Higher-order networks

An introduction to simplicial complexes

Lesson IV:

LTCC Course

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Lesson IV:
Higher-order network emergent geometry

- **Introduction to discrete (higher-order) geometry**
  - **Curvature**
  - **Spectral dimension**

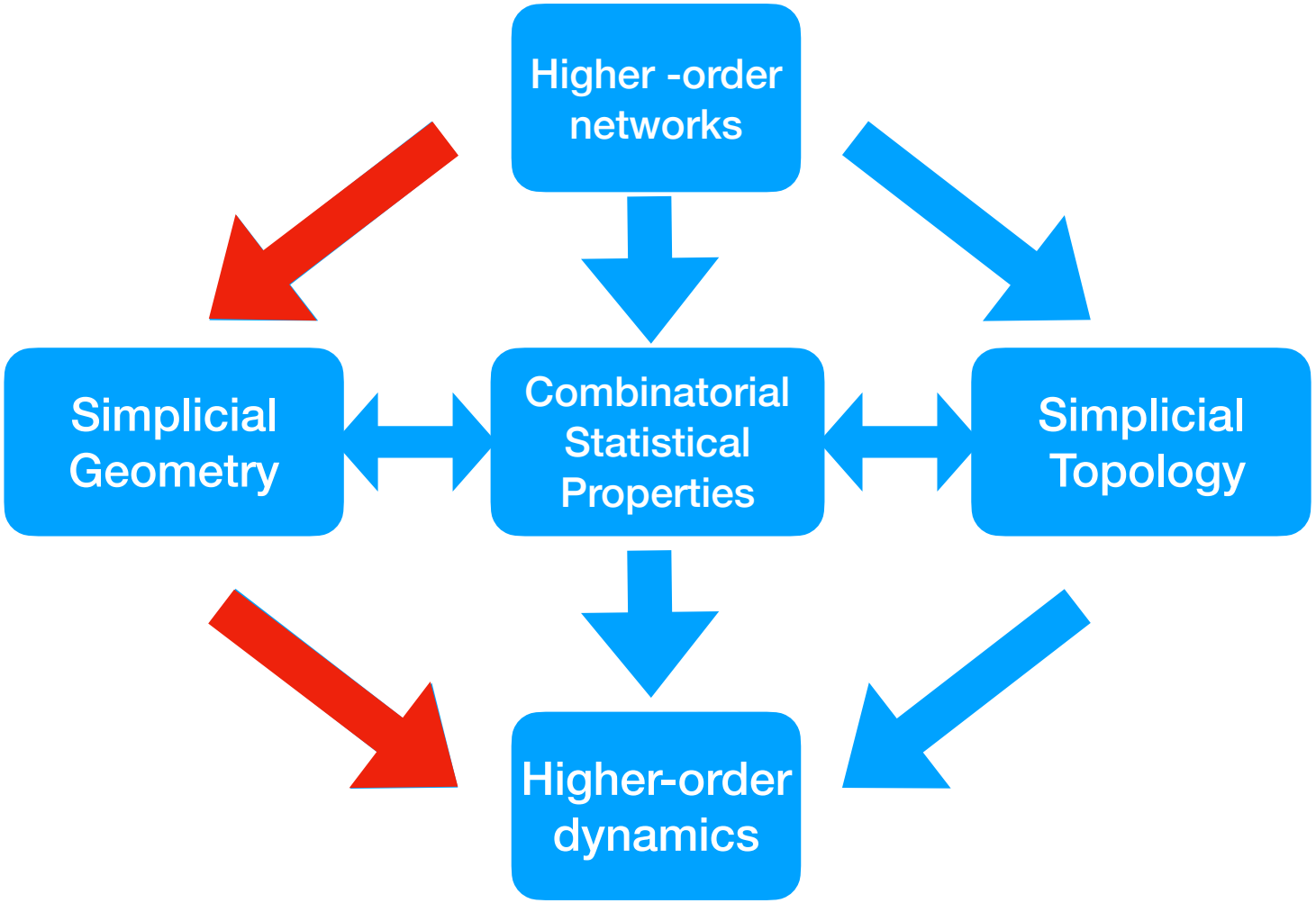
- **Emergent community structure**

- **Emergent geometry**

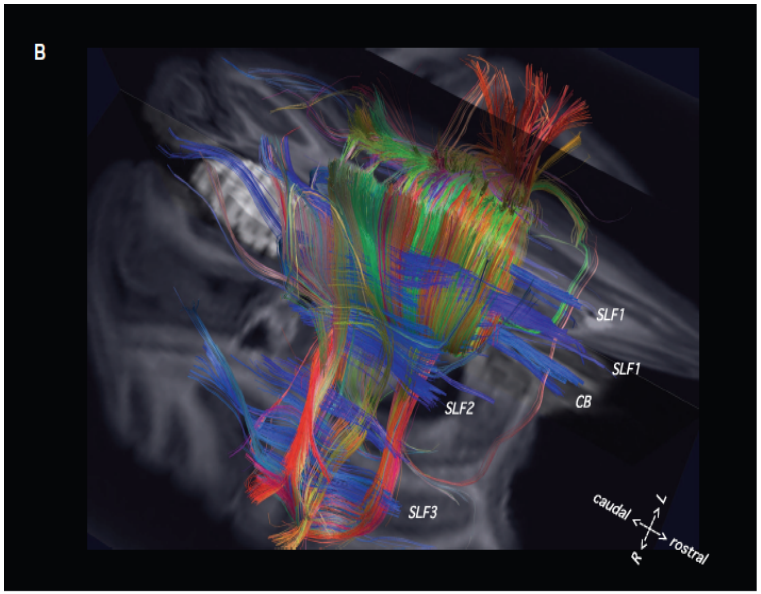
- **Network Geometry with Flavor (NGF)**

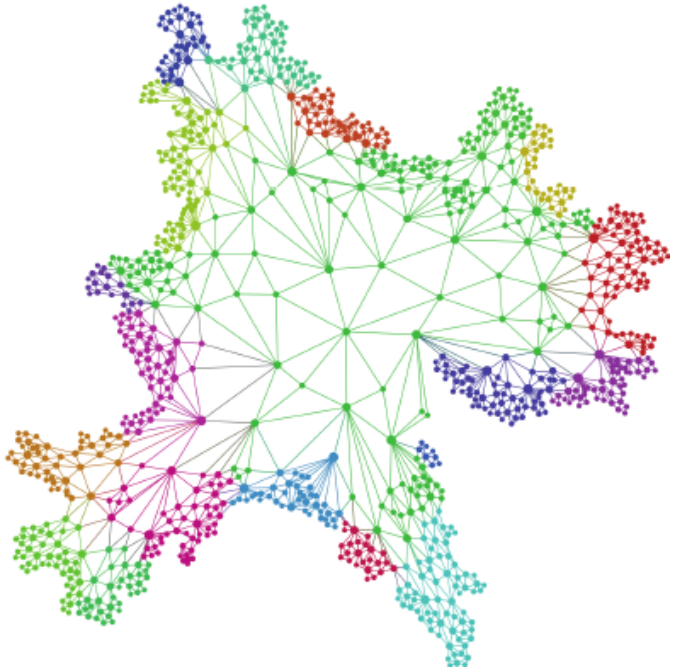
  Emergent Hyperbolic networks
  Emergent quantum statistics

Higher-order structure and dynamics



Network Topology and Geometry





are expected to have impact in a variety of applications,

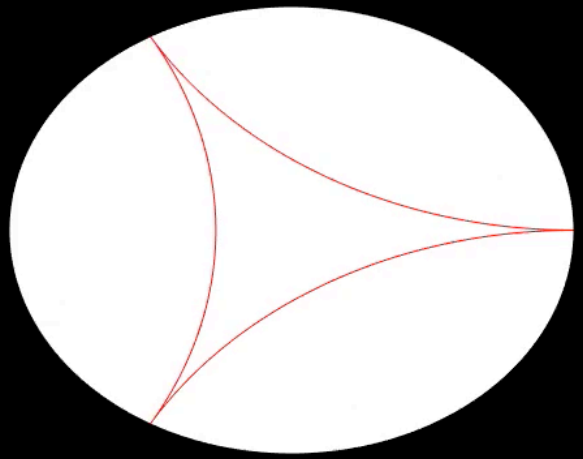
ranging from

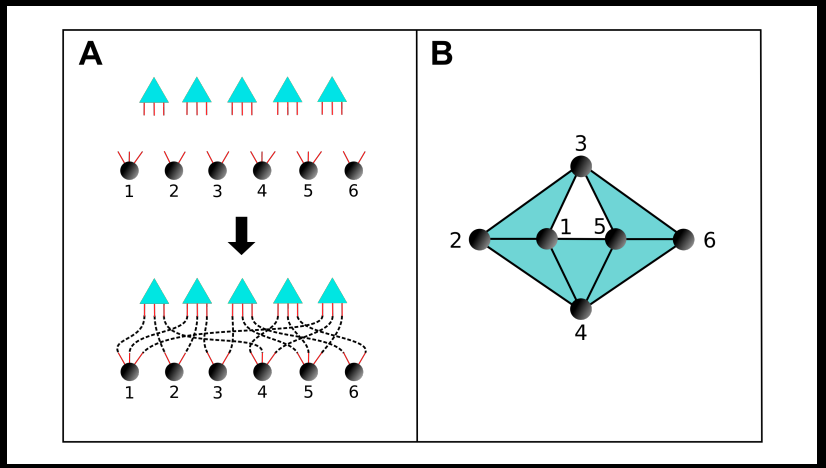
brain research to biological transportation networks

Simplicial complex models of arbitrary dimension

Emergent Hyperbolic Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





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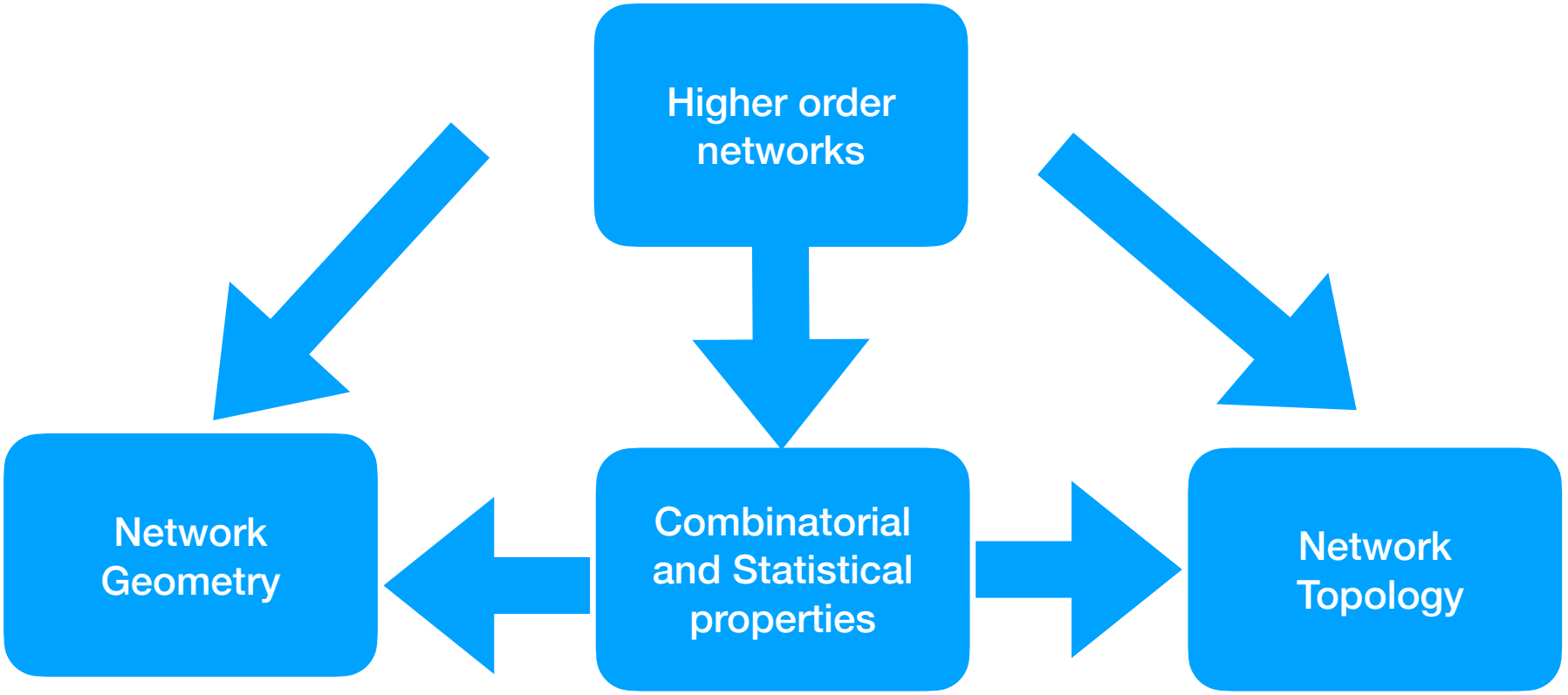
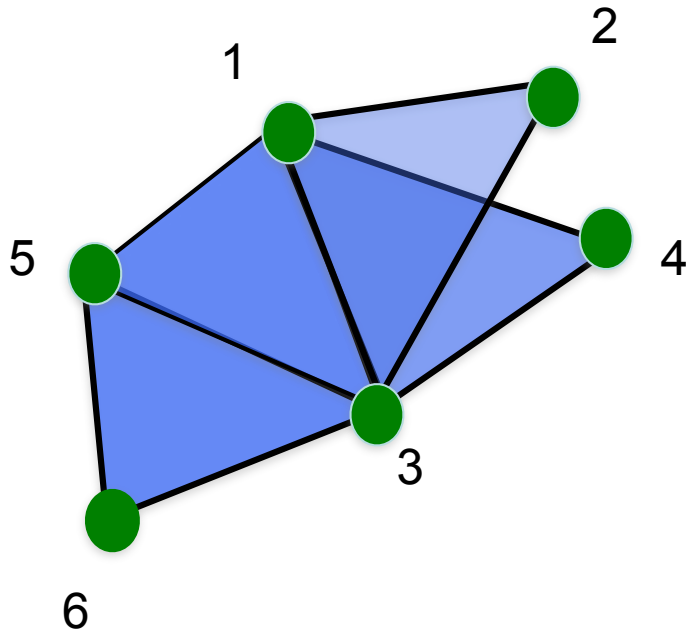
Higher order networks
Structure



Simplicial complex

SIMPLICIAL COMPLEX

A simplicial complex $\mathcal{K}$ is formed by a set of simplices that is closed under the inclusion of the faces of each simplex.
The dimension d of a simplicial complex is the largest dimension of its simplices.

