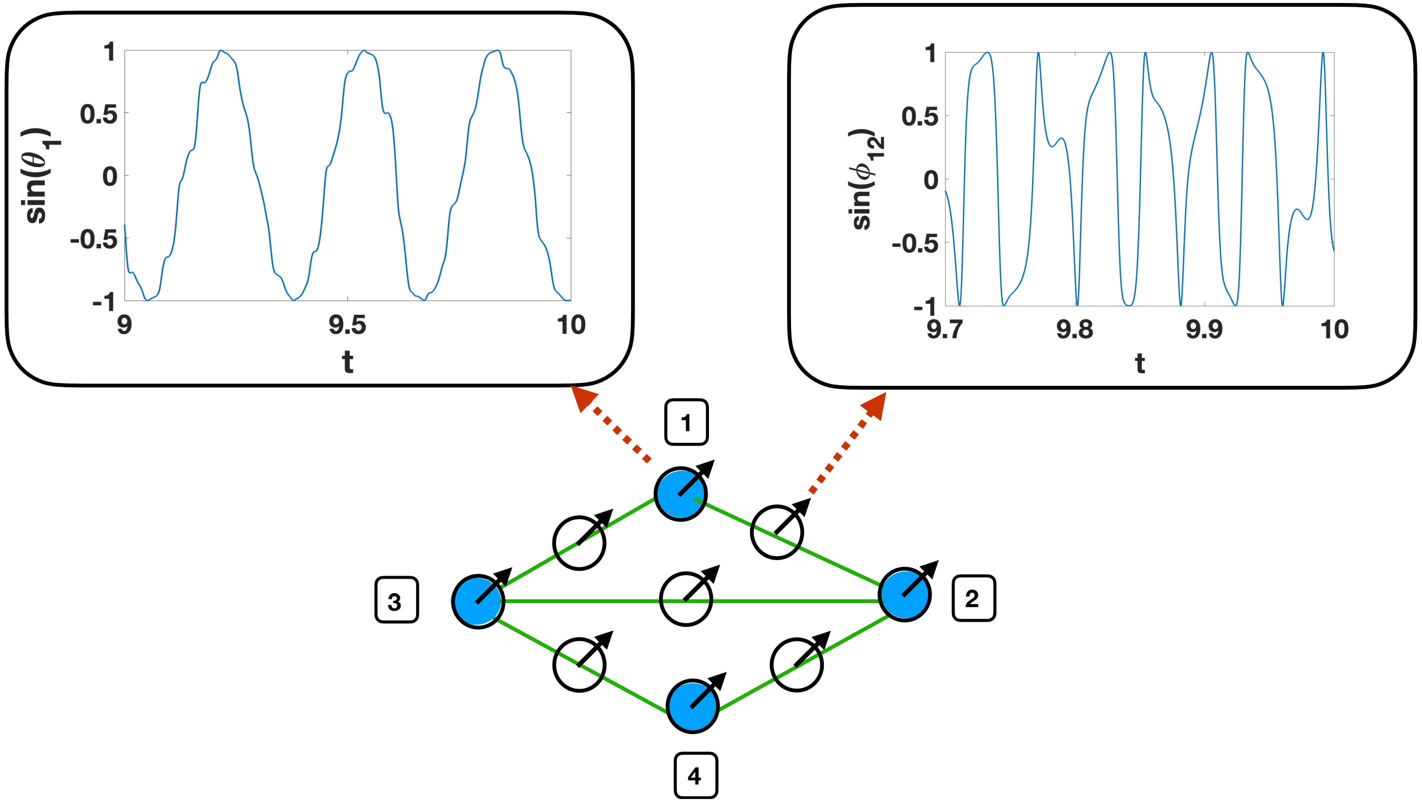


Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

Topological signals,
Hodge Laplacian

Topological signals

Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called *topological signals*



Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are co-chains or vector fields

Graph Laplacian in terms of the boundary matrix

The graph Laplacian of elements

$$\left(L_{[0]}\right)_{ij} = \delta_{ij} k_i - a_{ij}$$

Can be expressed in terms of the 1-boundary matrix

as

$$\boxed{\mathbf{L}_{[0]} = \mathbf{B}_{[1]} \mathbf{B}_{[1]}^{\top} .}$$

Graph Laplacian

The graph Laplacian matrix is defined as

$$L_{ij} = \delta_{ij}k_i - a_{ij}$$

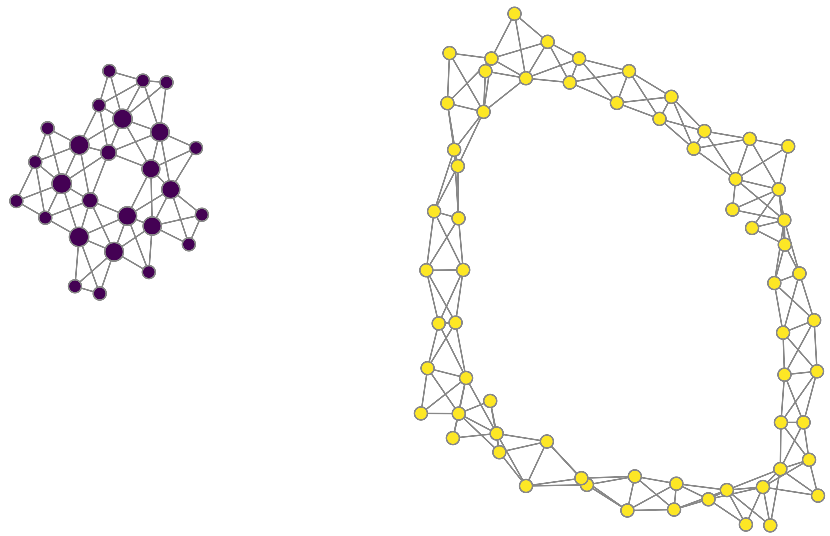
The graph Laplacian is a semi-definite positive matrix that in a connected network has eigenvalues

$$0 = \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_N$$

The Laplacian is key for describing diffusion processes and the Kuramoto model on networks and constitutes a natural link between topology and dynamics

The Fiedler eigenvalue μ_F is the smallest non-zero eigenvalue

Harmonic eigenvectors of the graph Laplacian



The quadratic form of the graph Laplacian reads

$$\mathbf{X}^\top \mathbf{L}_{[0]} \mathbf{X} = \frac{1}{2} \sum_{i,j} a_{ij} (X_i - X_j)^2$$

Therefore the harmonic eigenvectors of the graph Laplacian are constant on each connected component of the graph and zero everywhere else.

Harmonic homology of a graph

- The dimension of kernel of the graph Laplacian is given by the zero Betti number β_0.

- The Betti zero indicates the number of connected components of the graph. Note that since any non-empty graph has at least one connected component we have $\beta_0 \geq 1$.

- The harmonic eigenvectors are the eigenvectors that are constant on each connected component and zero everywhere else.

On a connected graph we have $\beta_0 = 1$ with corresponding harmonic eigenvector $\mathbf{u}^{harm} = \mathscr{C}\mathbf{1}$ where $\mathscr{C} = N^{-1/2}$ is the normalising constant

Higher-order Laplacian

The higher order Laplacians can be defined in terms of the incidence matrices as

$$\boxed{\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]} + \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}\,.}$$

The higher order Laplacian can be decomposed as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up},$$

with

$$\mathbf{L}_{[n]}^{down} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]},$$
$$\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}\,.$$

Fidler eigenvalues of up and down Hodge Laplacians

The up and down Hodge Laplacians
are positive semi-definite matrices that have eigenvalues

$$0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_M$$

There are two Fiedler eigenvalues

$$\mu_F^{down}, \mu_F^{up}$$

being the smallest non-zero eigenvalues of

$\mathbf{L}_{[n]}^{down}$ and $\mathbf{L}_{[n]}^{up}$

respectively

Harmonic eigenvectors of the Hodge Laplacian

The dimension of the kernel of the Hodge Laplacian

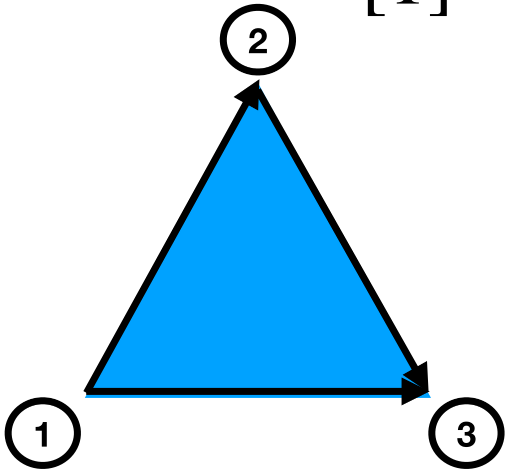
is given by the corresponding Betti number

$$\textbf{dim ker}(\mathbf{L}_{[n]}) = \beta_n$$

The harmonic eigenvectors

are associated to the generators of the homology

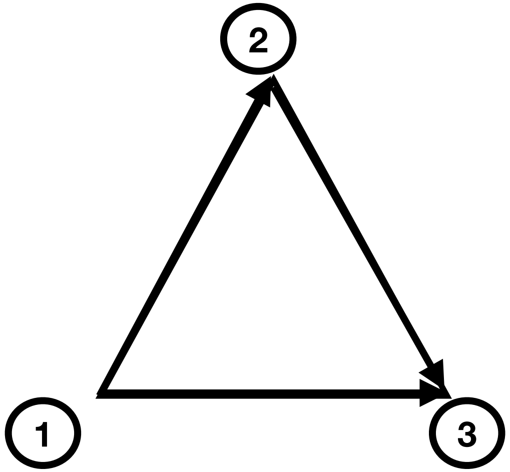
Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian



$$\mathbf{B}_{[1]} = \begin{array}{cccc} & [1,2] & [2,3] & [1,3] \\ [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \quad \mathbf{B}_{[2]} = \begin{array}{cc} & [1,2,3] \\ [1,2] & 1 \\ [2,3] & 1 \\ [1,3] & -1 \end{array}. \quad \mathbf{L}_{[1]} = \begin{array}{cccc} & [1,2] & [1,3] & [1,3] \\ [1,2] & 3 & 0 & 0 \\ [2,3] & 0 & 3 & 0 \\ [1,3] & 0 & 0 & 3 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{cccc} & \mu=3 & \mu=3 & \mu=3 \\ [1,2] & 1 & 0 & 0 \\ [2,3] & 0 & 1 & 0 \\ [1,3] & 0 & 0 & 1 \end{array}, \quad \mathbf{\Sigma}_{[1]} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian

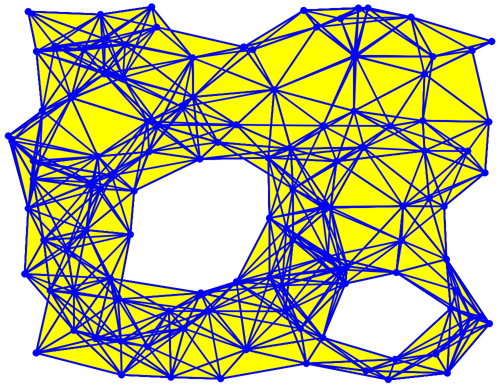


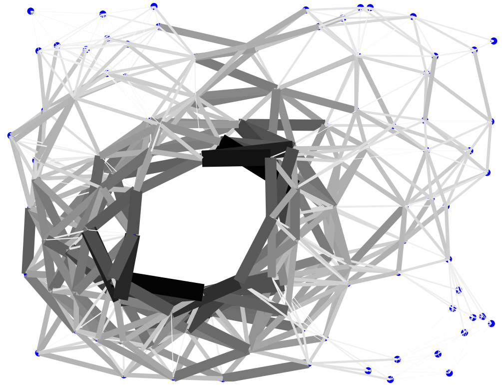
$$\mathbf{B}_{[1]} = \begin{array}{cccc} & [1,2] & [2,3] & [1,3] \\ [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \mathbf{0} \qquad \mathbf{L}_{[1]} = \begin{array}{cccc} & [1,2] & [1,3] & [1,3] \\ [1,2] & 2 & -1 & 1 \\ [2,3] & -1 & 2 & 1 \\ [1,3] & 1 & 1 & 2 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{cccc} & \mu = 0 & \mu = 3 & \mu = 3 \\ [1,2] & 1/\sqrt{3} & 1/\sqrt{2} & 0 \\ [2,3] & 1/\sqrt{3} & 0 & 1/\sqrt{2} \\ [1,3] & -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{2} \end{array}, \qquad \mathbf{\Sigma}_{[1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Visualisation of Hodge Laplacian harmonic eigenvectors