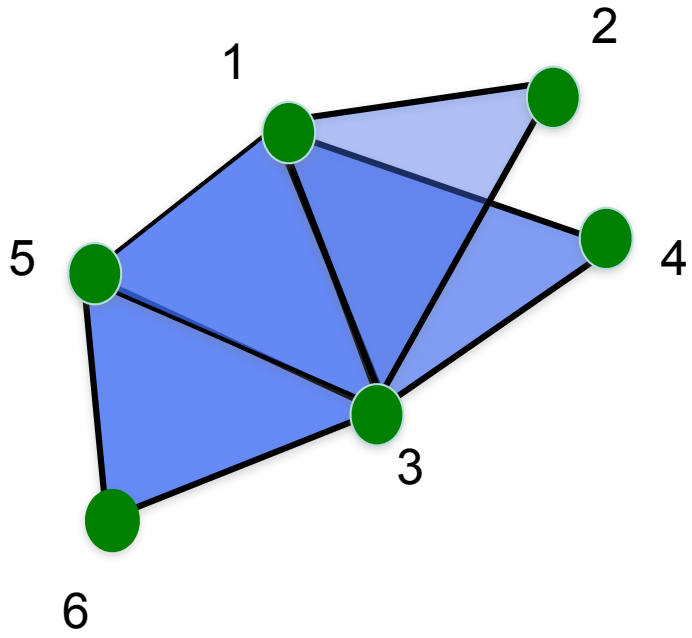


If a simplex α belongs to the simplicial complex $\mathcal{K}$ then every face of α must also belong to $\mathcal{K}$

$$\mathcal{K} = \{[1],[2],[3],[4],[5],[6],$$
$$[1,2],[1,3],[1,4],[1,5],[2,3],$$
$$[3,4],[3,5],[3,6],[5,6],$$
$$[1,2,3],[1,3,4],[1,3,5],[3,5,6]\}$$

Dimension of a simplicial complex

The dimension of a simplicial complex $\mathscr{K}$
is the largest dimension of its simplices



This simplicial complex has dimension 2

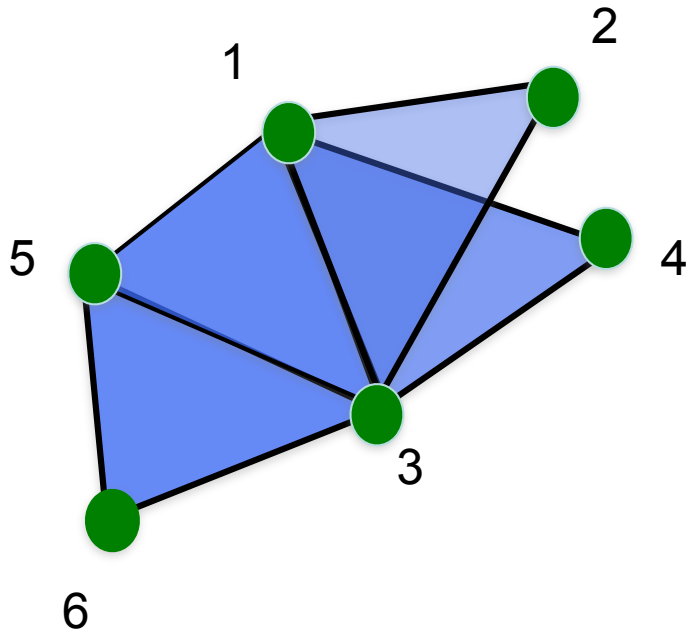
$$\mathscr{K} = \{[1],[2],[3],[4],[5],[6],$$
$$[1,2],[1,3],[1,4],[1,5],[2,3],$$
$$[3,4],[3,5],[3,6],[5,6],$$
$$[1,2,3],[1,3,4],[1,3,5],[3,5,6]\}$$

Pure simplicial complex

PURE SIMPLICIAL COMPLEXES

A pure d-dimensional simplicial complex is formed by a set of *d*-dimensional simplices and their faces.

Therefore pure *d*-dimensional simplicial complexes admit as facets only *d*-dimensional simplices.



A pure d-dimensional simplicial complex is fully determined by an adjacency matrix tensor with (d+1) indices.
For instance this simplicial complex is determined by the tensor

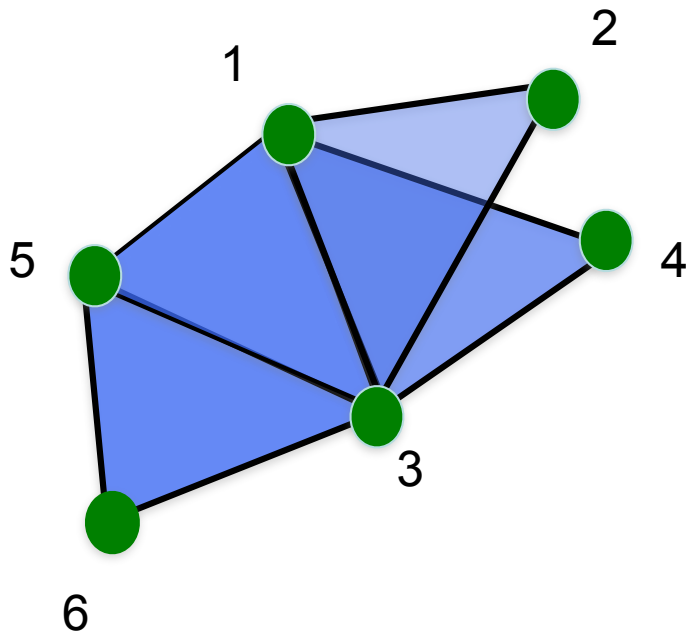
$$a_{rsp} = \begin{cases} 1 \text{ if } (r,s,p) \in \mathscr{K} \\ 0 \text{ otherwise} \end{cases}$$

Example

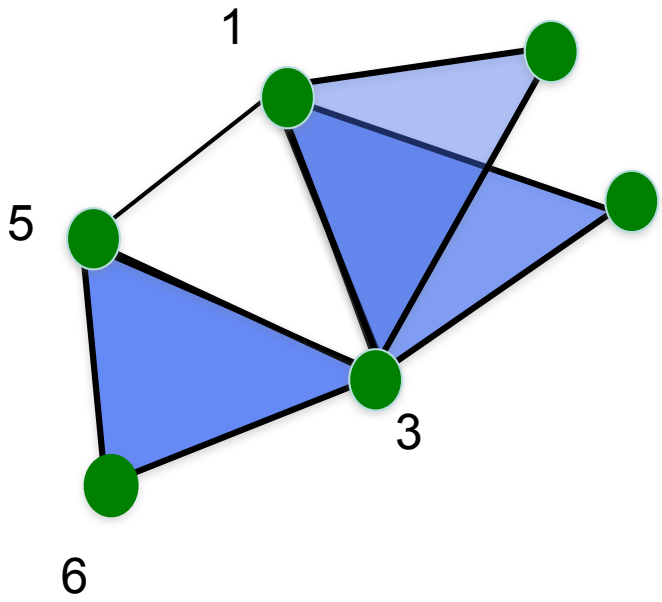
A simplicial complex $\mathscr{K}$ is pure

if it is formed by d-dimensional simplices

and their faces

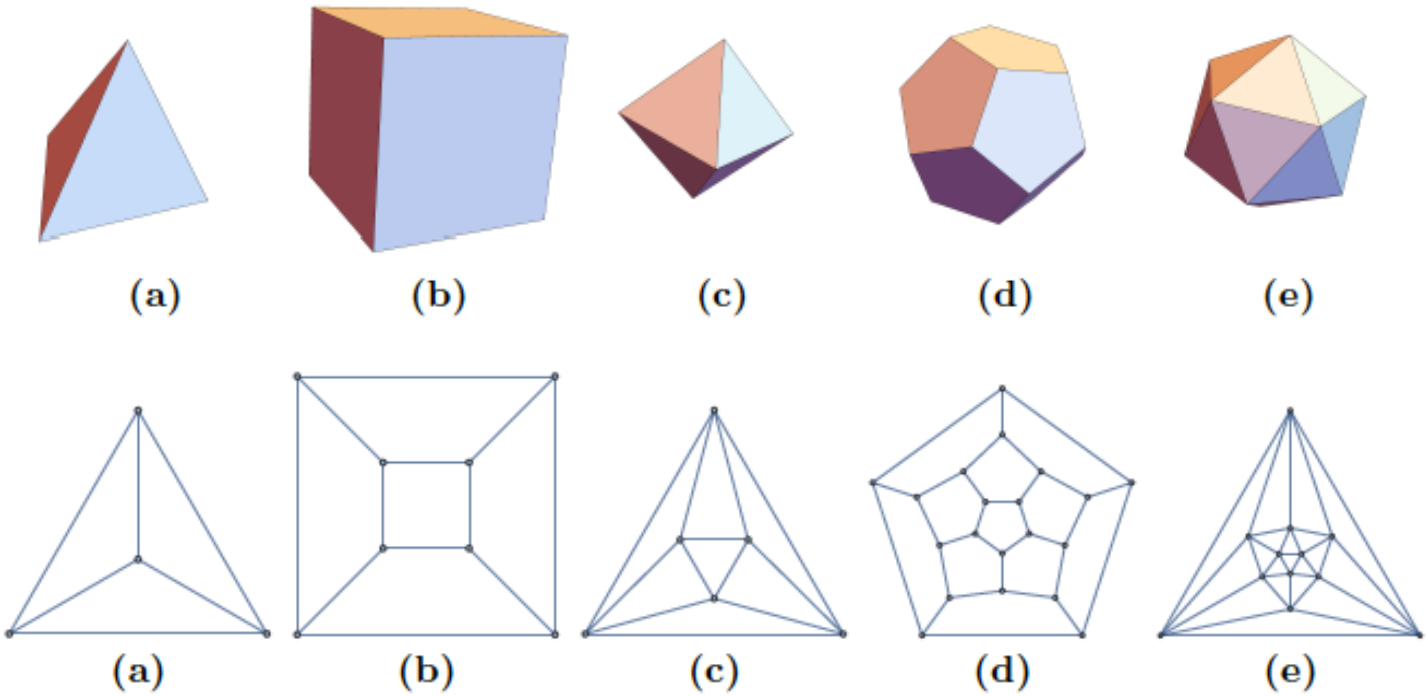


PURE SIMPLICIAL COMPLEX



SIMPLICIAL COMPLEX THAT IS NOT PURE

Cell complexes



A cell complex $\mathcal{K}$ has the following two properties:

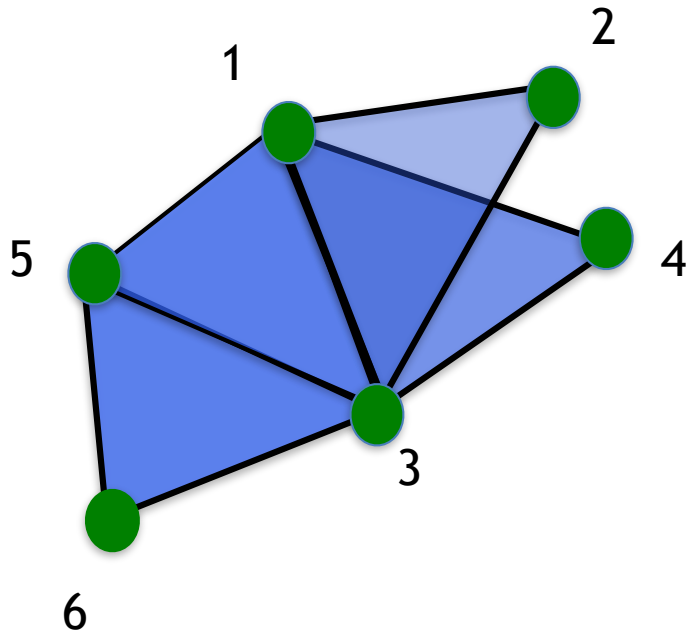
(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

Combinatorial and statistical properties of simplicial complexes

Generalized degrees

The generalized degree $k_{d,m}(\alpha)$ of a m-face α in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the m-face α.



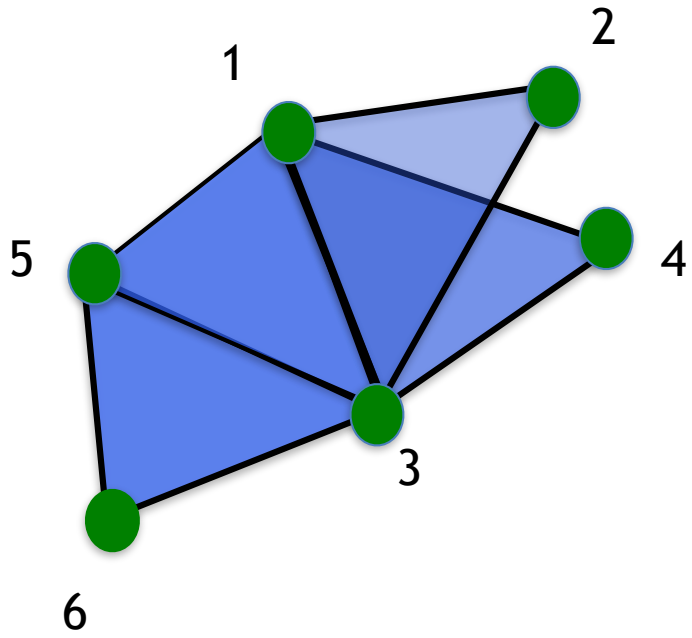
$k_{2,0}(\alpha)$ Number of triangles incident to the node α

$k_{2,1}(\alpha)$ Number of triangles incident to the link α

[Bianconi & Rahmede (2016)]

Generalized degree

The generalized degree $k_{d,m}(\alpha)$ of a m-face α in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the m-face α.