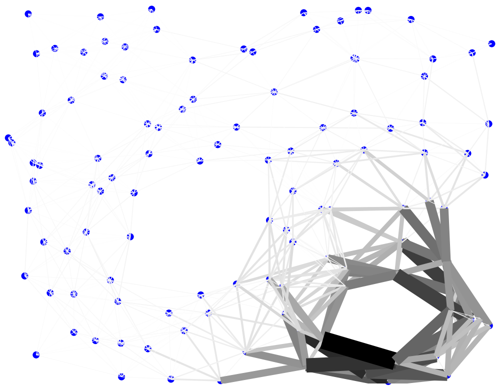
Harmonic eigenvectors localize around the cavities of the simplicial complex



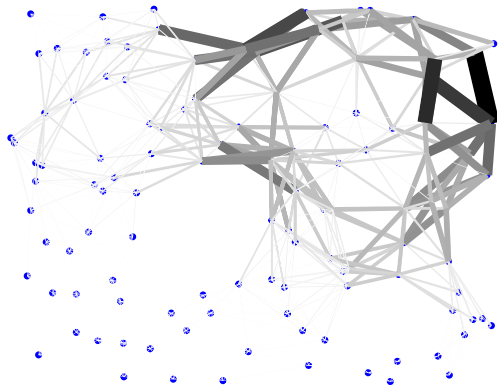


Harmonic eigenvector

Muhammad, A. and Egerstedt, M.,
Control using higher order Laplacians
in network topologies.
In *Proc. of 17th International Symposium on Mathematical Theory of Networks and Systems* (pp. 1024-1038) 2006



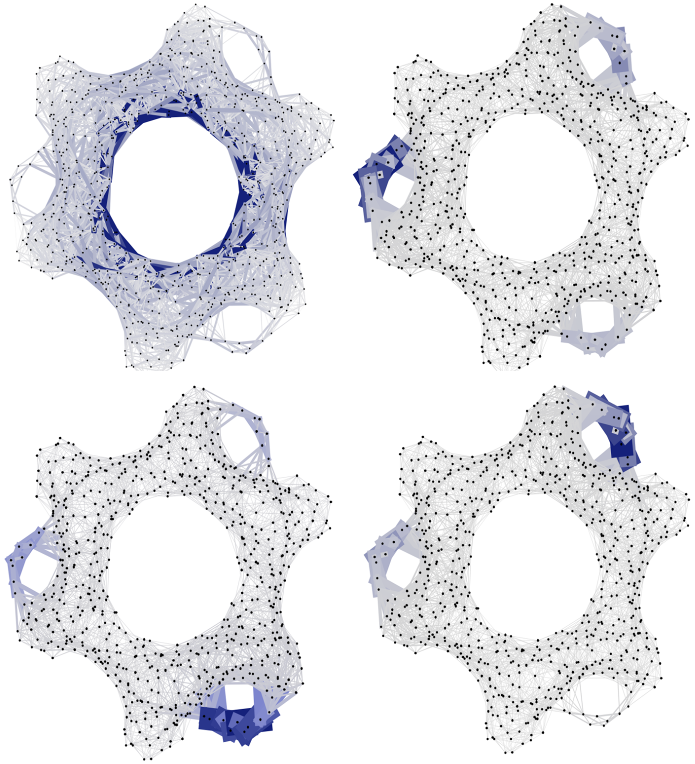
Harmonic eigenvector



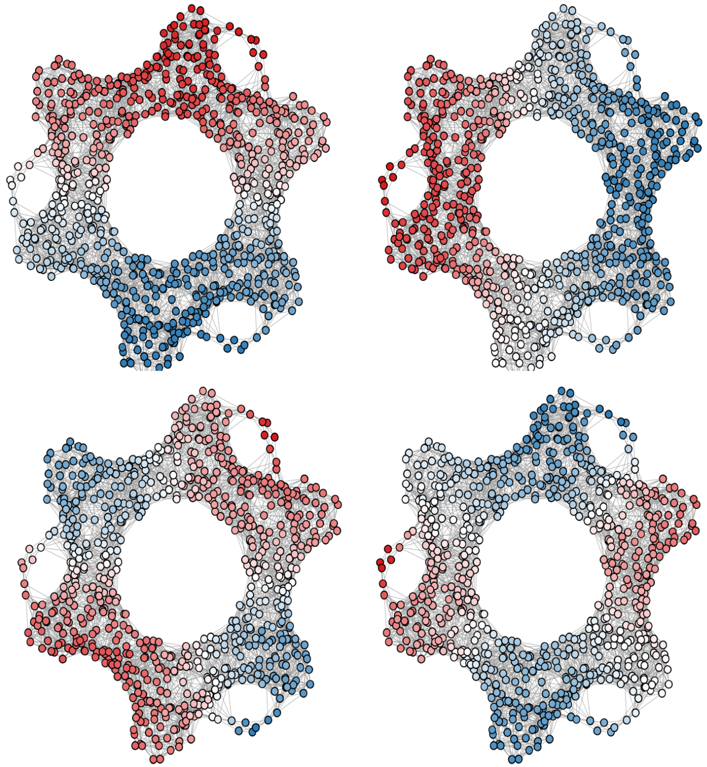
Non-Harmonic eigenvector

Clustering of molecular molecules based on homology

Clustering based on harmonic eigenvectors



Clustering based on non-harmonic eigenvectors



Wee et al. (2023)

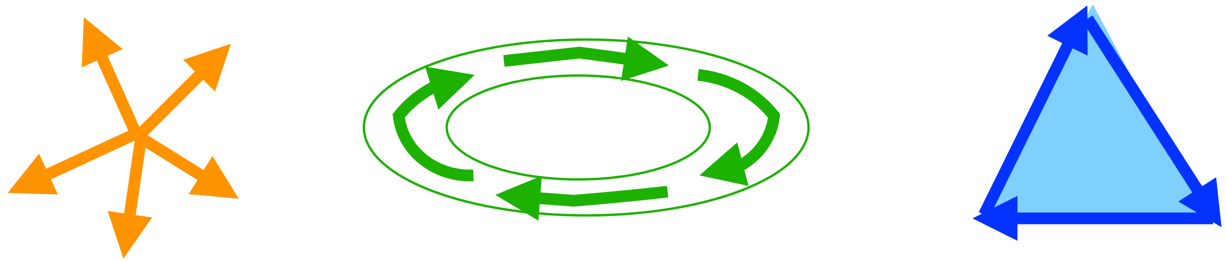
Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

in a irrotational, harmonic and solenoidal components

$$\boxed{\mathbb{R}^{D_n} = \mathbf{im}(\mathbf{B}_{[n]}^\top) \oplus \mathbf{ker}(\mathbf{L}_{[n]}) \oplus \mathbf{im}(\mathbf{B}_{[n+1]})}$$

which in the case of topological signals of the links can be sketched as



Irrotational component
Gradient Flow

Harmonic component

Solenoidal component
Curl Flow

Hodge decomposition

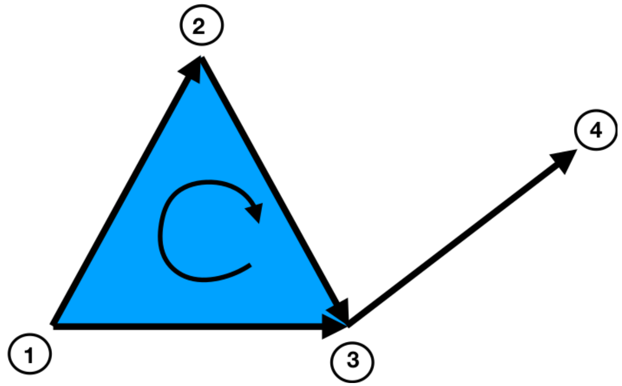
- Every n cochain $\mathbf{x} \in C^n$ can be decoposed in a unique way into three components

$$\boxed{\mathbf{x} = \mathbf{x}^{[1]} + \mathbf{x}^{[2]} + \mathbf{x}^{harm} \qquad \text{With} \qquad \begin{aligned} \mathbf{x}^{[1]} &= \mathbf{L}_{[1]}^{down}(\mathbf{L}_{[1]}^{down})^{+}\mathbf{x} \\ \mathbf{x}^{[2]} &= \mathbf{L}_{[1]}^{up}(\mathbf{L}_{[1]}^{up})^{+}\mathbf{x} \end{aligned}}$$

- where for $n = 1$ we have that $\mathbf{x}^{[1]}, \mathbf{x}^{[2]}$ are the irrotational and solenoidal components respectively and $\mathbf{x}^{harm}$ is the harmonic component.

- Note that in the above formula $\mathbf{A}^{+}$ indicates the pseudo-inverse of the matrix $\mathbf{A}$

Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

$\mathbf{B}_{[1]}$	**Discrete divergence**
$\mathbf{B}_{[1]}^{\top}$	**Discrete gradient**
$\mathbf{B}_{[2]}^{\top}$	**Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}, \quad \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n-1]}^{\top} = \mathbf{0}$$

Higher-order
Diffusion of topological signals

Heat diffusion on a graph

Given a graph G and a node signal $\mathbf{x} \in C^0$ defined on it, the heat diffusion process $\mathbf{x} = \mathbf{x}(t)$ obeys

$$\dot{\mathbf{x}} = -\mathbf{L}_{[0]}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

- This process characterises the relaxation of the dynamics toward the harmonic (constant) eigenvector of the graph Laplacian
- This process is important in many applications and its relevance is also enhanced by the fact that it provides a linearised dynamics of more complex nonlinear processes

Heat diffusion on a graph

Consider the heat diffusion process

$$\dot{\mathbf{x}} = -\mathbf{L}_{[0]}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

Where $\mathbf{x}(t) = \sum_{\mu} c_\mu(t)\mathbf{u}_\mu$ is decomposed into the eigenvectors of the graph Laplacian.

Expressing the heat diffusion equation into the basis of the eigenvector of the graph Laplacian we obtain

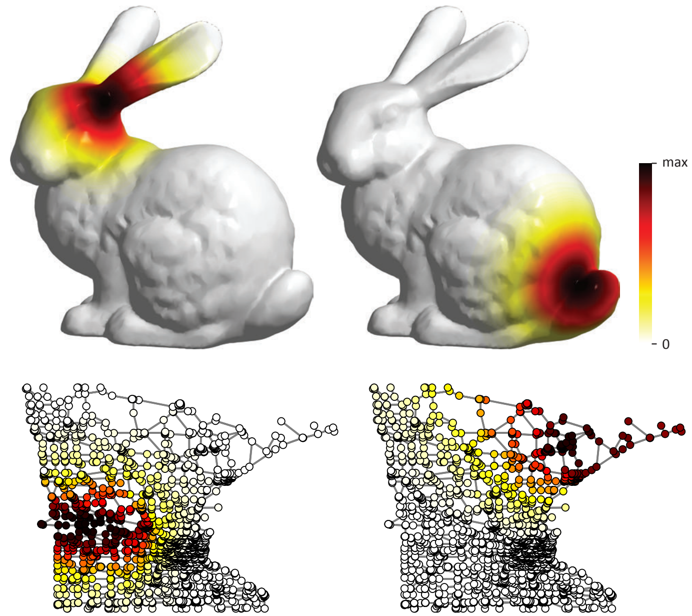
$\dot{c}_\mu = -\mu c_\mu$ with solution $c_\mu(t) = c_\mu(0)e^{-\mu t}$

Heat diffusion on a graph

Assume that $\mathbf{x}_0 = \mathbf{e}_i$ is localised on node i then we have $c_\mu(0) = u_\mu(i)$

$$x_j(t) = \sum_\mu e^{-\mu t} u_\mu(i) u_\mu(j) = h_t(i,j)$$

Which is called the heat kernel.



In the figure visualisation of heat kernels

Bronstein, et.al, P., 2017. Geometric deep learning: going beyond euclidean data. *IEEE Signal Processing Magazine*, *34*(4), pp.18-42.

Heat diffusion on a graph

Given the dynamics

$$x_j(t) = \sum_\mu e^{-\mu t} u_\mu(i) u_\mu(j)$$

The relaxation is toward to homogenous state

$$\lim_{t\to\infty} x_j(t) = u_0(i)u_0(j)$$

since $\mathbf{u}_0 \propto \mathbf{1}$ this limit does not depend on j!

If there is a spectral gap,

the characteristic temporal scale for the relaxation to equilibrium is given

$$\tau = 1/\mu_F$$

Diffusion of topological signals

Given a simplicial complex $\mathscr{K}$ and a n-cochain $\mathbf{x} \in C^n$ we define the higher-order diffusion as

$$\dot{\mathbf{x}} = -\mathbf{L}_{[n]}\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Diffusion of topological signals

According to Hodge decomposition we can decompose the signal as

$$\mathbf{x} = \mathbf{x}^{[1]} + \mathbf{x}^{[2]} + \mathbf{x}^{harm}$$

where

$$\mathbf{L}_{[n]}^{up}\mathbf{x}^{[1]} = \mathbf{L}_{[n]}^{down}\mathbf{x}^{[2]} = \mathbf{L}_{[n]}\mathbf{x}^{harm} = \mathbf{0}$$

The diffusion dynamics $\dot{\mathbf{x}} = -\mathbf{L}_{[n]}\mathbf{x}$ therefore reads for these components

$$\dot{\mathbf{x}}^{harm} = \mathbf{0} \qquad \mathbf{x}^{harm}(0) = \mathbf{x}_0^{harm}$$

$$\dot{\mathbf{x}}^{[1]} = -\mathbf{L}_{[n]}^{down}\mathbf{x}^{[1]} \qquad \mathbf{x}^{[1]}(0) = \mathbf{x}_0^{[1]}$$

$$\dot{\mathbf{x}}^{[2]} = -\mathbf{L}_{[n]}^{up}\mathbf{x}^{[2]} \qquad \mathbf{x}^{[2]}(0) = \mathbf{x}_0^{[2]}$$

Higher-order diffusion

Higher-order diffusion relaxes to the harmonic eigenvectors

In presence of a spectral gap

in the spectrum of $\mathbf{L}_{[n]}^{down}$ and of $\mathbf{L}_{[n]}^{up}$

the irrotational and the solenoidal components

relax to the steady state with characteristic scale

$$\tau^{[1]} = 1/\mu_F^{down}, \tau^{[2]} = 1/\mu_F^{up}$$

correspondingly.

1-Laplacian flow

1-Laplacian flow
stabilises to the
single non-trivial co-homology
eigenvector
on this simplicial complex

Muhammad, A. and Egerstedt, M.,
Control using higher order Laplacians
in network topologies.
In *Proc. of 17th International Symposium on*
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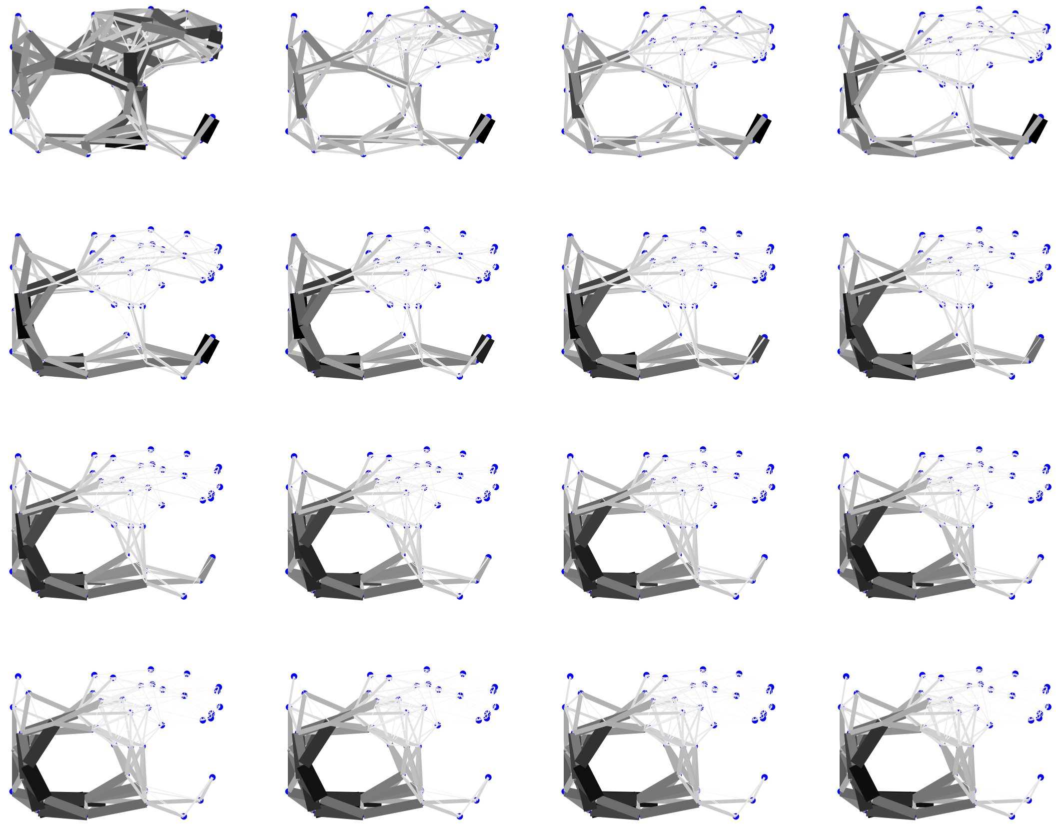


Properties of higher-order diffusion

Higher-order diffusion stabilises on the homological eigenvectors

- The homological eigenvectors are localised on holes
- The Betti number can be zero or greater than one.

Therefore a steady state is reached only if the Betti number is positive.

In presence of more than one hole the stabilisation of the flow on one more more holes will depend on the initial condition

We distinguish two Fiedler eigenvalues,

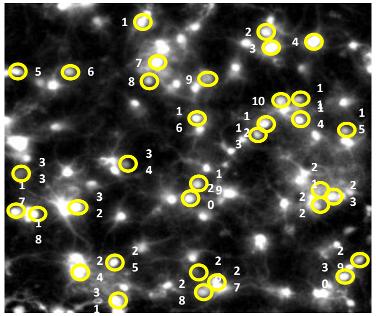
one for the up one for the down Hodge Laplacian

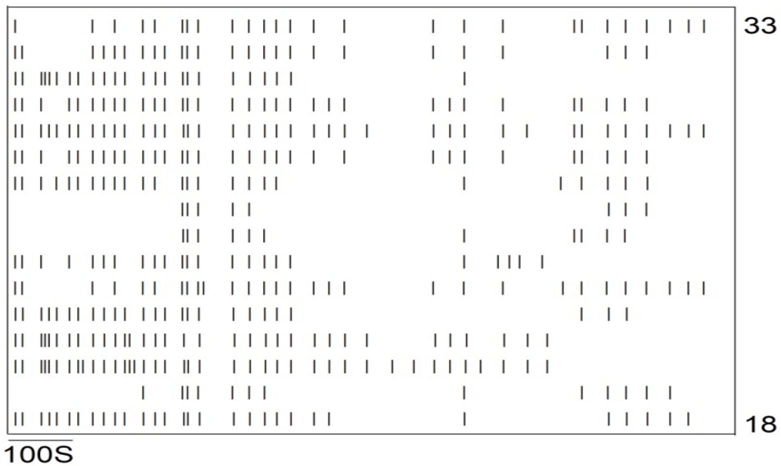
In presence of a spectral gap these Fiedler eigenvalues characterise the characteristic scale of relaxation of the irrotational and solenoidal component correspondigly

Kuramoto model on a graph

Synchronization is a fundamental dynamical process

NEURONS

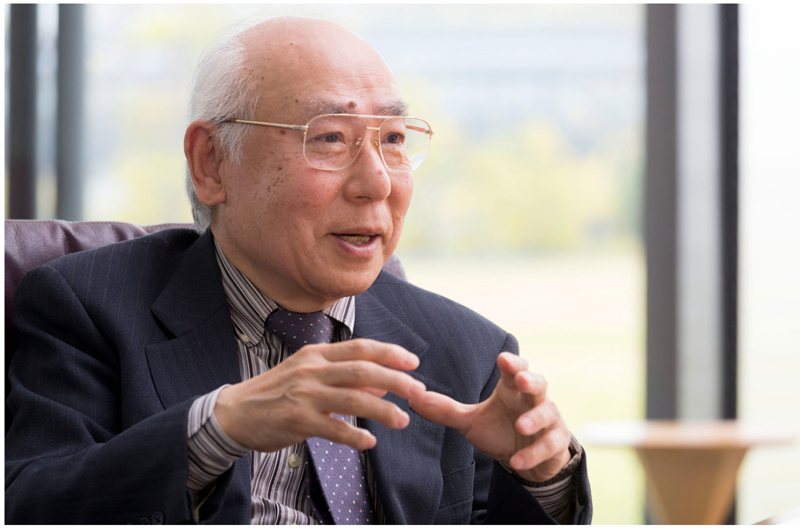




FIREFLIES

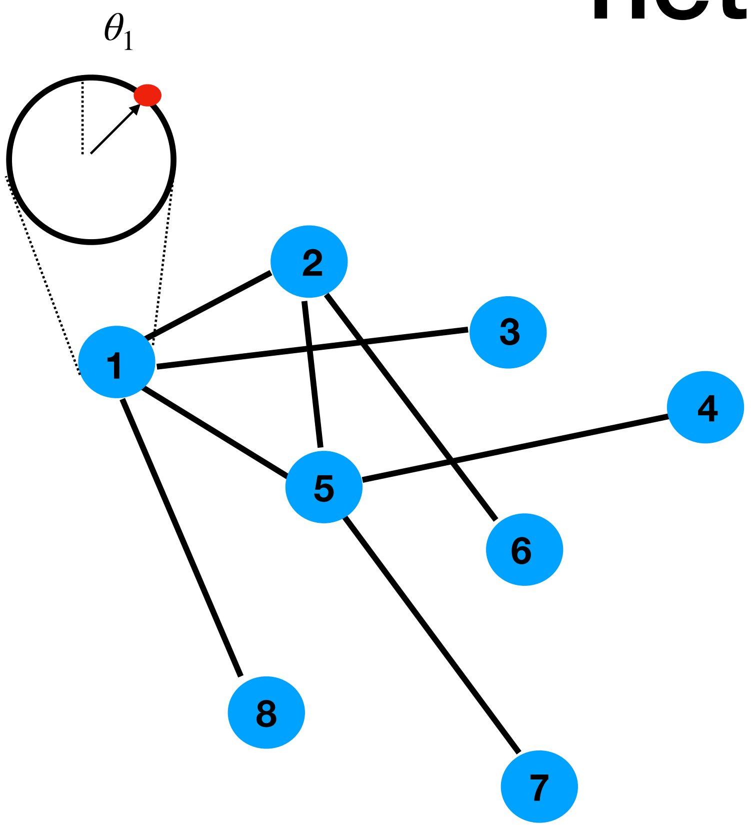
Founding fathers of synchronisation

Christiaan Huygens



Yoshiki Kuramoto

Kuramoto model on a network



Given a network of N nodes defined by an adjacency matrix a we assign to each node a phase obeying

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right)$$

where the internal frequencies of the nodes are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