i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

Combinatorial properties of the generalised degrees

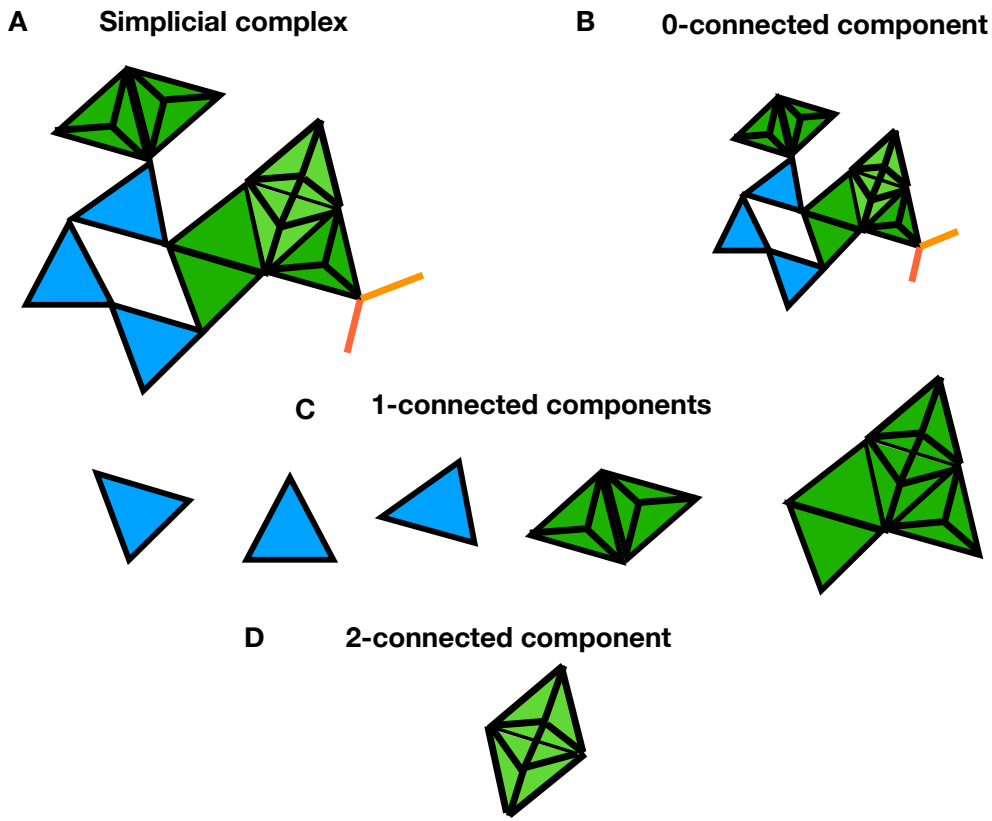
The generalized degrees $k_{d,m}(\alpha)$ of a pure d-dimensional simplicial complex can be defined in terms of the adjacency tensor $\mathbf{a}$ as

$$k_{d,m}(\alpha) = \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supseteq \alpha} a_{\alpha'}$$

The generalized degrees obey a nice combinatorial relation as they are not independent of each other.
In fact for $m'>m$ we have

$$k_{d,m}(\alpha) = \frac{1}{\binom{d-m}{m'-m}} \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supseteq \alpha} k_{d,m'}(\alpha') \,.$$

m-connected components

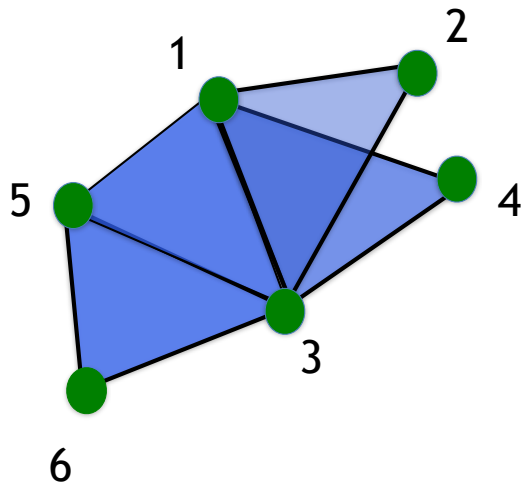


Geometrical properties of simplicial complexes

Incidence number

To each (d-1)-face α we associate the incidence number

$$n_\alpha = k_{d,d-1}(\alpha) - 1$$



[Bianconi & Rahmede (2016)]

(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

Discrete manifolds

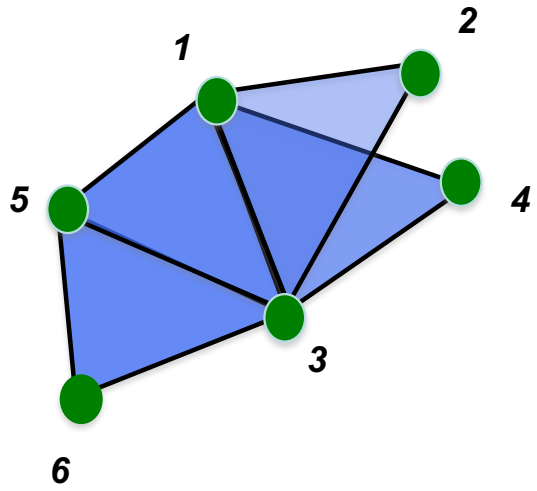
COMBINATORIAL CONDITIONS FOR DISCRETE MANIFOLDS

A discrete manifold $\mathcal{M}$ of dimension d is a pure simplicial complex that satisfies the following two conditions:

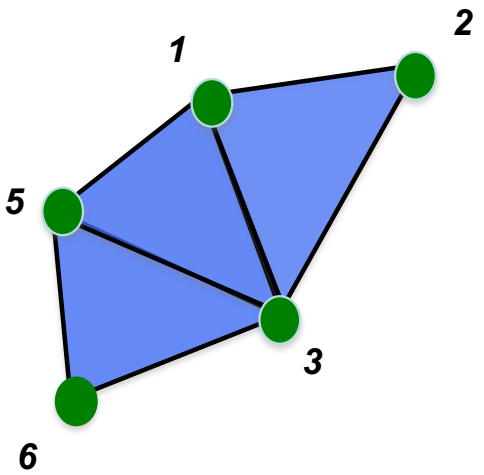
- it is $(d-1)$-connected;
- every two d-simplices α, α' belonging to the simplicial complex $\mathcal{K}$ either overlap on a $(d-1)$-face of $\mathcal{K}$, i.e. $\alpha \cap \alpha' \in S_{d-1}(\mathcal{K})$ or do not overlap, i.e. $\alpha \cap \alpha' = \emptyset$.
- all its $(d-1)$-faces α have an incidence number $n_\alpha \in \{0, 1\}$.

Discrete manifolds

If n_α takes only values $n_\alpha \in \{0,1\}$
each (d-1)-face is incident at most to two
d-dimensional simplices.



NOT A MANIFOLD



MANIFOLD

Bulk and area of a discrete manifold

Boundary area and bulk of a discrete manifold

The boundary $\mathcal{A}$ of a d-dimensional discrete manifold $\mathcal{M}$ is formed by the set of all $(d-1)$-dimensional faces $\alpha \in \mathcal{M}$ with incidence number $n_\alpha = 0$ and by all their faces. The area A is the number of $(d-1)$-dimensional faces in the boundary $\mathcal{A}$. The bulk $\mathcal{B}$ of a discrete manifold $\mathcal{M}$ is formed by the set of all the faces that are not in the boundary $\mathcal{A}$.

Regge curvature

Extends the notion of Ricci curvature of simplicial complexes
Applies to discrete manifolds

REGGE CURVATURE

The Regge curvature (Regge (1961)) is associated to each $(d-2)$-dimensional face $\alpha \in S_{d-2}(\mathcal{M})$ of a discrete d dimensional manifold $\mathcal{M}$. The Regge curvature R_α for a face $\alpha \in S_{d-2}(\mathcal{M})$ is defined as

$$R_\alpha = \begin{cases} 2\pi - \theta_\alpha & \text{if } \alpha \in \mathcal{B}, \\ \pi - \theta_\alpha & \text{otherwise,} \end{cases} \tag{42}$$

where θ_α is the sum of all dihedral angles of the d-dimensional simplices incident to the face α.

Regge curvature and generalized degrees

If the discrete manifold is formed
by a set of geometrically identical d-simplices
the Regge curvature
is simply related to the generalized degree of the (d-2)-faces, i.e.

$$R_\alpha = \begin{cases} 2\pi - \theta_0 k_{d,d-2}(\alpha) & \text{if } \alpha \in \mathscr{B}, \\ \pi - \theta_0 k_{d,d-2}(\alpha) & \text{otherwise,} \end{cases}$$

where θ_0 indicates the dihedral angle of each d-simplex.

Combinatorial curvature for planar triangulations

For planar triangulations the curvature is localised on the nodes. Assuming all triangles equilateral, and the curvature expressed in units 2π we obtain

- For a node in the **bulk** we have $R_i = \frac{6 - k_{2,0}(i)}{6}$

The node has zero curvature for $k_{2,0} = 6$ (the node is incident to 6 triangles), negative and positive curvature of $k_{2,0}(i) > 6, k_{2,0}(i) < 6$ respectively

- For a node in the **boundary** we have $R_i = \frac{3 - k_{2,0}(i)}{6}$

The node has zero curvature for $k_{2,0}(i) = 3$ (3 triangles), negative and positive curvature of $k_{2,0}(i) > 3, k_{2,0}(i) < 3$ respectively

Gauss-Bonnet theorem

GAUSS-BONNET THEOREM

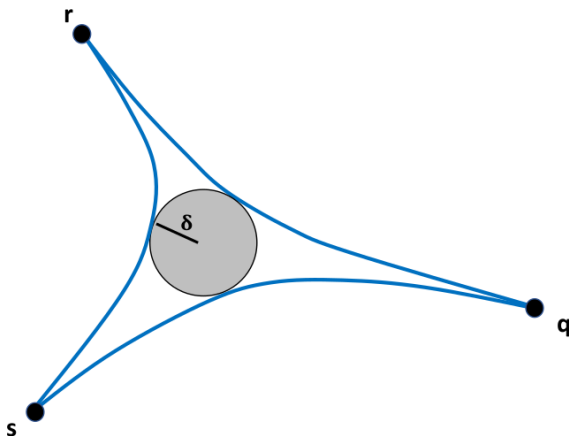
According to the Gauss-Bonnet theorem the sum over the Regge curvatures R_α of all the $(d-2)$-faces α of the discrete d-dimensional manifold $\mathcal{M}$ is a topological invariant proportional to the Euler characteristic χ of the manifold, i.e.

$$\sum_{\alpha \in S_{d,d-1}(\mathcal{M})} R_\alpha = 2\pi\chi. \tag{4.5}$$

The Gauss-Bonnet theorem relates geometry to topology
As it states that that the sum of all the curvatures of a simplicial complex
Are invariant under stretching and deformation of the underlying manifold

Gromov hyperbolicity

Global notion of curvature for a networks
Applies to general networks (also not manifolds)



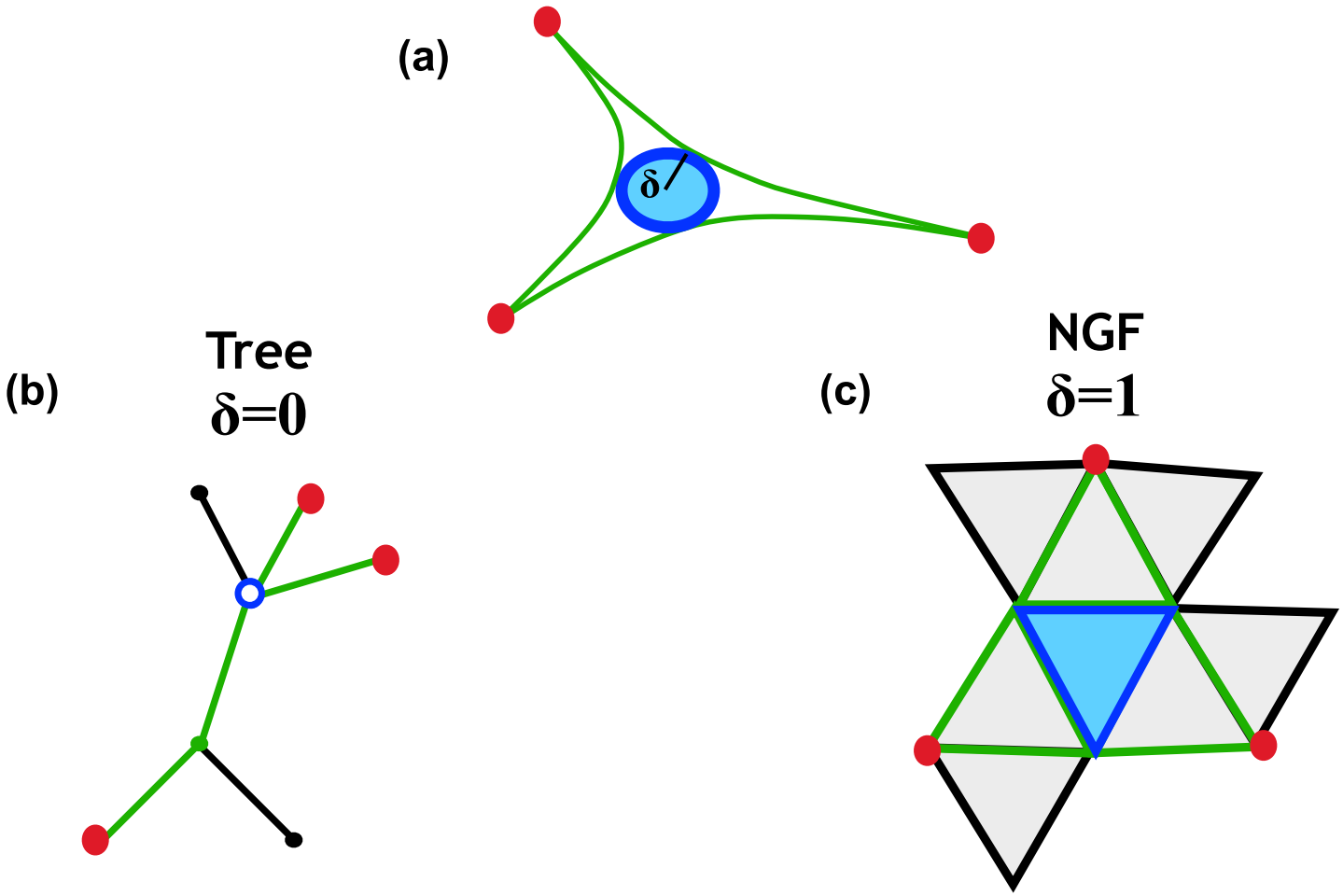
GROMOV δ-HYPERBOLICITY

A network is said to be δ-hyperbolic, if it obeys the δ-slim property, i.e. if there is a $\delta > 0$ such that for any triple of nodes r, s, q connected by the shortest paths $\mathcal{P}_{rs}, \mathcal{P}_{sq}, \mathcal{P}_{rq}$ the union of the δ-neighbourhood of any pair of shortest paths, say $N_\delta(\mathcal{P}_{rs}) \cup N_\delta(\mathcal{P}_{sq})$ includes nodes belonging to the third path, i.e. $\mathcal{P}_{rq}$.

For the actual algorithm see

Albert, R., DasGupta, B. and Mobasheri, N., 2014.
Topological implications of negative curvature for biological and social networks.
Physical Review E, *89*(3), p.032811.

Examples of δ-hyperbolic networks



Ollivier-Ricci curvature

The Ollivier-Ricci curvature is a local curvature defined on each link/edge of a network

The Ollivier-Ricci curvature is defined as

$$\kappa^{w}(i,j) = 1 - \frac{W(m_i, m_j)}{d(i,j)}$$

Where $d(i,j)$ is the distance between the nodes i and j and where $W(m_i, m_j)$ is the optimal transport distance between the two distributions m_i, m_j localised on the neighbours of node i and the node j respectively.