The oscillators are non-identical

Order parameter for synchronization

We consider the global order parameter R

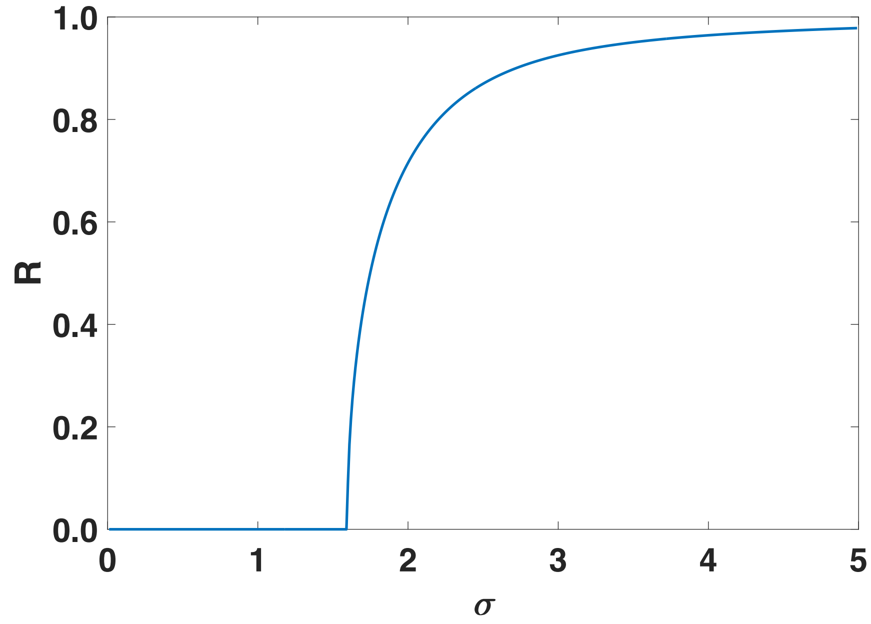
$$R = \frac{1}{N}\left|\sum_{i=1}^{N} e^{\mathrm{i}\,\theta_i}\right|$$

which indicates the

synchronisation transition such that for

$$|\sigma - \sigma_c| \ll 1$$

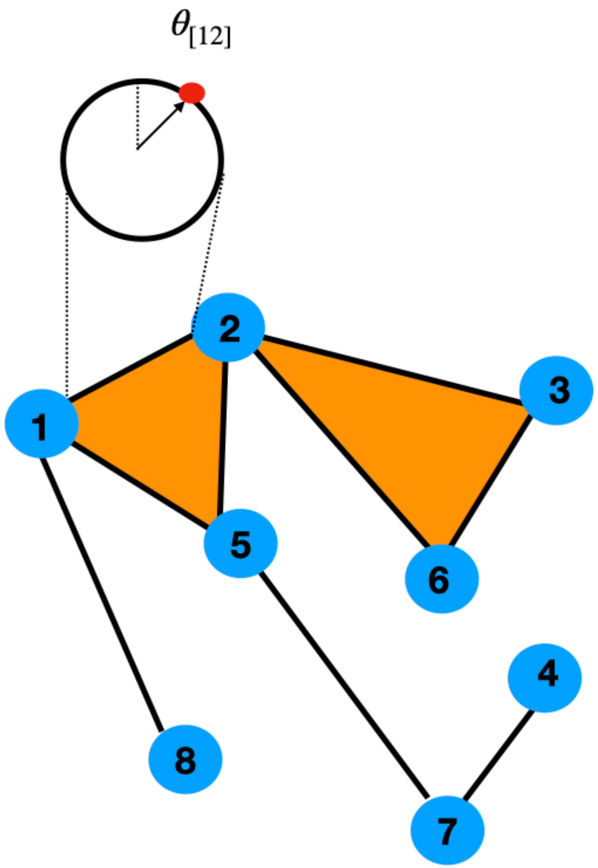
$$R = \begin{cases} 0 & \textbf{for } \sigma < \sigma_c \\ c(\sigma - \sigma_c)^{1/2} & \textbf{for } \sigma \geq \sigma_c \end{cases}$$



Kuramoto (1975)

Topological Kuramoto model on simplicial complexes

The higher-order simplicial Kuramoto model

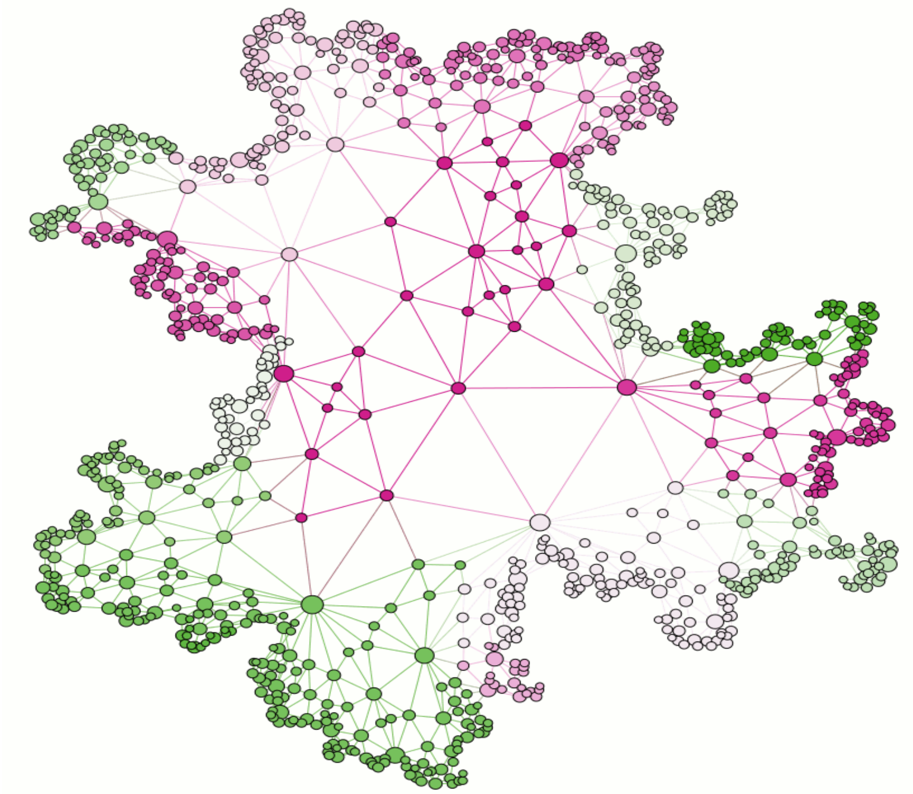


How to define the higher-order Kuramoto model coupling higher dimensional topological signals?

A. P. Millán, J. J. Torres, and G.Bianconi,
***Physical Review Letters*, *124*, 218301 (2020)**

Topological signals

Simplicial complexes can sustain dynamical variables (signals)
not only defined on nodes but also defined on higher order simplices
these signals are called *topological signals*



Standard Kuramoto model in terms of boundary matrices

The standard Kuramoto model, can be expressed in terms

of the boundary matrix $\mathbf{B}_{[1]}$ as

$$\boxed{\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

The standard Kuramoto model in terms of boundary matrices

Let us show that the Kuramoto equations

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right)$$

can be also written in matrix form as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using the explicit expression of the elements of the boundary matrix $\mathbf{B}_{[1]}$

$$[B_{[1]}]_{i\ell} = \begin{cases} -1 & \text{if } \ell = [i,j] \\ 1 & \text{if } \ell = [j,i] \\ 0 & \text{otherwise} \end{cases}$$

Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_i = \omega_i - \sigma \sum_{\ell} [B_{[1]}]_{i\ell} \sin \left(\sum_j [B_{[1]}]_{\ell j} \theta_j \right)$$

For the link $\ell = [i, j]$ we obtain

$$[B_{[1]}]_{i\ell} \sin \left(\sum_j [B_{[1]}]_{\ell j} \theta_j \right) = - a_{ij} \sin(\theta_j - \theta_i)$$

Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_i = \omega_i - \sigma \sum_{\ell} [B_{[1]}]_{i\ell} \sin \left(\sum_{j} [B_{[1]}]_{\ell j} \theta_j \right)$$

For the link $\ell = [j, i]$ we obtain

$$[B_{[1]}]_{i\ell} \sin \left(\sum_{j} [B_{[1]}]_{\ell j} \theta_j \right) = a_{ij} \sin(\theta_i - \theta_j) = - a_{ij} \sin(\theta_j - \theta_i)$$

Linearised dynamics

Let us study the linearisation of the Kuramoto dynamics.

Let us start from the nonlinear system

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using $\sin \mathbf{x} \simeq \mathbf{x}$ we get the linearised dynamics

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{L}_{[0]} \boldsymbol{\theta}$$

Linearised Dynamics

The linearised dynamics is dictated by the graph

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{L}_{[0]} \boldsymbol{\theta} \,.$$

The phases and the intrinsic frequencies can be decomposed in the basis of the eigenvectors of the graph Laplacian

$$\boldsymbol{\theta}(t) = \sum_{\mu} c_{\mu}(t) \mathbf{u}_{\mu}$$

$$\boldsymbol{\omega} = \sum_{\mu} \omega_{\mu} \mathbf{u}_{\mu}$$

The dynamical equation in this basis reduce to

$$\dot{c}_{\mu} = \omega_{\mu} - \sigma \mu c_{\mu}$$

Linearised Dynamics (continuation)

The dynamical equations

$$\dot{c}_\mu = \omega_\mu - \sigma\mu c_\mu$$

have solution

$$c_{harm}(t) = c_{harm}(0) + \omega_{harm} t$$
$$c_\mu(t) = \frac{\omega_\mu}{\sigma\mu}\left(1 - e^{-\sigma\mu t}\right) + c_\mu(0) e^{-\sigma\mu t}$$

Therefore the harmonic mode undergoes an unperturbed motion,

while the non-harmonic modes are freezing with time.

The harmonic mode of the non-linear Kuramoto model

Let us now study the full nonlinear Kuramoto equation

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta} \qquad (1)$$

Let us consider the harmonic eigenvector $\mathbf{u}_{harm}^{\top} \propto \mathbf{1}^{T}$ of the graph Laplacian

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]} \mathbf{B}_{[1]}^{\top}.$$

Since the graph Laplacian is symmetric we have $\mathbf{u}_{harm}^{\top} \mathbf{B}_{[1]} = \mathbf{0}$

By multiplying (1) by $\mathbf{u}_{harm}^{\top}$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\theta} \rangle}{dt} = \langle \mathbf{u}_{harm}, \boldsymbol{\omega} \rangle$

Therefore the harmonic mode oscillates at constant frequency also in the nonlinear Kuramoto model.

Topological signals

We associate to each

n-dimensional simplex α a phase ϕ_α

For instance for n=1 we might associate to each link a oscillating flux

The vector of phases is indicated by

$$\boxed{\boldsymbol{\phi} = (\ldots, \phi_\alpha \ldots)^\top}$$

Topological synchronisation

We propose to study the higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},}$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

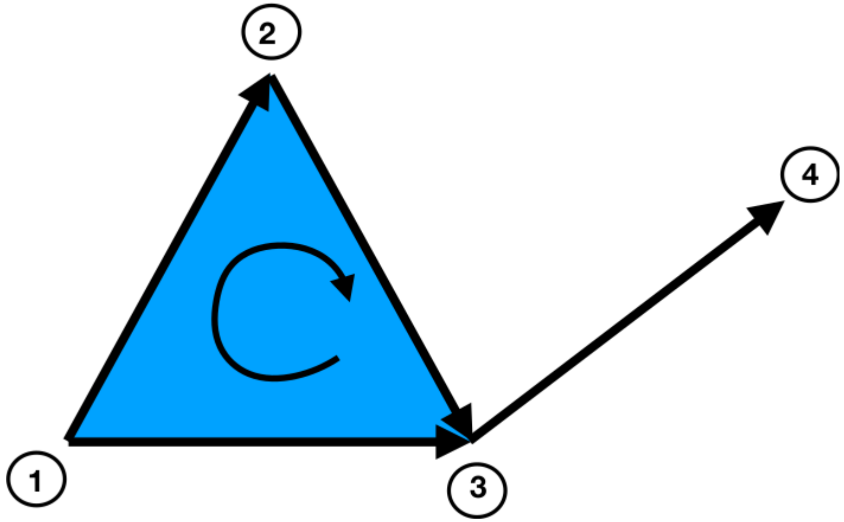
$$\boldsymbol{\phi} = (\ldots, \theta_{\alpha} \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_{\alpha} \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_{\alpha} \sim \mathcal{N}(\Omega, 1)$$

Topologically induced many-body interactions



$$\dot{\phi}_{[12]} = \hat{\omega}_{[12]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[12]} - \phi_{[23]}) + \sin(\phi_{[13]} + \phi_{[12]})\right],$$

$$\dot{\phi}_{[13]} = \hat{\omega}_{[13]} + \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[13]} + \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[23]} = \hat{\omega}_{[23]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[23]} - \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[34]} = \hat{\omega}_{[34]} - \sigma \left[\sin(\phi_{[34]}) - \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

Linearised Dynamics

The linearised dynamics is dictated by the Hodge-Laplacian

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n]} \boldsymbol{\phi},$$

The harmonic component of the signal oscillates freely

The other modes freeze asymptotically in time

In the Topological Kuramoto model the dynamics of the synchronised state is localised on the n-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

The harmonic mode of the non-linear Kuramoto model

Let us now study the full nonlinear Topological Kuramoto equation

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi}, \quad (2)$$

Let us consider any harmonic eigenvector $\mathbf{u}_{harm}^{\top}$ of the Hodge Laplacian

$$\mathbf{L}_{[n]} = \mathbf{B}_{[n+1]} \mathbf{B}_{[n+1]}^{\top} + \mathbf{B}_{[n]}^{\top} \mathbf{B}_{[n]}.$$

Since Hodge decomposition applies $\mathbf{u}_{harm}^{\top} \mathbf{B}_{[n+1]} = \mathbf{u}_{harm}^{\top} \mathbf{B}_{[n]}^{\top} = \mathbf{0}$

By multiplying (2) by $\mathbf{u}_{harm}^{\top}$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$

Therefore the harmonic modes oscillate at constant frequency also in the nonlinear Topological Kuramoto model.

If we define a higher-order Kuramoto model on

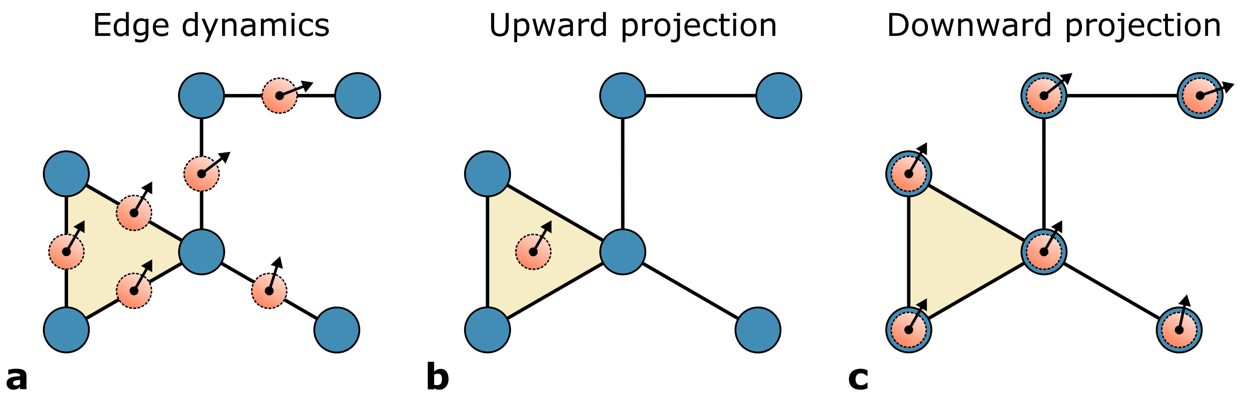
n-simplices,

(let us say links, n=1) a key question is:

What is the dynamics induced

on (n-1) faces and (n+1) faces?

i.e. what is the dynamics induced on nodes and triangles?



Projected dynamics on n-1 and n+1 faces

A natural way to project the dynamics is to use the incidence matrices obtaining

$$\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi}$$ **Discrete curl**

$$\boldsymbol{\phi}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\phi}$$ **Discrete divergence**

Projected dynamics on n-1 and n+1 faces

Thanks to Hodge decomposition,

the projected dynamics

on the (n-1) and (n+1) faces

decouple

$$\boxed{\begin{aligned}\dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma\mathbf{L}_{[n+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]})\\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma\mathbf{L}_{[n-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]})\end{aligned}}$$

Proof

Starting from the Topological Kuramoto dynamics

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},$$

We apply $\mathbf{B}_{[n+1]}^{\top}$ to both sides of the equations getting for $\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi}$

$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[n+1]}^{\top} \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[n+1]}^{\top} \sin \mathbf{B}_{[n+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[n+1]}^{\top} \mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]} \boldsymbol{\phi},$$

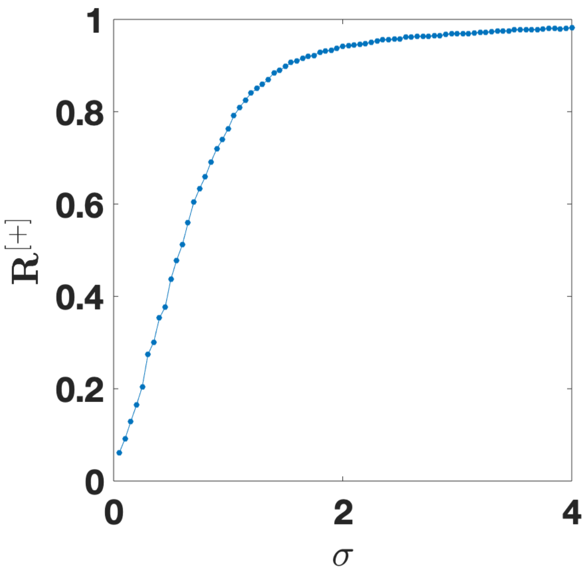
Using $\mathbf{B}_{[n+1]}^{\top} \mathbf{B}_{[n+1]} = \mathbf{L}_{n+1}^{down}, \mathbf{B}_{[n+1]}^{\top} \mathbf{B}_{[n]}^{\top} = \mathbf{0}$ we get

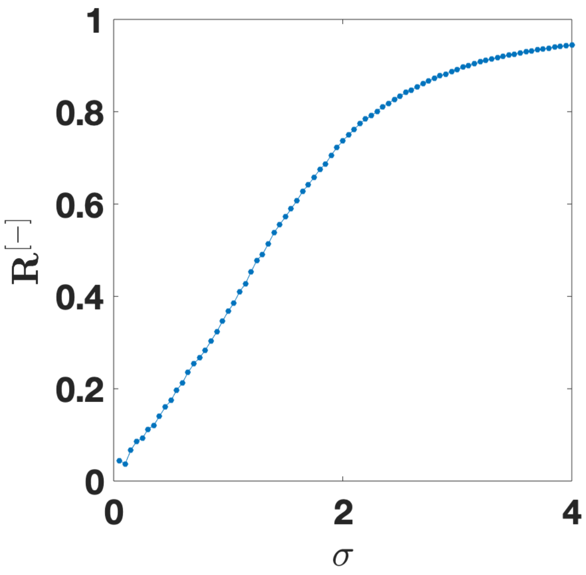
$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[n+1]}^{\top} \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[n+1]}^{down} \sin \boldsymbol{\phi}^{[+]}$$

A similar derivation holds for getting the equation for $\boldsymbol{\phi}^{[-]}$

Simplicial Synchronization transition

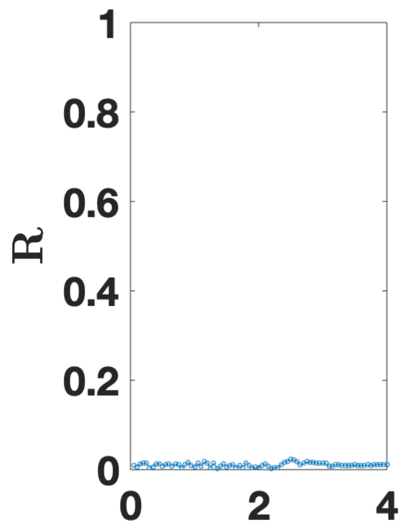
$$R^{[+]} = \frac{1}{N_{n+1}} \left| \sum_{\alpha=1}^{N_{n+1}} e^{i\phi_\alpha^{[+]}} \right| \qquad R^{[-]} = \frac{1}{N_{n-1}} \left| \sum_{\alpha=1}^{N_{n-1}} e^{i\phi_\alpha^{[-]}} \right|$$





Order parameters using the n-dimensional phases

$$\boxed{R = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{i\phi_\alpha}\right|}$$



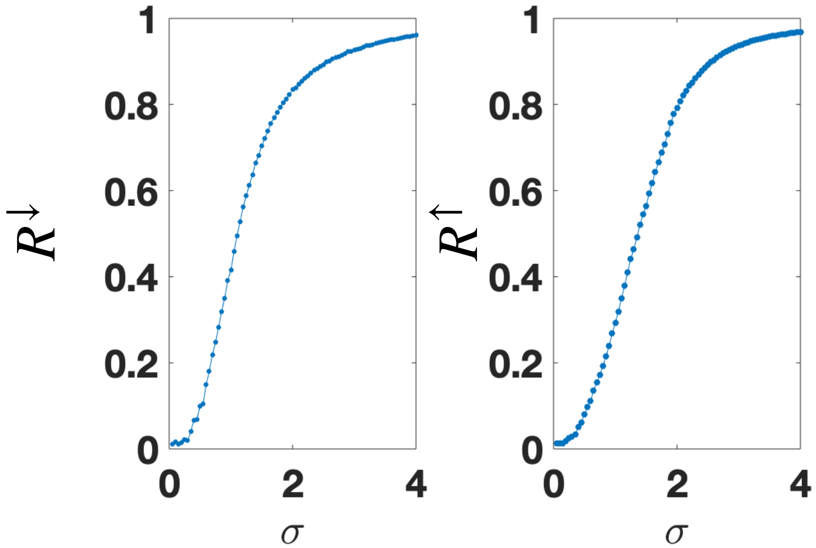
Order parameters using the n-dimensional phases

$$\boldsymbol{\phi}^{\downarrow} = \mathbf{L}_{[n]}^{down}\boldsymbol{\phi}$$

$$\boxed{R^{\downarrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{\mathrm{i}\,\phi_\alpha^{\downarrow}}\right|}$$

$$\boldsymbol{\phi}^{\uparrow} = \mathbf{L}_{[n]}^{up}\boldsymbol{\phi}$$

$$\boxed{R^{\uparrow} = \frac{1}{N_n}\left|\sum_{\alpha=1}^{N_n} e^{\mathrm{i}\,\phi_\alpha^{\uparrow}}\right|}$$



Only if we perform

the correct topological filtering

of the topological signal

we can reveal higher-order topological synchronisation

Explosive topological synchronisation

We propose the Explosive Topological Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\phi} - \sigma R^{[+]}\mathbf{B}_{[n]}^{\top} \sin \mathbf{B}_{[n]}\boldsymbol{\phi}}$$