Ollivier-Ricci curvature

The distribution m_i can be for instance defined as (m_j is defined similarly)

$$m_i(i') = \begin{cases} \alpha & \textbf{if } i = i' \\ (1-\alpha)/k_i & \textbf{if } i' \in N(i) \\ 0 & \textbf{otherwise} \end{cases}$$

The optimal transportation distance is defined as

$$W(m_i^\alpha, m_j^\alpha) = \textbf{inf}_M \sum_{i',j' \in V} d(i',j')M(i',j')$$

where $M(i',j')$ is the mass transported between node i' and node j' and M is the optimised transport function among all functions that satisfy

$$\sum_{j'} M(i',j') = m_i(i') \qquad \sum_{i'} M(i',j') = m_j(j')$$

Hausdorff dimension

HAUSDORFF DIMENSION OF NETWORK MODELS

Given a class of network models, such as d-dimensional lattices or ensembles of networks with given degree distribution, for which we can consider a series of models with increasing network size N, the Haudorff dimension d_H scales for $N \gg 1$

$$D \sim N^{1/d_H}, \tag{4.6}$$

where D is the diameter of the network model with N nodes, i.e. the maximum among all the shortest distances between any two pair of nodes in the network.

Graph Laplacian

The graph Laplacian matrix is defined as

$$L_{ij} = \delta_{ij} k_i - a_{ij}$$

The graph Laplacian is a semi-positive matrix that in a connected network has eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \lambda_N$$

The Laplacian is key for describing diffusion processes and the Kuramoto model on networks and constitutes a natural link between topology and dynamics

The Fiedler eigenvalue λ_2 is also called **spectral gap**

Spectral dimension

In geometrical network models

$$\lambda_2 \to 0 \text{ for } N \to \infty$$

and we say that the spectral gap "closes"

If the density of eigenvalues $\rho(\lambda)$ scales like

$$\rho(\lambda) \sim \lambda^{d_S/2-1} \text{ for } \lambda \ll 1$$

d_S is called the ***spectral dimension***

Relation among the spectral and the Hausdorff dimension

The spectral and the Hausdorff dimension are distinct notions.

The Hausdorff and the spectral dimensions are related by the inequalities

$$d_H \geq d_S \geq 2\frac{d_H}{d_H + 1}$$

A small-world network with infinite Hausdorff dimension can have a finite spectral dimension with $d_S \geq 2$

Square d-dimensional lattice

The eigenvalues λ of the Laplacian

of a d-dimensional lattice are given by

$$\lambda = \sum_{i \in \{1,2,3,\ldots,d\}} 4 \sin^2(q_i/2) \simeq |\mathbf{q}|^2$$

where $\mathbf{q}$ is the wave-number characterising the eigenvectors of the Laplacian (Fourier basis) with

$$q_i = \frac{2\pi n_i}{L}$$

The spectral dimension of a d-dimensional lattice is $d_S = d$

Square 1-dimensional lattice

The eigenvalues λ of the Laplacian

of a 1-dimensional lattice are given by

$$\lambda = 2(1 - \cos q) = 4 \sin^2(q/2) \simeq q^2$$

where q is the wave-number characterising the eigenvectors of the Laplacian (Fourier basis) with

$$q = \frac{2\pi n}{L}$$

Spectrum of a 1D lattice

The Laplacian of a 1D lattice with periodic boundary conditions (a cycle) has elements

$$L_{ij} = 2\delta_{ij} - a_{ij} = 2\delta_{ij} - \delta_{x_j, x_i+1} - \delta_{x_j, x_i-1}$$

The eigenvector of the Laplacian are given by the Fourier modes with wave-number q, i.e.

$$u_i = e^{\mathrm{iq}\cdot \mathrm{x_i}}$$

The eigenvalue problem $\sum_j L_{ij} u_j = \lambda u_i,$ reads

$$\sum_j L_{ij} u_j = 2u_i - u_{i+1} + u_{i-1} = (2 - e^{\mathrm{iq}} - e^{-\mathrm{iq}}) u_i = \lambda u_i$$

with $\lambda = [2 - e^{\mathrm{iq}} - e^{-\mathrm{iq}}] = 2(1 - \cos(q))$

For $|q| \ll 1$ we have $\cos(q) = 1 - \frac{1}{2} q^2$ and hence $\lambda \simeq q^2$

Periodic boundary conditions 1D lattice

The periodic boundary conditions imposed

$$u_L = e^{\mathrm{iq}\cdot\mathrm{L}} = u_0 = 1$$

Therefore they imply

$$qL = 2\pi n \quad n \in \mathbb{N}$$

The only admitted wavenumber are given by

$$q = \frac{2\pi}{L} n \quad n \in \{0,1,2,\ldots,L-1\}$$

Square 2-dimensional lattice

The eigenvalues λ of the Laplacian

of a 2-dimensional lattice are given by

$$\lambda = 4\sin^2(q_x/2) + 4\sin^2(q_y/2) \simeq |\mathbf{q}|^2$$

where $\mathbf{q}$ is the wave-number characterising the eigenvectors of the Laplacian (Fourier basis) with

$$q_i = \frac{2\pi n_i}{L}$$

Spectrum of a 2D lattice

For a 2D lattice with periodic boundary conditions (a torus)

the Laplacian has matrix elements

$$L_{ij} = 4\delta_{ij} - a_{ij}$$

The eigenvectors of the Laplacian has Fourier modes $u_i = e^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}_\mathrm{i}} = e^{\mathrm{i}(\mathrm{q_x x_i + q_y y_i})}$

From the eigenvalue problem $\sum_j L_{ij} u_j = \lambda u_i$

We deduce the eigenvalue

$$\lambda(\mathbf{q}) = [2 - e^{\mathrm{iq_x}} - e^{-\mathrm{iq_x}}] + [2 - e^{\mathrm{iq_y}} - e^{-\mathrm{iq_y}}] = 2(1 - \cos(q_x)) + 2(1 - \cos(q_y))$$

Therefore for $|\,\mathbf{q}\,| \ll 1$ we obtain $\lambda(\mathbf{q}) \simeq q_x^2 + q_y^2 = |\,\mathbf{q}\,|^2$

Periodic boundary conditions

Imposing the periodic boundary conditions the only admitted wavenumber have components

$$q_x = \frac{2\pi}{L} n_x \quad n \in \{0,1,2,\ldots,L-1\}$$

$$q_y = \frac{2\pi}{L} n_y \quad n \in \{0,1,2,\ldots,L-1\}$$

Square d-dimensional lattice

The eigenvalues λ of the Laplacian

of a d-dimensional lattice are given by

$$\lambda = \sum_{i \in \{1,2,3,\ldots,d\}} 4 \sin^2(q_i/2) \simeq |\mathbf{q}|^2$$

where $\mathbf{q}$ is the wave-number characterising the eigenvectors of the Laplacian (Fourier basis) with

$$q_i = \frac{2\pi n_i}{L}$$

Spectral dimension of a d-dimensional lattice

The number Δn of models corresponding to wavenumber of absolute value $|\mathbf{q}|, |\mathbf{q}| + \Delta|\mathbf{q}|$ is given by $\Delta n = \left(\frac{L}{2\pi}\right)^d \Omega_d |\mathbf{q}|^{d-1} \Delta|\mathbf{q}|$

In the limit $L \to \infty$ we have $\left(\frac{L}{2\pi}\right)^d \Omega_d |\mathbf{q}|^{d-1} d|\mathbf{q}| = N\rho(\lambda)d\lambda$

Using $|\mathbf{q}| = \lambda^{1/2} \quad d|\mathbf{q}| = \frac{1}{2}\lambda^{-1/2}d\lambda$ we obtain

$$\rho(\lambda) \propto \lambda^{d/2-1}$$

Therefore the spectral dimension of a d-dimensional lattice is d

Fiedler eigenvalue of a network with spectral dimension

The Fiedler eigenvalue of a connected network with spectral dimension can be estimated from

$$\int_{\lambda<\lambda_2} \rho(\lambda) d\lambda = \frac{1}{N}$$

Assuming $\rho(\lambda) \propto \lambda^{(d_S-2)/2}$ we get

$$\lambda_2 \propto N^{-2/d_S}$$

Therefore the **spectral gap closes**, i.e.

$$\lambda_2 \to 0, \text{ as } N \to \infty$$

Heat diffusion on a graph

Consider the heat diffusion process

$$\dot{\mathbf{x}} = -\mathbf{L}_{[0]}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

Where $\mathbf{x}(t) = \sum_{\lambda} c_\lambda(t)\mathbf{u}_\lambda$

is decomposed into the eigenvectors of the graph Laplacian.

Expressing the heat diffusion equation into the basis of the eigenvector of the graph Laplacian we obtain

$\dot{c}_\lambda = -\lambda c_\lambda$ with solution $c_\lambda(t) = c_\lambda(0)e^{-\lambda t}$

Heat diffusion on a graph

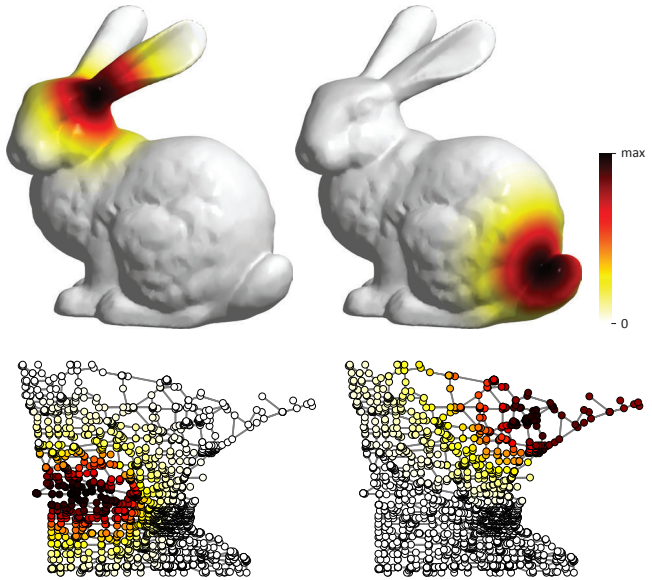
Assume that $\mathbf{x}_0 = \mathbf{e}_i$ is localised on node i then

we have $c_\lambda(0) = u_\lambda(i)$

$$x_j(t) = \sum_\lambda e^{-\lambda t} u_\lambda(i) u_\lambda(j) = h_t(i,j)$$

Which is called the heat kernel.

In the figure visualisation of heat kernels



Bronstein, et.al, P., 2017. Geometric deep learnin
going beyond euclidean data.
***IEEE Signal Processing Magazine*, *34*(4), pp.18-4**

Heat diffusion on a graph

Given the dynamics

$$x_j(t) = \sum_{\lambda} e^{-\lambda t} u_\lambda(i) u_\lambda(j)$$

The relaxation is toward to homogenous state

$$\lim_{t \to \infty} x_j(t) = u_0(i) u_0(j)$$

since $\mathbf{u}_0 \propto \mathbf{1}$ this limit does not depend on j!

If there is a spectral gap,

the characteristic temporal scale for the relaxation to equilibrium is given

$$\tau = 1/\lambda_2$$

However what happens if $\lambda_2 \to 0$?

Return-time probability

Given an initial condition localised at node i

We have

$$x_i(t) = \sum_{\lambda} e^{-\lambda t} u_\lambda(i) u_\lambda(i)$$

Averaging over all possible initial conditions we get that the probability of return at time t is given by

$$p(t) = \frac{1}{N} \sum_{\lambda} \sum_{i} e^{-\lambda t} u_\lambda(i) u_\lambda(i) \simeq \int \rho(\lambda) e^{-\lambda t} d\lambda$$

Assuming that the network has a spectral dimension the relaxation is power-law rather than exponential!

$$p(t) \propto t^{-d_S/2}$$

A drunken man always finds his was home while a drunken bird may be lost forever

Higher-order spectral dimension

Some geometrical simplicial complexes

do not have just a single spectral dimension

but they display a vector of spectral dimensions

$$\mathbf{d_S} = (d_S^{[0]}, d_S^{[1]}, \ldots, d_S^{[d-2]})$$

with one spectral dimension for each m-order up-Laplacian

Higher-order spectral dimension

In particular the higher-order spectral dimension

is observed when the density of non-zero eigenvalues of

the m-order up Hodge Laplacian $\mathbf{L}_{[m]}$

scales as

$$\rho_m(\lambda) \propto \lambda^{d_s^{[m]}/2-1}$$

The possible coexistence

of several different higher-order spectral dimensions

implies that the diffusion

taking place on simplices of different order

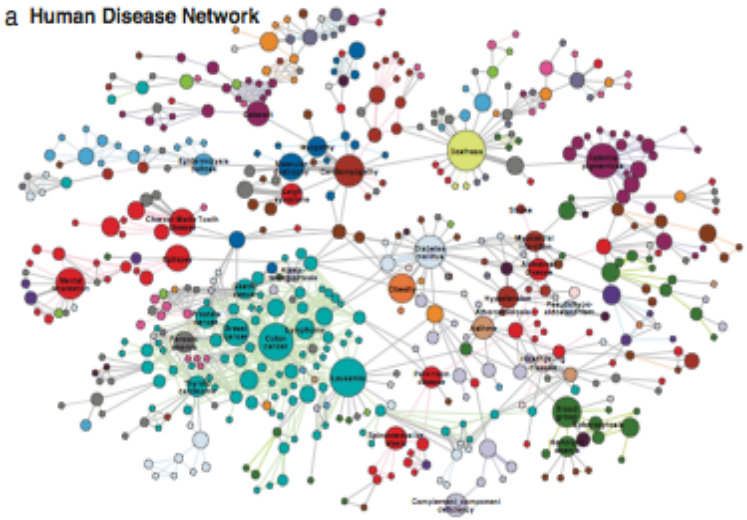
can have significant differences

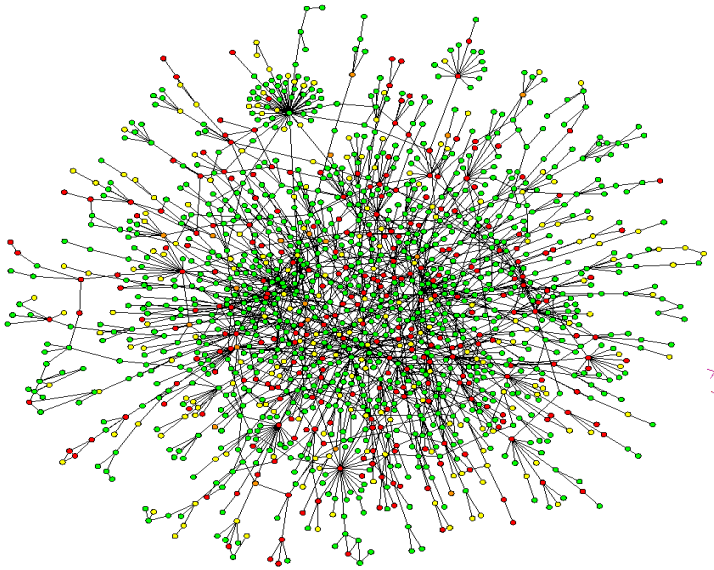
although the simplices belong to the same simplicial complex

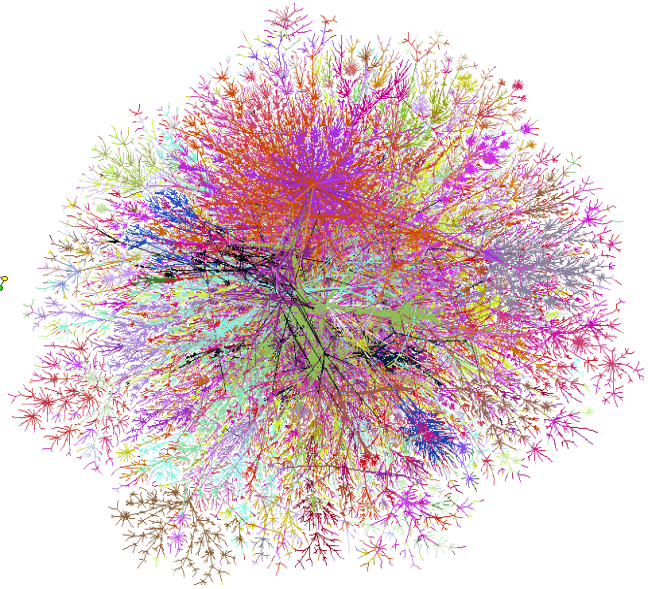
Which structural properties determine the presence of a spectral dimension?