Projected dynamics

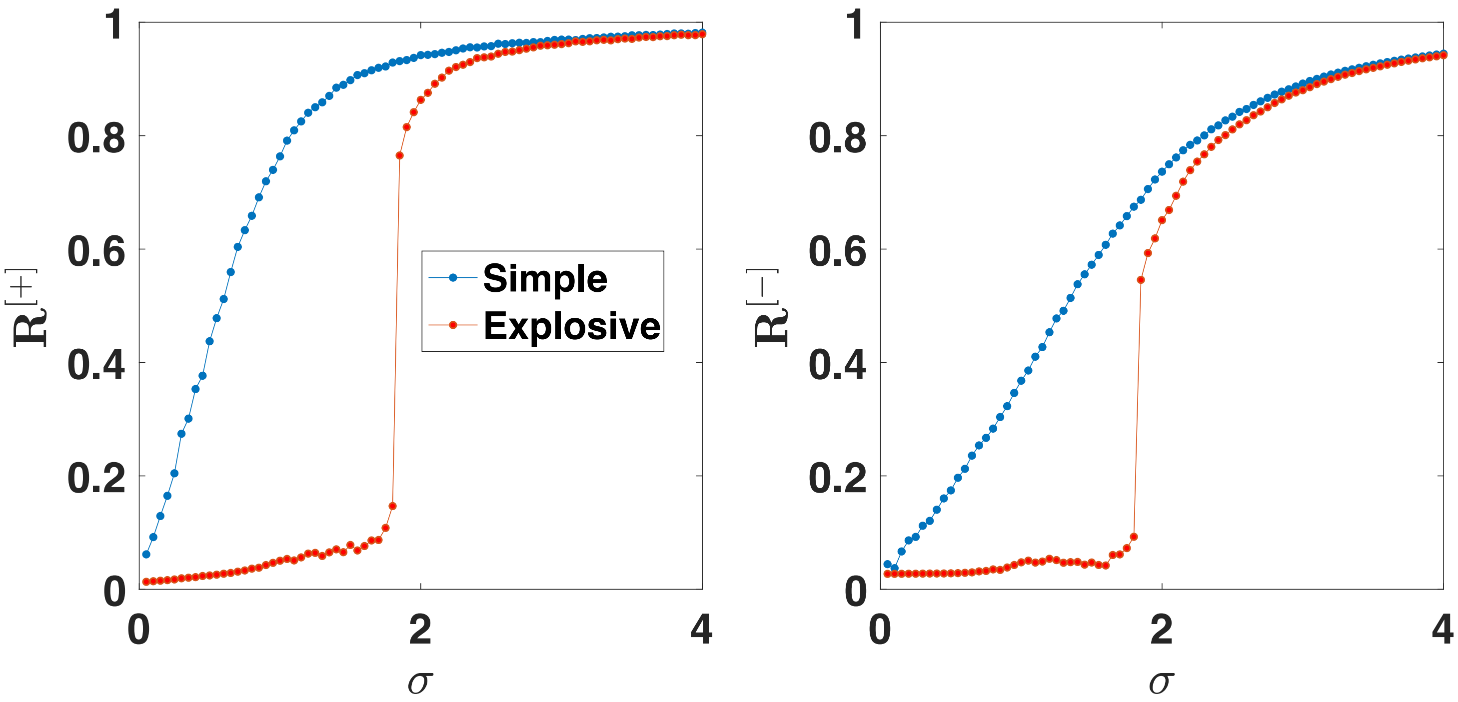
The projected dynamics on

(n+1) and (n-1) are now coupled

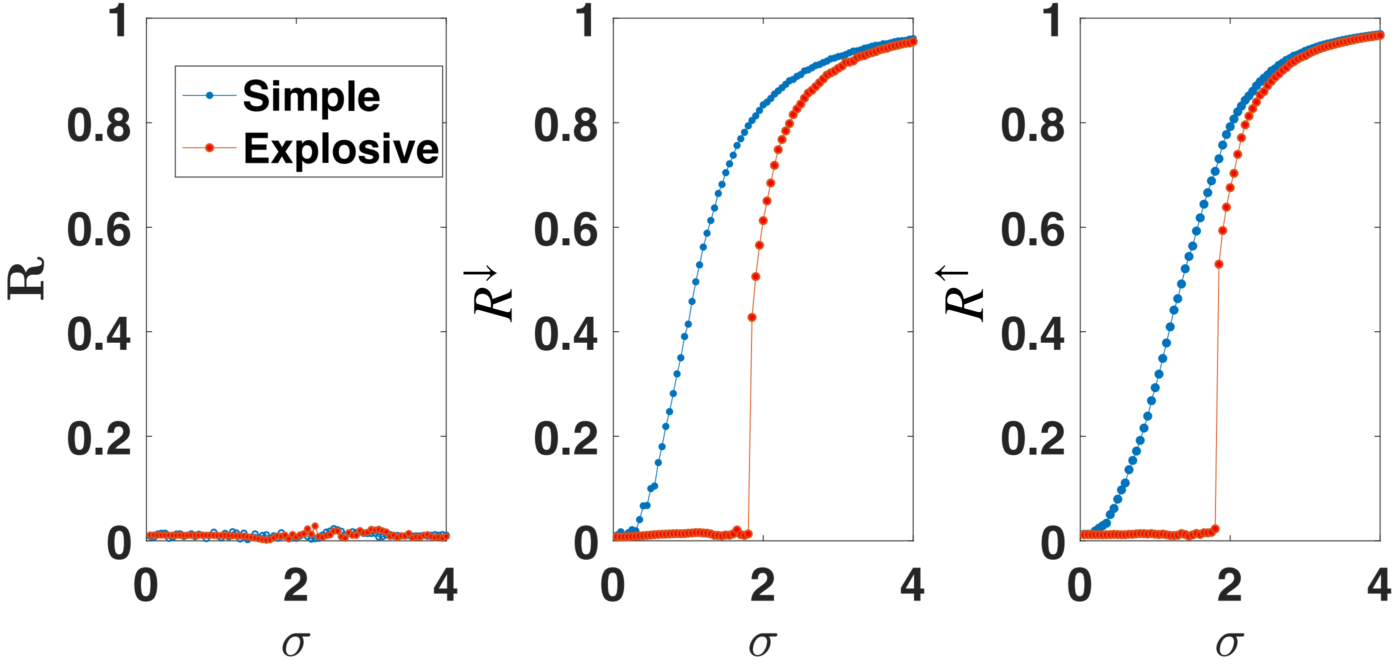
by their order parameters

$$\boxed{\begin{aligned}\dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[n+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{L}_{[n+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]})\\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[n]}\hat{\boldsymbol{\omega}} - \sigma R^{[+]}\mathbf{L}_{[n-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]})\end{aligned}}$$

The explosive simplicial synchronisation transition



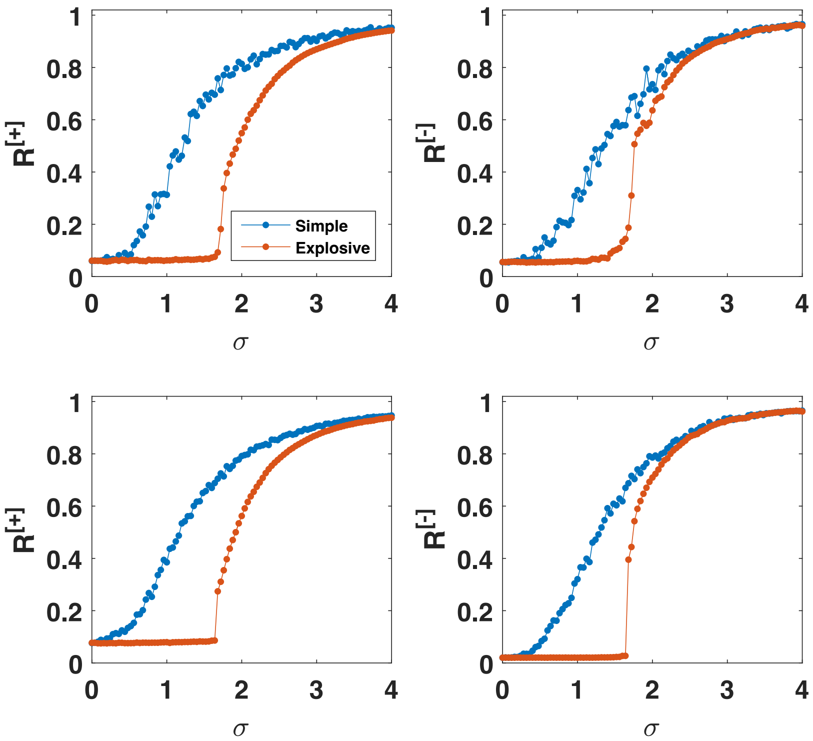
Order parameters associated to n-faces



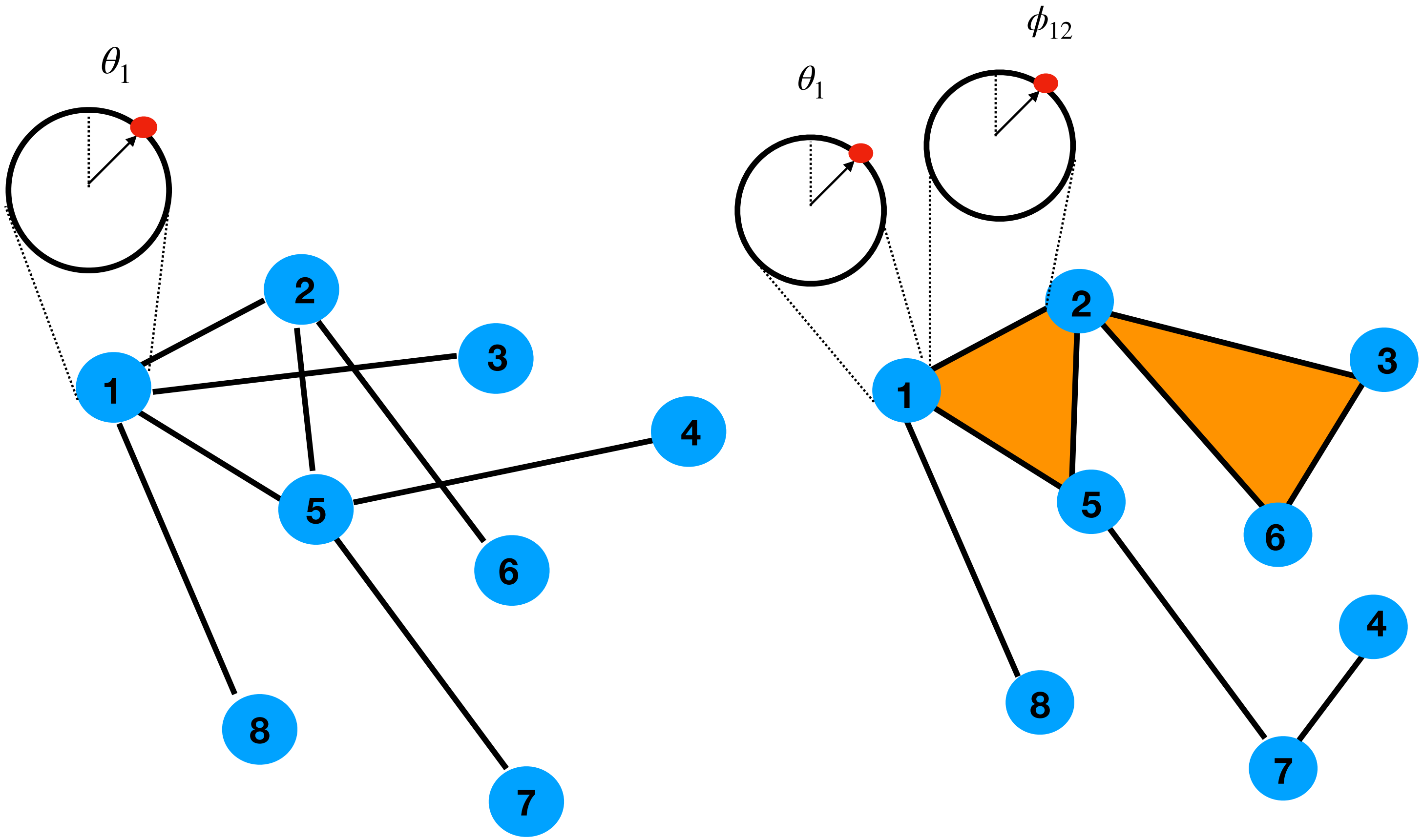
Higher-order synchronisation on real Connectomes

Homo sapiens Connectome

C.elegans Connectome



Coupling topological signals of different dimension



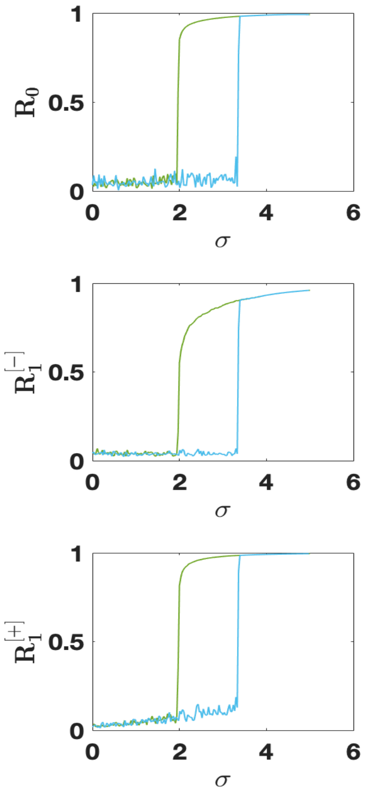
R. Ghorbanchian, J. Restrepo, J.J. Torres and G. Bianconi (2020)