Growing network models

Networks







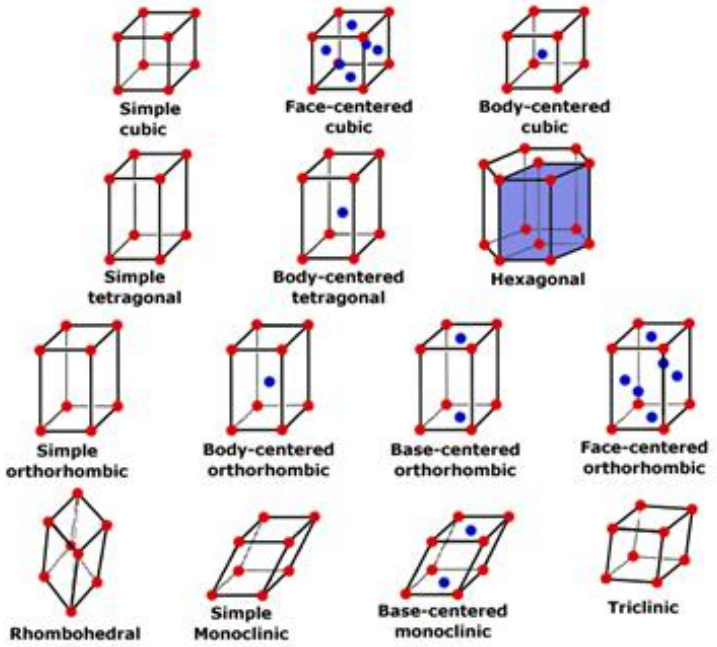
describe

the interactions between the elements

of large complex systems.

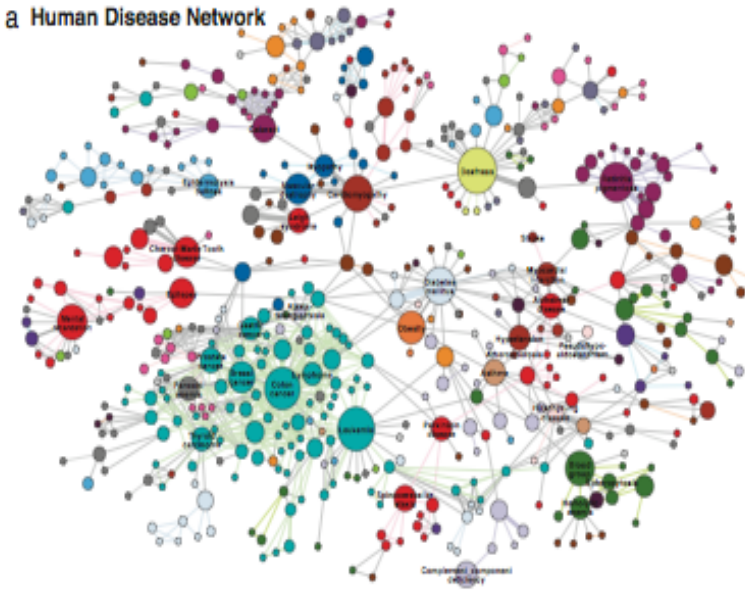
Randomness and order
Complex networks

LATTICES



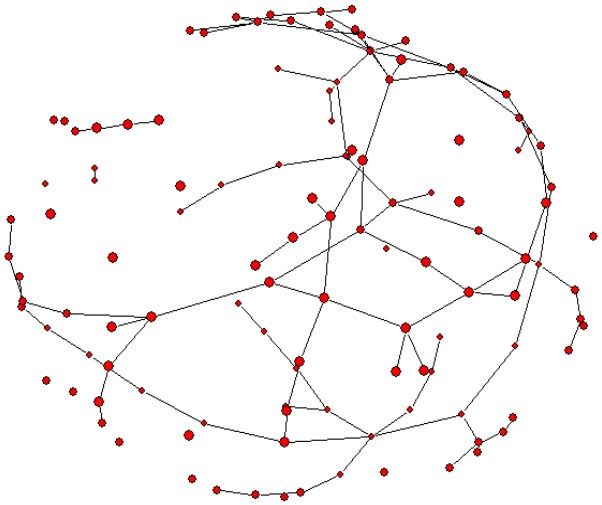
Regular networks
Symmetric

COMPLEX NETWORKS



Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

RANDOM GRAPHS



Totally random
Binomial degree distribution

Universalities

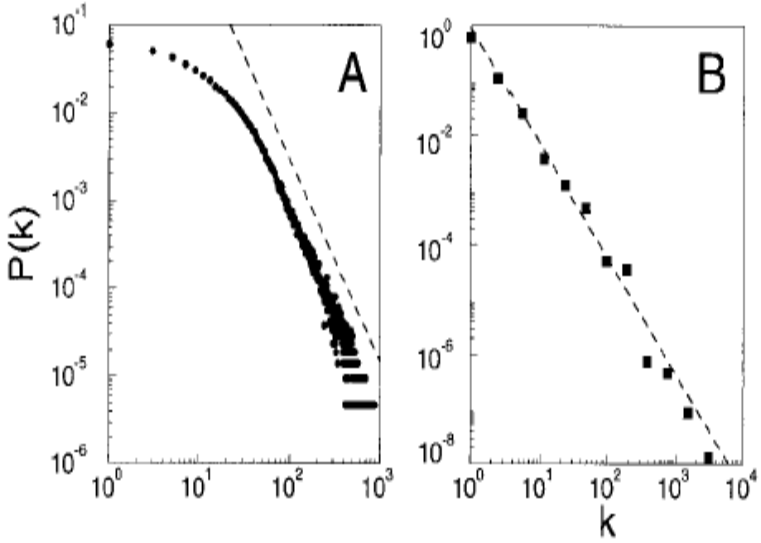
- **Small-world:** $d_H = \infty$

[Watts & Strogatz 1998]

- **Scale-free:** $P(k) \sim k^{-\gamma}$ **for** $k \gg 1$

[Barabasi & Albert 1999]

with $\gamma \in (2,3]$

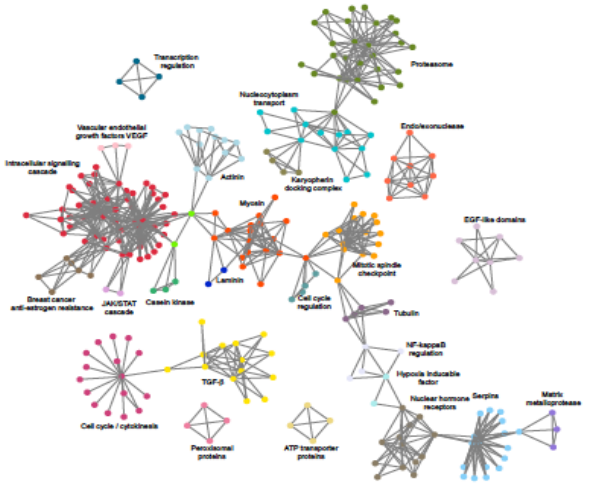


$\langle k \rangle \to$ **const** $\langle k^2 \rangle \to \infty$

for $N \to \infty$

- **Modularity:** Local communities of nodes

[Fortunato 2010]



Models

- Non-equilibrium growing network models:

  Explanatory of emergent properties of complex networks

  -BA model, BB model

- Deterministic models:

  Hierarchical models

  -Apollonian network, Pseudo-fractal network

- Maximum entropy ensembles:

  Maximum random graphs satisfying a set of constraints

  -Configuration model, Exponential Random Graphs

Growth by uniform attachment of links

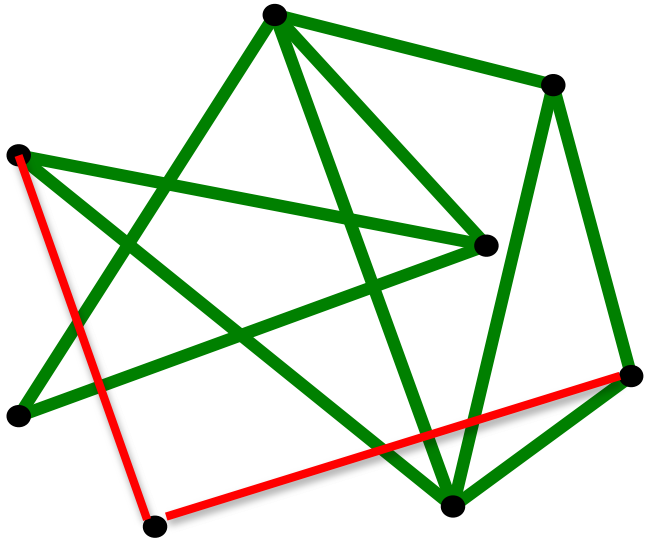
GROWTH :

At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

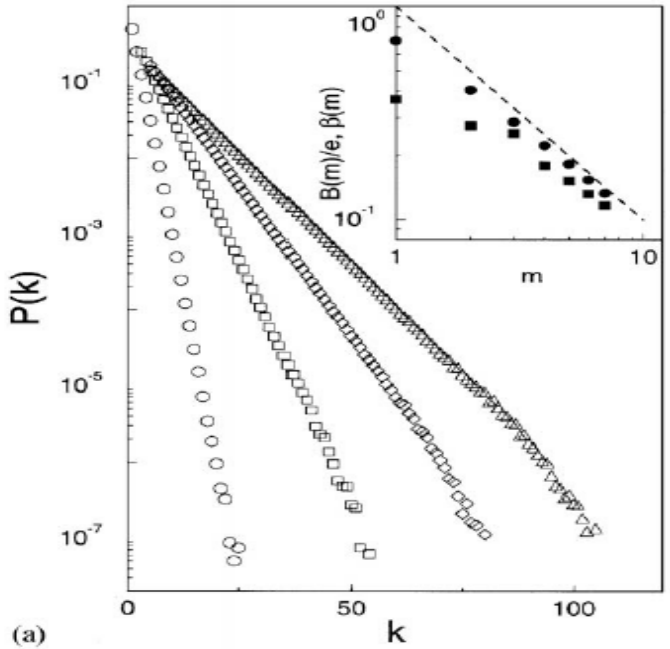
UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$



Exponential



[Barabási & Albert, Physica A (1999)]

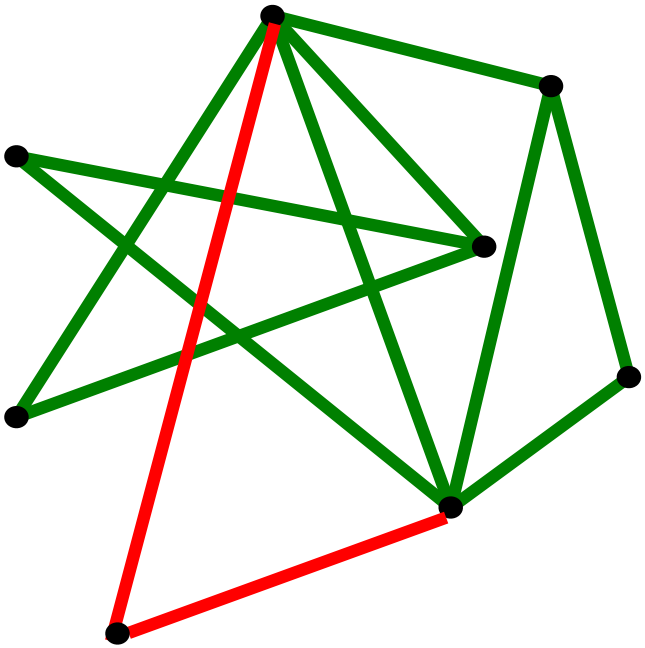
Barabasi-Albert model

GROWTH :

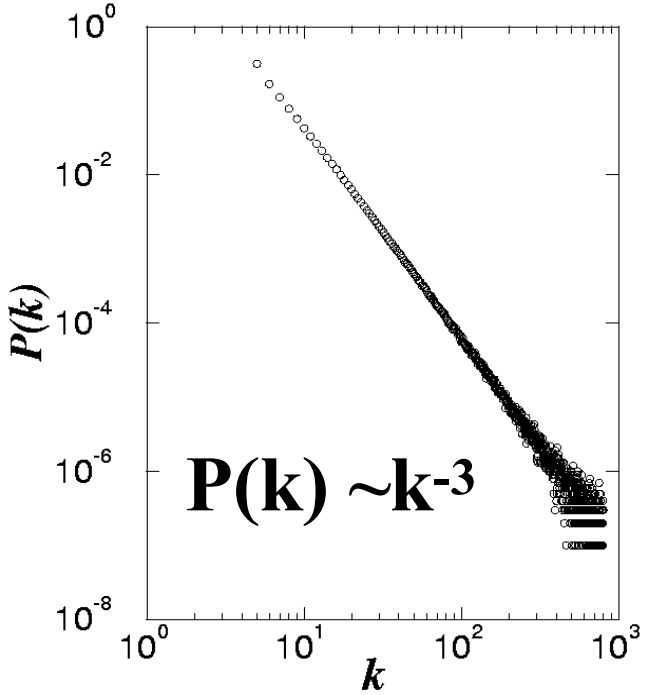
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