Explosive synchronisation of globally coupled topological signals

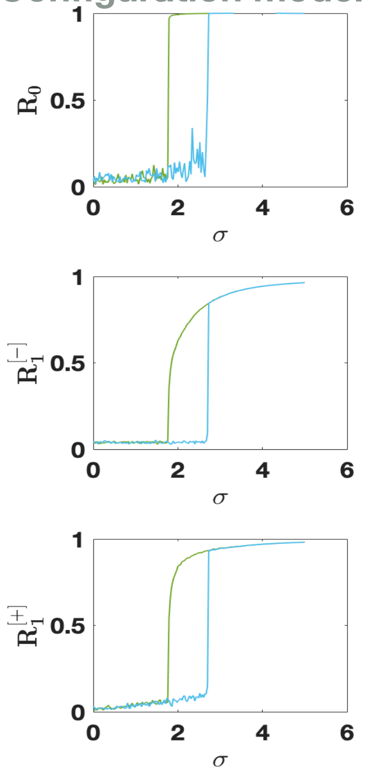
$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]}\mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta}$$

$$\begin{aligned}\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} &- \sigma R_1^{[+]} R_0 \mathbf{B}_{[n]}^\top \sin \mathbf{B}_{[n]} \boldsymbol{\phi} \\ &- \sigma R_1^{[-]} \mathbf{B}_{[n+1]} \sin \mathbf{B}_{[n+1]}^\top \boldsymbol{\phi}\end{aligned}$$

NGF s=-1 d=3



Simplicial Configuration model



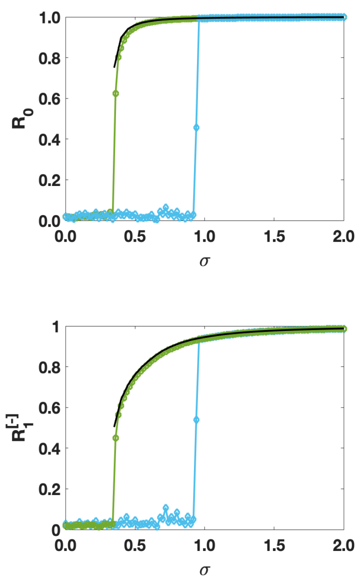
Annealed solution on random networks

The annealed solution captures the backward transition

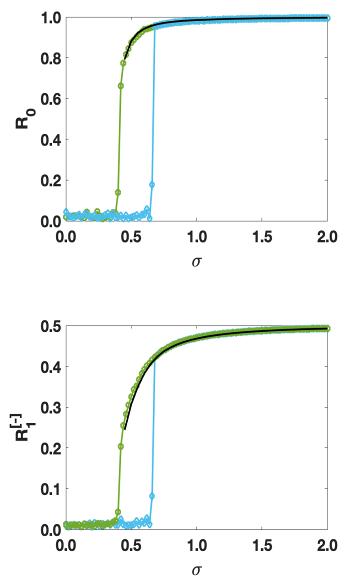
Reveals that the transition is discontinuous

Gives very reliable results for connected networks that are not too sparse

Poisson network



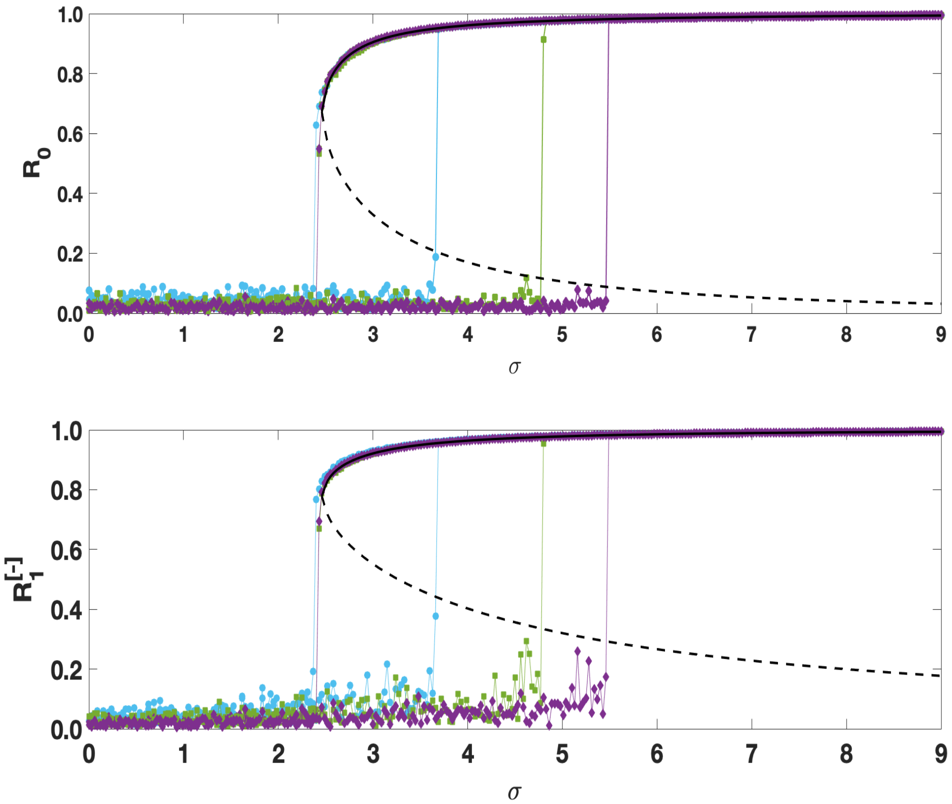
Power-law network



Solution on a fully connected network

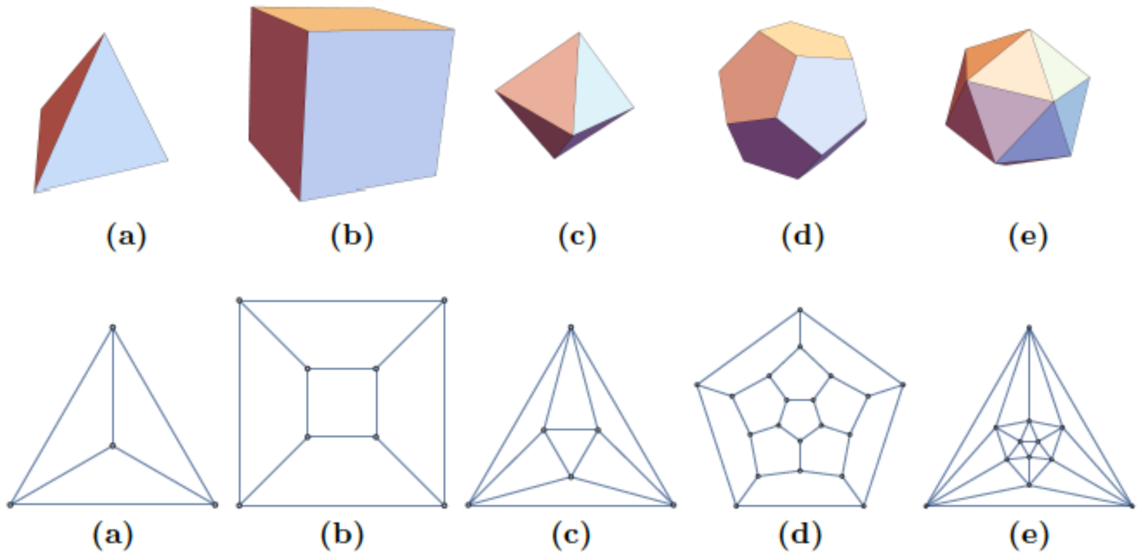
Fully connected networks undergo a discontinuous synchronisation transition of topological signals defined on nodes and links

The hysteresis loop is driven by finite size effects



Global synchronisation of topological signals on simplicial and cell complexes

Cell complexes



A cell complex $\hat{\mathcal{K}}$ has the following two properties:

(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

Global synchronisation on graphs

Uncoupled dynamics of identical node oscillators

Consider coupled identical oscillators defined on the nodes, captured by the 0-cochain $\mathbf{X} \in C^0$ with value $\mathbf{x}_i \in \mathbb{R}^d$ on each node i .

In absence of interactions these nodes obey the same dynamics

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i)$$

with arbitrary non-linear function $\mathbf{f}(\mathbf{x})$.

Global synchronisation on graphs

Consider the coupling of the oscillators implemented with the graph Laplacian leading to the coupled dynamics

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{\beta} [L_{[0]}]_{ij}\, \mathbf{h}(\mathbf{x}_j)$$

with arbitrary non-linear functions $\mathbf{f}(\mathbf{x}), \mathbf{h}(\mathbf{x})$.

The global synchronisation is a state in which

$$\mathbf{x}_i = \mathbf{x}_j \ \forall i, j \in Q_0(\mathcal{K})$$

Global synchronisation state of topological signals

The global synchronisation is a state in which

$$\mathbf{x}_i = \mathbf{x}_j \ \forall i, j \in Q_0(\mathscr{K})$$

The coupled dynamics

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{\beta} [L_{[0]}]_{ij} \ \mathbf{h}(\mathbf{x}_j)$$

admits always a global synchronisation state in which all the node haves the same dynamics.

In fact the harmonic eigenvector of the graph Laplacian is constant

$$\mathbf{u}^{harm} \propto \mathbf{1}$$

Master Stability Function for graphs

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the graph Laplacian.
- It is based on an expansion around a stable solution of the uncoupled dynamics.

Global synchronisation of higher-order topological signals

Uncoupled dynamics of topological signals

Consider coupled identical oscillators defined on the n-simplices, captured by the n-cochain $\mathbf{X} \in C^n$ with $n > 0$ and values $\mathbf{x}_r \in \mathbb{R}^d$ on each n-simplex r .

In absence of interactions these simplices obey the same dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

To insure invariance of the uncoupled equations upon change of orientation of each simplex we must impose that $\mathbf{f}(\mathbf{x})$ is an **odd** function, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Proof

Consider the uncoupled dynamics $$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

Upon change of orientation of the simplex r we have $\mathbf{x}_r \to -\mathbf{x}_r$.

Therefore the dynamics becomes $\frac{d\mathbf{x}_r}{dt} = -\mathbf{f}(-\mathbf{x}_r)$

Imposing invariance of the dynamics under this change of orientation implies that the function $\mathbf{f}(\mathbf{x})$ must be odd, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Coupled identical topological signals

- The coupled dynamics obeys

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[n]}]_{rq}\, \mathbf{h}(\mathbf{x}_q)$$

- where in order to ensure invariance under change of orientation of the simplifies $\mathbf{h}(\mathbf{x})$ should be an **odd** function.

Global synchronisation state of topological signals

Recall that for higher order topological signals, the signs of the signal is determined by the orientation of the simplex, i.e.

$$\mathbf{x}(\alpha_r) = -\mathbf{x}(-\alpha_r)$$

For instance a positive sign of an edge flux is relative to the orientation chosen for that edge.

It follows that the state of global synchronisation is a state in which

$$\mathbf{x}_r = u_r \bar{\mathbf{x}} \text{ with } u_r \in \{1, -1\} \ \forall r \in Q_n(\mathscr{K})$$

Global topological synchronisation

- It follows that the coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_q [L_{[n]}]_{rq} \, \mathbf{h}(\mathbf{x}_q)$$

- can lead to global synchronisation only if the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$ admits an eigenvector $\mathbf{u}$ with elements of constant absolute value.