PREFERENTIAL ATTACHMENT :

The probability Π_i that a new node will be connected to node *i* depends on the degree k_i of that node



$$\Pi_i = \frac{k_i}{\sum_j k_j}$$



[Barabási et al. Science (1999)]

Master equation approach for the Barabasi-Albert model

We write the master equation for the average number of nodes $N^t(k)$ that have degree *k* at time *t*

$$N^{t+1}(k) = N^t(k) + m\Pi(k-1)N^t(k-1)[1-\delta_{k,m}] - m\Pi(k)N^t(k) + \delta_{k,m}$$

where

$$\Pi(k) = \frac{k}{Z} \text{ with } Z = \sum_j k_j \simeq 2mt$$

We assume that asymptotically

$$N^t(k) \simeq tP(k) \text{ for } t \gg 1$$

Master equation approach for the Barabasi-Albert model

In the asymptotic limit we have therefore

$$(t+1)P(k) = tP(k) + m\frac{(k-1)}{2m}P(k-1)[1-\delta_{k,m}] - m\frac{k}{2m}P(k) + \delta_{k,m}$$

Which lead to the time independent equations

$$P(k) = \frac{(k-1)}{2}P(k-1)[1-\delta_{k,m}] - \frac{k}{2}P(k) + \delta_{k,m}$$

Whose solution is

$$P(k) = \frac{k-1}{k+2}P(k-1) \text{ for } k > m$$

$$P(m) = \frac{2}{m+2}$$

Leading to

$$P(k) = 2m\frac{\Gamma(m+1)}{\Gamma(m)}\frac{\Gamma(k)}{\Gamma(k+2)} \simeq k^{-\gamma} \quad \text{with} \quad \gamma = 3$$

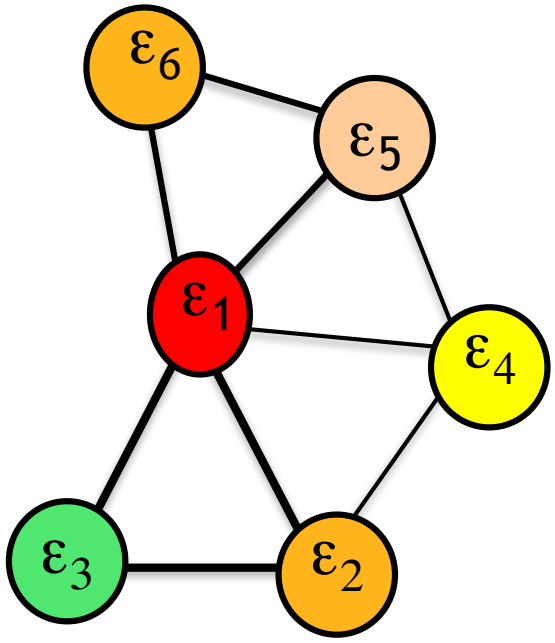
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

$g(\varepsilon)$ *distribution*



The Bianconi-Barabasi model

Growth:

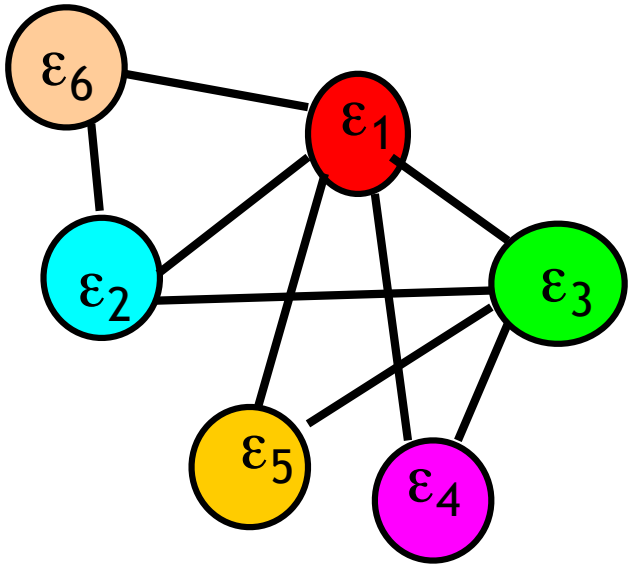
–At each time a new node and *m* links are added to the network.

–To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

Preferential attachment towards

high degree low energy nodes:

–Each node connects to the rest of the network by *m* links attached preferentially to well connected, low energy nodes.



$$\Pi_i = \frac{e^{-\beta \epsilon_i k_i}}{\sum_j e^{-\beta \epsilon_j k_j}}$$

[G. Bianconi, A.-L. Barabási 2001]

Self-consistent solution of the Bianconi-Barabasi model

We write the master equation for the average number of nodes with energy ε that have degree *k* at time *t* $N_\epsilon^t(k)$

$$N_\epsilon^{t+1}(k) = N_\epsilon^t(k) + m\Pi(k-1,\epsilon)N_\epsilon^t(k-1)[1-\delta_{k,m}] - m\Pi(k,\epsilon)N_\epsilon^t(k) + g(\epsilon)\delta_{k,m}$$

where

$$\Pi(k,\epsilon) = \frac{e^{-\beta\epsilon}k}{Z} \text{ with } Z = \sum_j k_j e^{-\beta\epsilon_j}$$

We assume self consistently that

$$\lim_{t\to\infty} \frac{Z}{2mt} = C = e^{-\beta\mu}$$

and that

$$N_\epsilon^t(k) \simeq tP_\epsilon(k) \text{ for } t \gg 1$$

Self-consistent solution of the Bianconi-Barabasi model

We obtain that

$$P_\epsilon(k) = e^{\beta(\epsilon-\mu)} \frac{\Gamma(m + e^{\beta(\epsilon-\mu)})}{\Gamma(m)} \frac{\Gamma(k)}{\Gamma(k+1+e^{\beta(\epsilon-\mu)})} \simeq k^{-\gamma_\epsilon} \quad \textbf{with} \quad \gamma_\epsilon = 1 + e^{\beta(\epsilon-\mu)}$$

Therefore we can evaluate the degree distribution

$$P(k) = \int d\epsilon g(\epsilon) P_\epsilon(k)$$

By imposing the self-consistent equation we find that

$$1 = \int d\epsilon g(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

Where the Bose-Einstein occupation

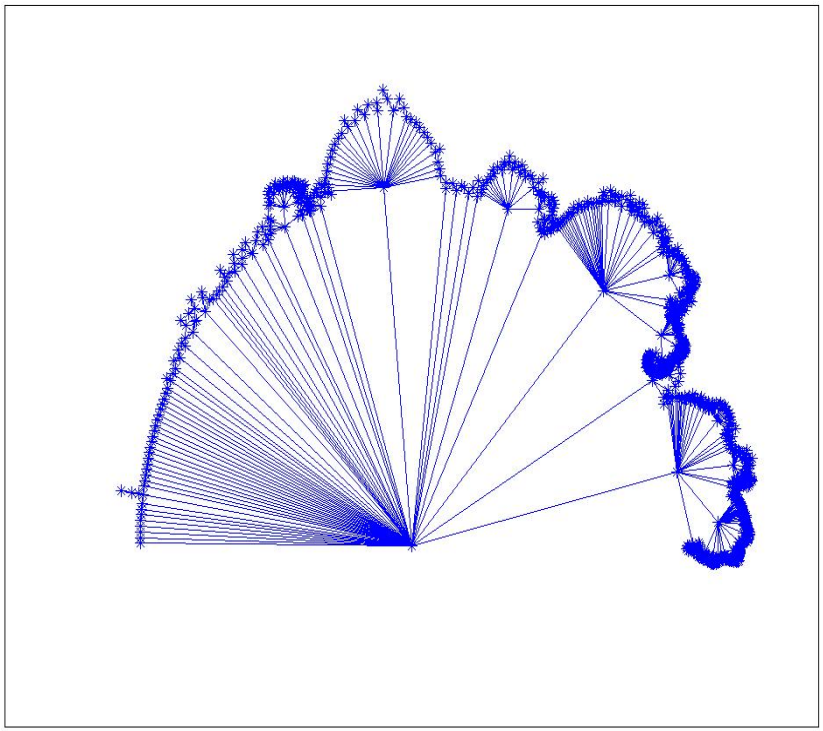
$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

indicates the number of links attached to nodes of energy ε

Bose-Einstein condensation in complex networks

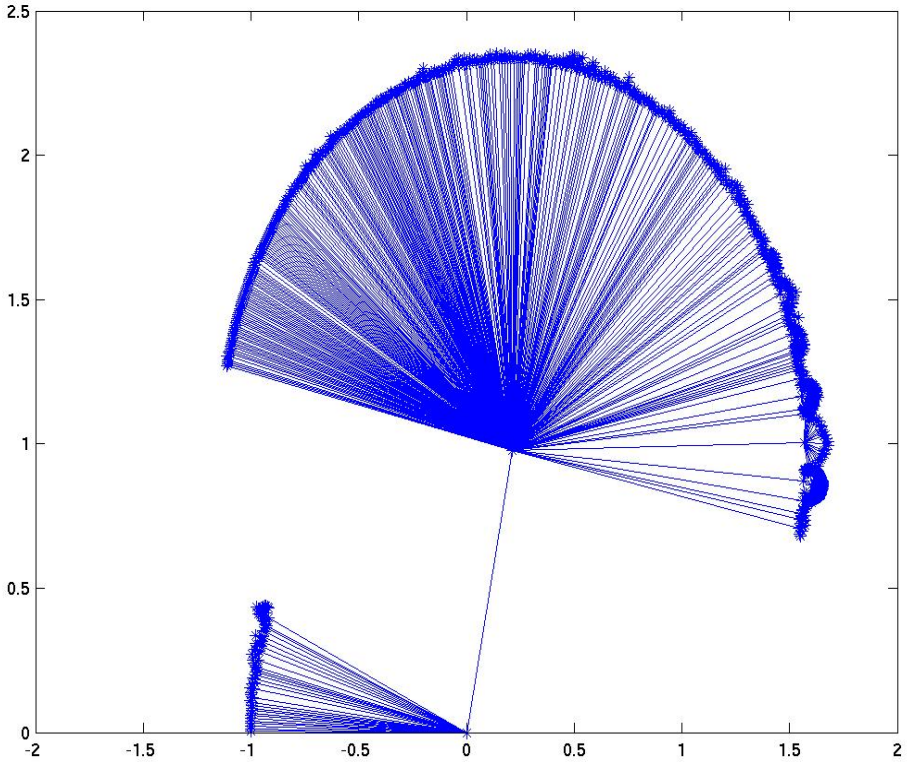
Scale-Free Phase

$$\beta < \beta_c$$



Bose-Einstein Condensate Phase

$$\beta > \beta_c$$

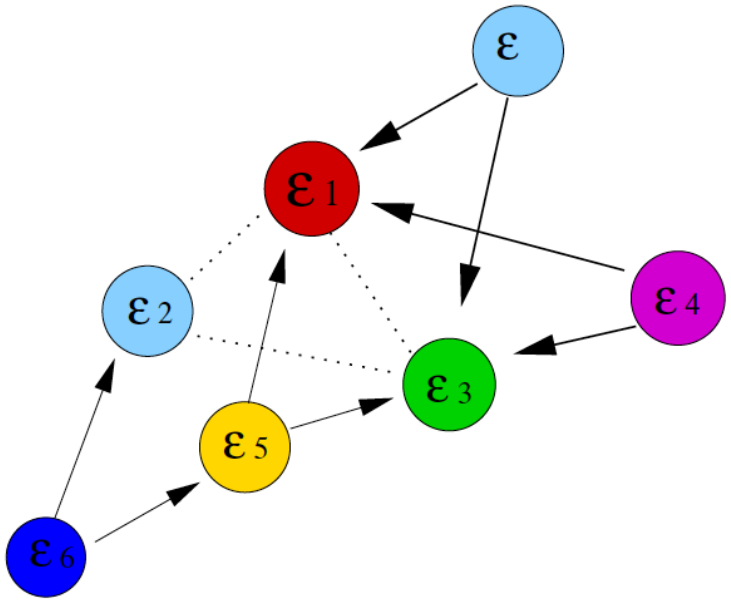


[G. Bianconi, A.-L. Barabási 2001]

Quantum statistics in growing networks

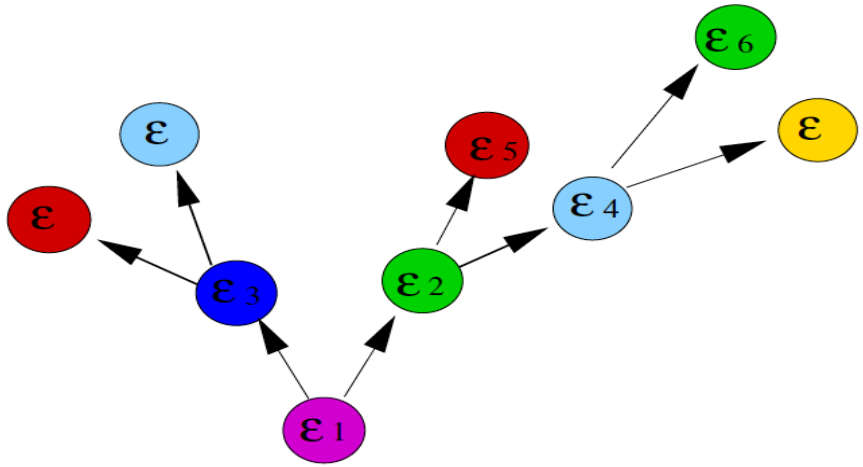
Scale-free network

Bianconi-Barabasi model (2001)



Complex Cayley tree

Bianconi (2002)



Bose Einstein statistics

Fermi statistics

The Complex Growing Cayley tree model

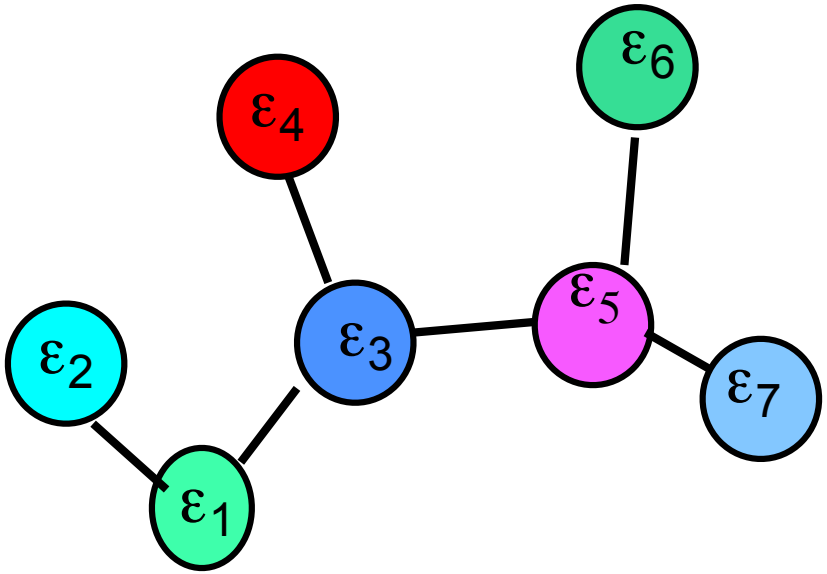
Growth:

–At each time attach a old node with n_i=0 to *m* links are added to the network and then we set n_i=1.

–To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

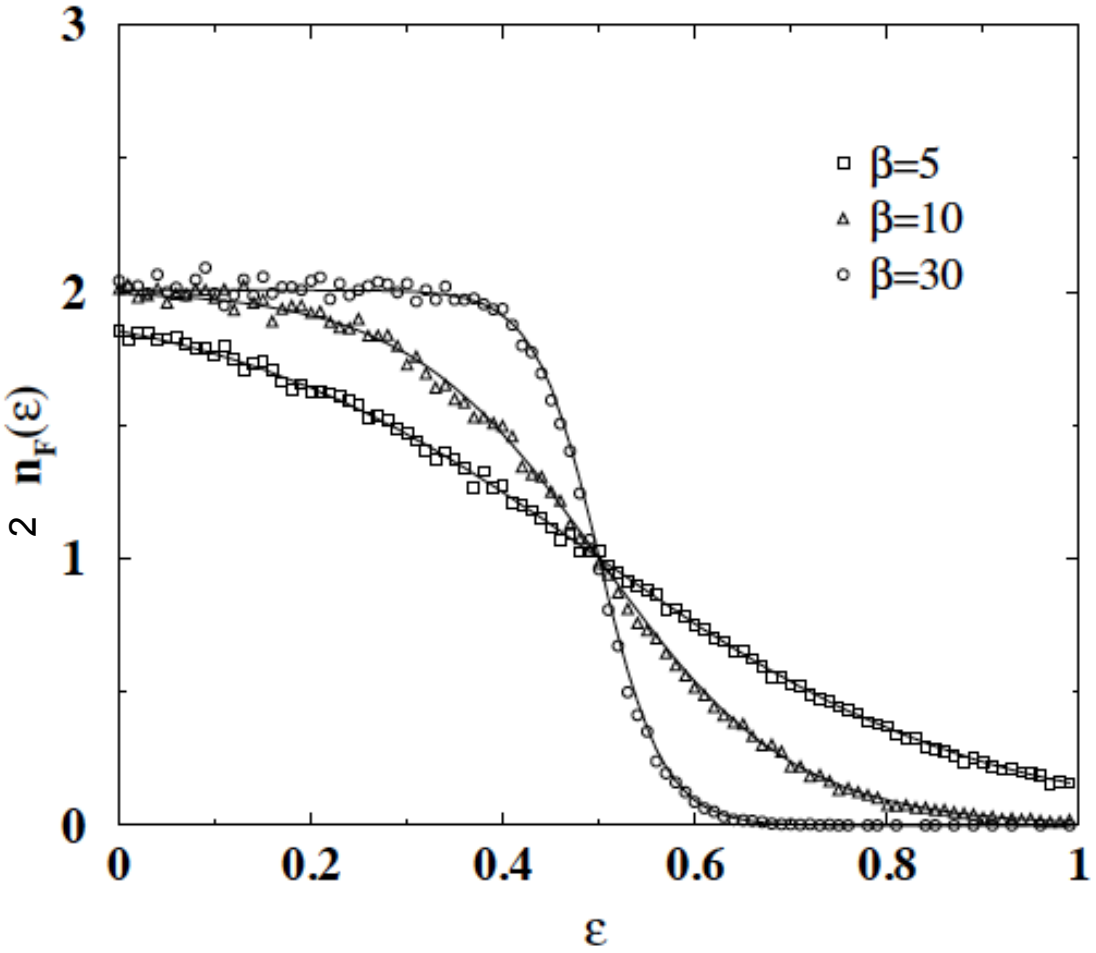
Attachment towards low energy nodes:

–The node *i* to which we attach the new "unitary cell" is chosen with probability



$$\Pi_i = \frac{e^{-\beta\epsilon_i}\,(1-n_i)}{\sum_j e^{-\beta\epsilon_j}\,(1-n_j)}$$

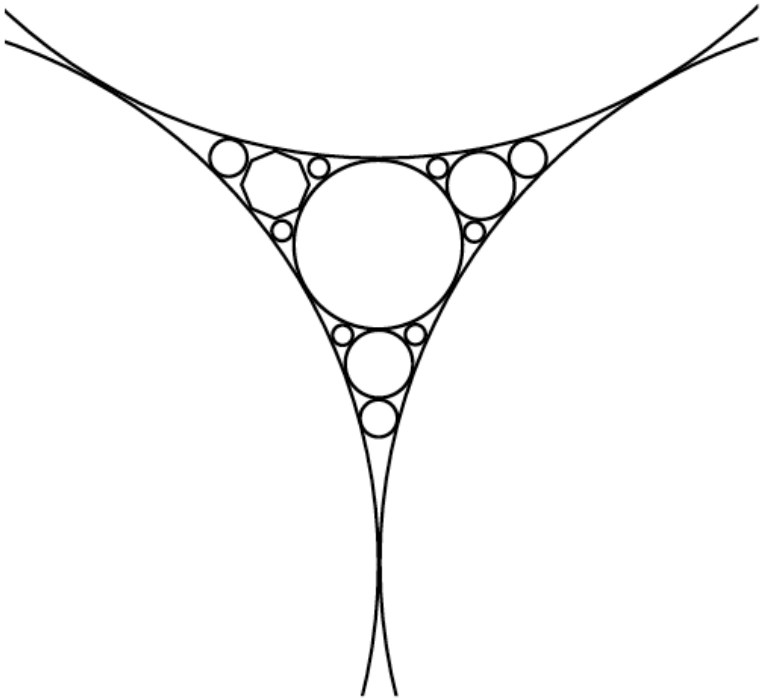
Energy distribution of the nodes at the bulk of the growing Cayley tree network

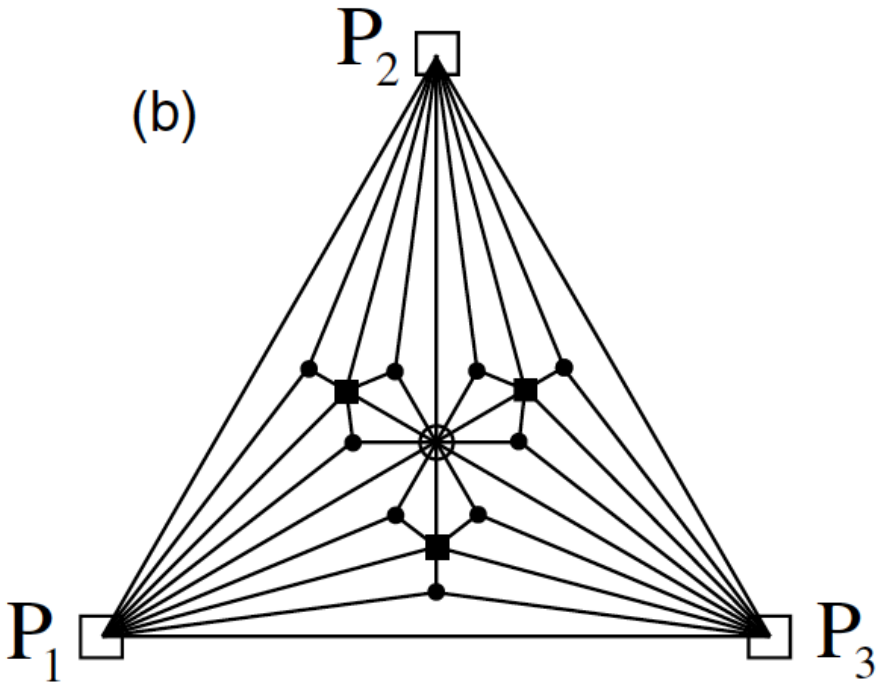


Apollonian networks

Apollonian networks are formed by linking the centers of an Apollonian sphere packing

They are scale-free and are described by the Apollonian group





[Andrade et al. PRL 2005]
[Soderberg PRA 1992]

Modularity

Modularity is a measure to characterise the significance of a given community assignment in a graph.

In particular it measures whether nodes belonging to the same community
are more connected among themselves than in a null hypothesis.

Given a network of N nodes with each node i assigned to the community c_i
and L links the modularity M is defined as

$$M = \frac{1}{2L} \sum_{i,j} [A_{ij} - p_{ij}] \delta(c_i, c_j)$$

where p_{ij} is the probability that in the null model node i and node j are linked.
The typical choice for the null model is the configuration model with

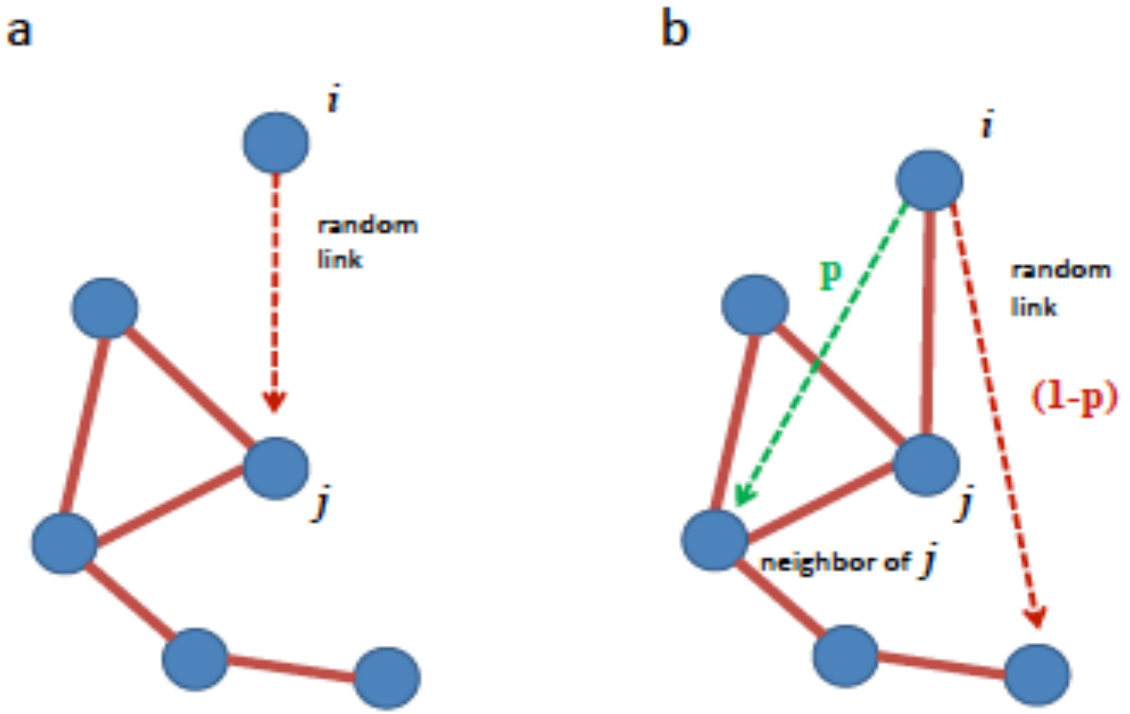
$$p_{ij} = \frac{k_i k_j}{2L}$$

Emergent properties of simplicial complexes

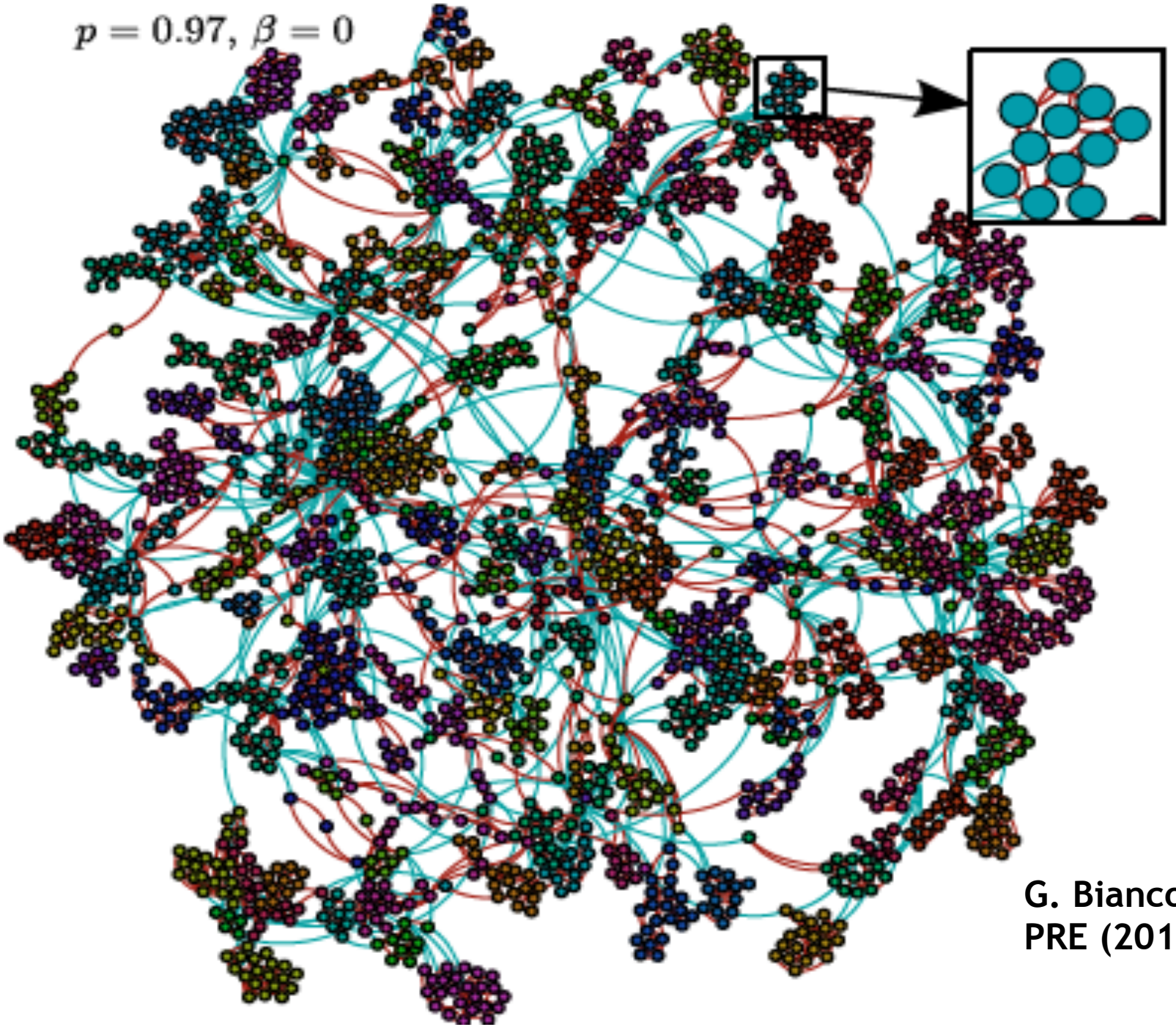
Emergence of communities

Triadic closure

- Starting from a finite connected network with $n_0>2$ nodes
- **(1) GROWTH** : At every timestep we add a new node with 2 edges (connected to the nodes already present in the system).
- **(2) TRIADIC CLOSURE**: The first link is attached to a random node, the second link with probability p closes a triangle and with probability 1-p is connected randomly