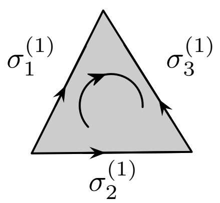
- Therefore for identical higher-order oscillators there are not only dynamical but also topological constraints to global synchronisation

Topological conditions for global synchronisation

- Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.
- Global synchronisation can only happen if there is one such vector $\mathbf{u}$ in the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$.
- Therefore we must have $\mathbf{B}_{[n]}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$
- This implies that:
- **On simplicial complexes topological signals of odd dimension can never achieve global synchronisation**
- **Cell complexes of any dimension can achieve global synchronisation overcoming topological obstruction**

Topological constraints for global synchronisation

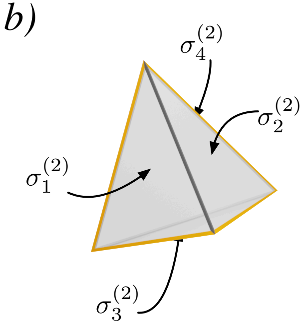
a)



$$\mathbf{B}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(1,1,1)\mathbf{B}_2 = -1 \neq 0$$

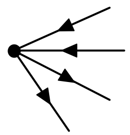
b)



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_3 = 0$$

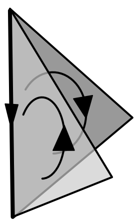
c)



$$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

$$\mathbf{B}_1 \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \\ \vdots \end{pmatrix}$$

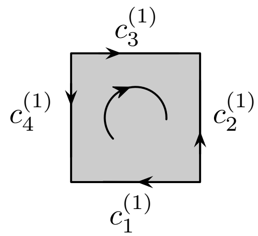
d)



$$\mathbf{B}_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\mathbf{B}_2 \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \\ \vdots \end{pmatrix}$$

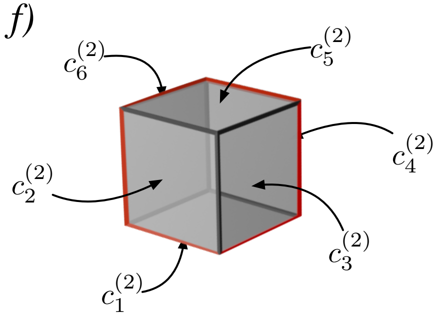
e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_2 = 0$$

f)

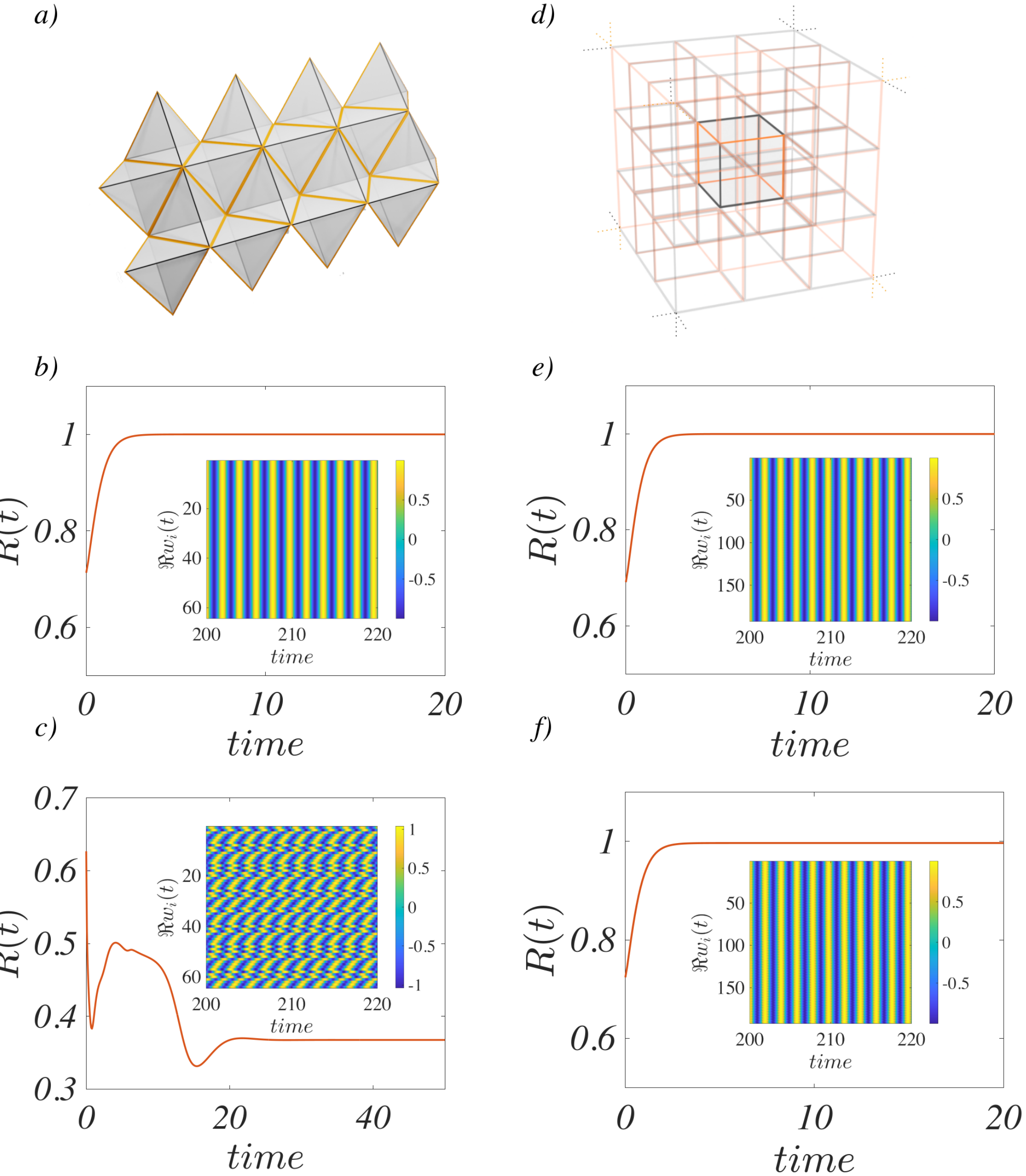


$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$

Master Stability Function for simplicial and cell complexes

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the Hodge Laplacian.
- It should account for the possible degeneracy of the zero eigenvalue (a dimension of the kernel greater than one)
- It is based on an expansion around a stable solution of the uncoupled dynamics.



Global Topological synchronisation

- On cell complexes forming square lattices topological signals of any dimension can achieve global synchronisation
- On simplicial complexes topological signals of odd dimension can never achieve global synchronisation

Carletti, Giambagli, Bianconi (2022)

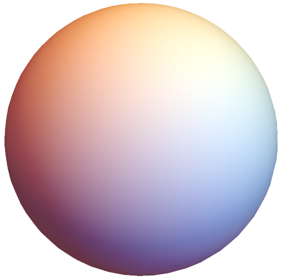
Properties of global synchronisation of topological signals

- The globally synchronised state is aligned with an harmonic eigenvector of the Hodge Laplacian, i.e. **requires topologies with holes that span the entire simplicial or cell complex**.

- Since the Hodge Laplacian has an harmonic space with dimension given by the Betti number, the same simplicial or cell complex can sustain different globalised states (see tori)

Example of manifolds sustaining global synchronisation

Synchronisation of $(n-1)$-dimensional topological signal

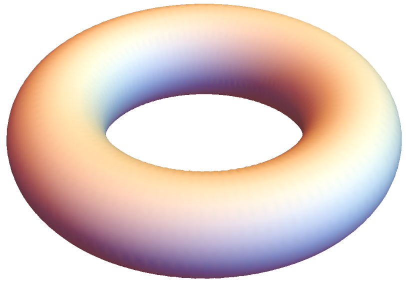


n-dimensional hypersphere

Betti numbers

$$\beta_0 = \beta_{n-1} = 1$$
$$\beta_k = 0 \text{ for } 0 < k < n-1$$

Synchronisation of any k-dimensional topological signal

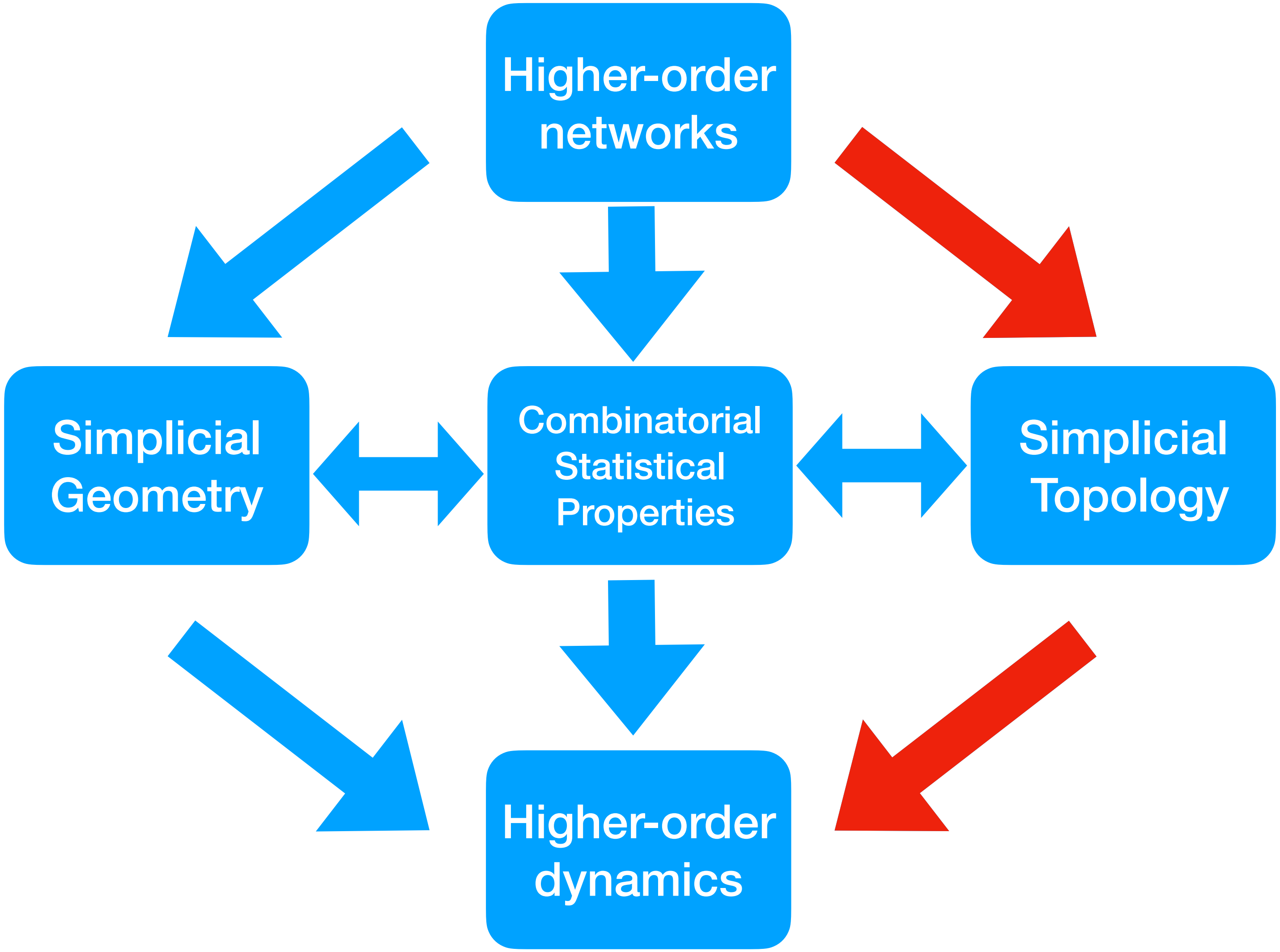


n-dimensional torus (cell complex)

Betti numbers

$$\beta_k = \binom{n-1}{k}$$

Higher-order structure and dynamics



Lesson III:
Topology and higher-order dynamics

- **Spectral properties of the Laplacians**

- **Diffusion**
  - **Heat diffusion on graphs**
  - **Higher-order diffusion of topological signals**

- **Topological Kuramoto model**
  - **The Kuramoto model on graphs**
  - **The Topological Kuramoto model**

- **Global synchronisation of topological signals**
  - **Global synchronisation ongraphs**
  - **Global synchronisation on simplicial and cell complexes**

References

Higher-order topological Kumamoto model
Millán, A.P., Torres, J.J. and Bianconi, G., 2020. Explosive higher-order Kuramoto dynamics on simplicial complexes. *Physical Review Letters*, *124*(21), p.218301.

Globally Coupled dynamics of nodes and links
Ghorbanchian, Reza, Juan G. Restrepo, Joaquín J. Torres, and Ginestra Bianconi. "Higher-order simplicial synchronization of coupled topological signals." *Communications Physics* 4, no. 1 (2021): 1-13.