Emergence of communities

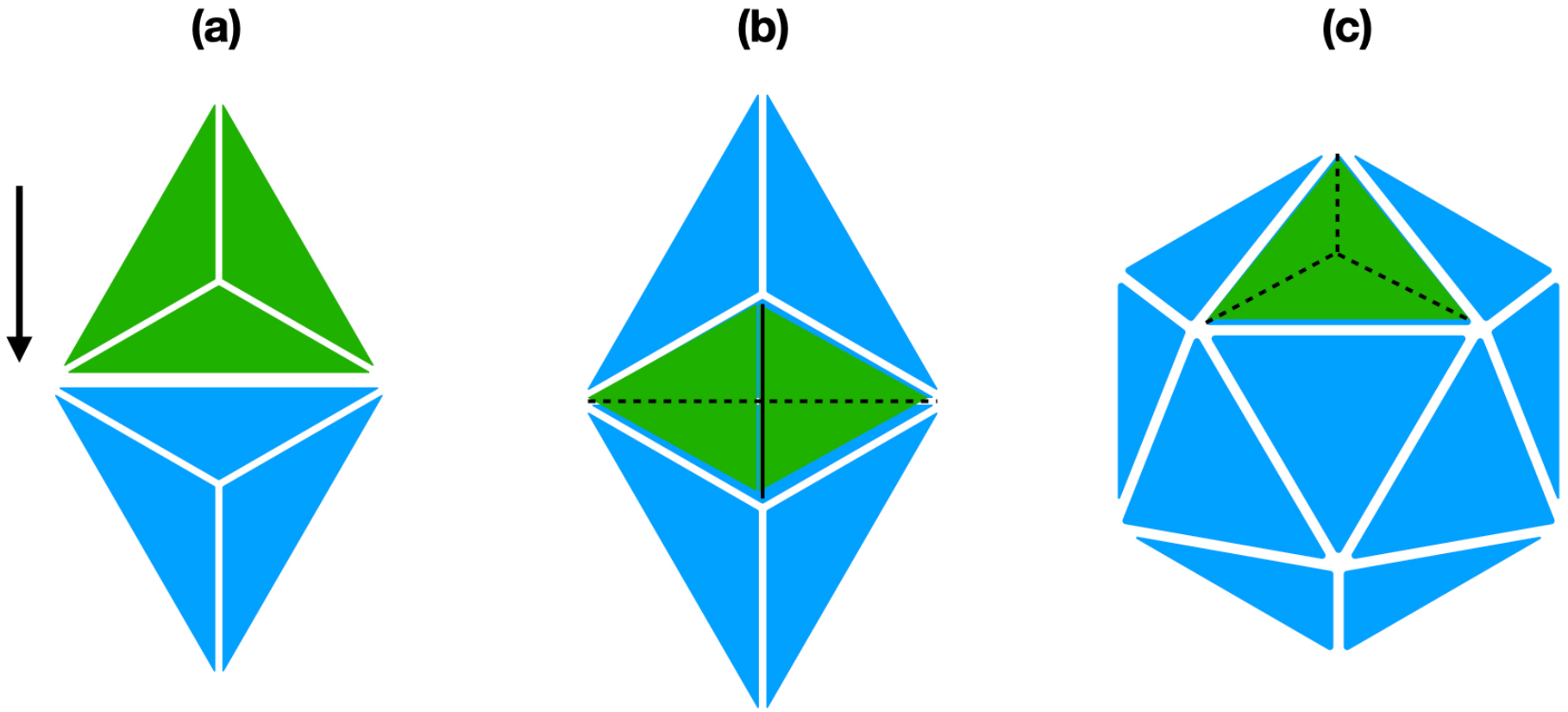


G. Bianconi et al.
PRE (2014)

Topological moves

Topological moves

Topological moves enumerate the ways of adding/removing simplices without changing the topology of a discrete manifold

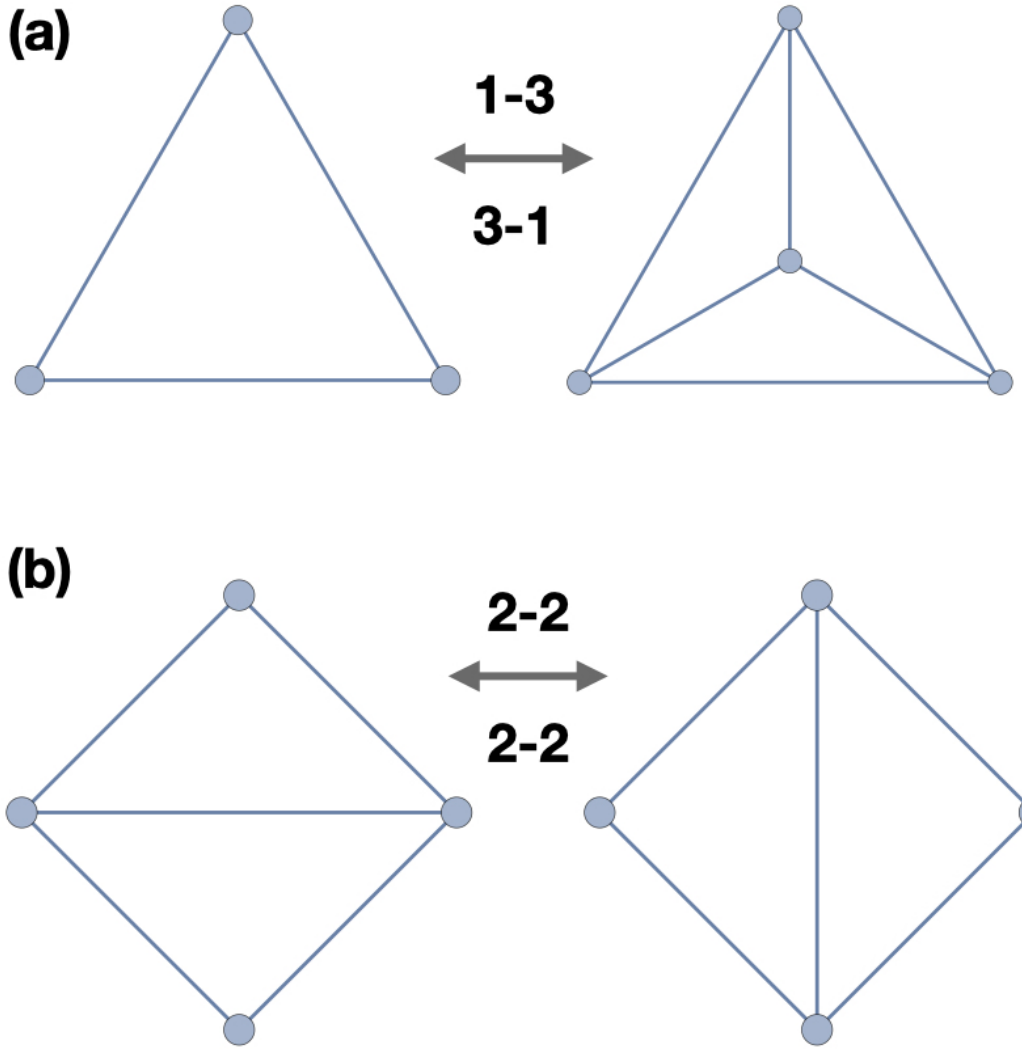


Topological moves in 3D

Topological moves

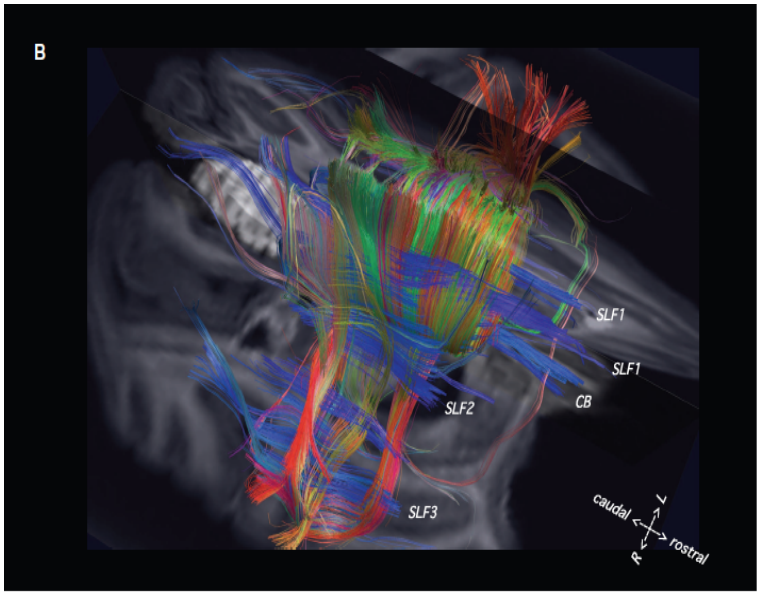
3D Topological moves

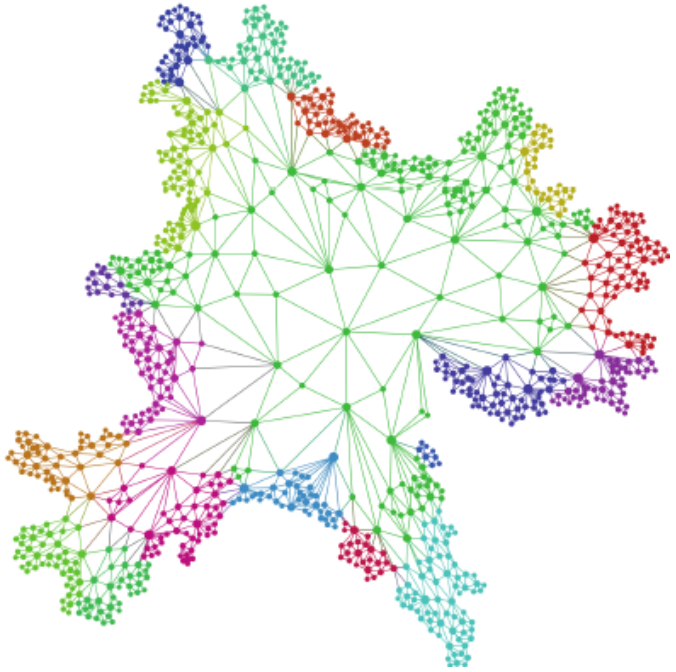
Projected in 2D



Emergent geometry

Network Topology and Geometry





are expected to have impact in a variety of applications,

ranging from

brain research to biological transportation networks

Is the network geometry of complex systems
an a priori pre-requisite
for the network evolution
or is an emergent phenomenon of the
network dynamics?

Emergent geometry

In the framework of emergent geometry
networks with a geometry
are generated
by non-equilibrium dynamics
that is purely combinatorial,
i.e. is independent of the network geometry

Emergent geometry in 2-dimensional simplicial complexes

Emergent network geometry

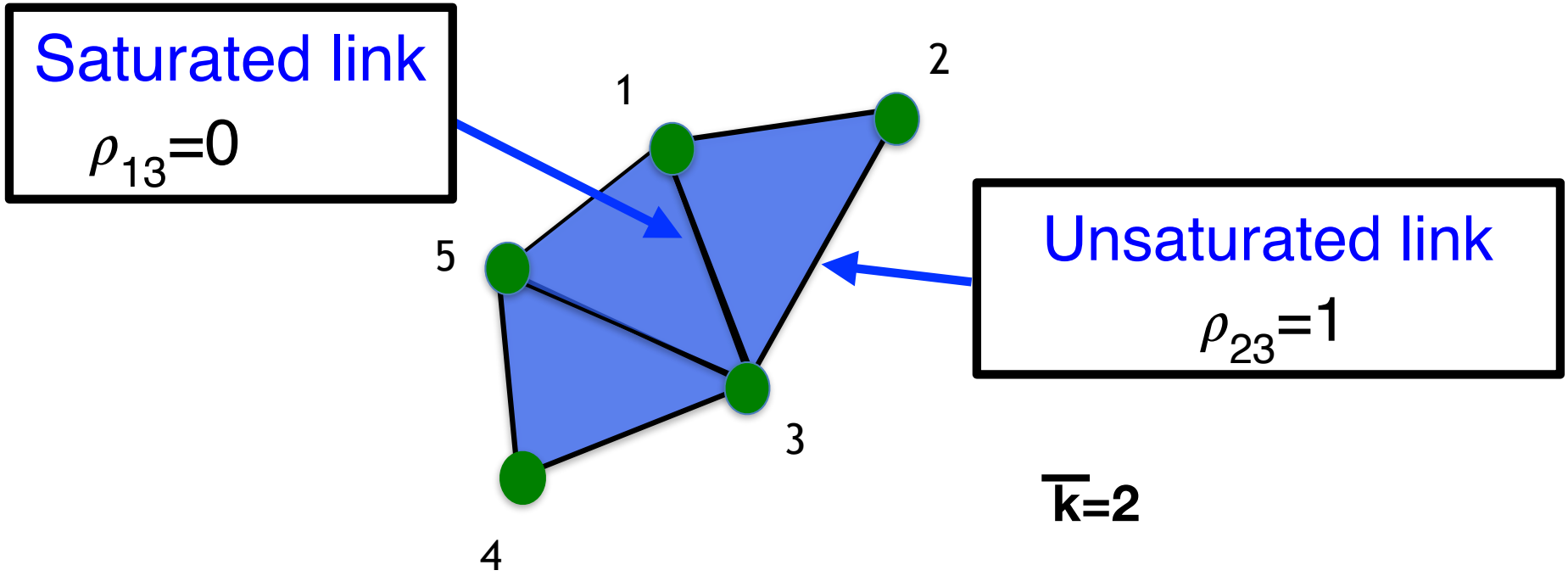
The model describes
the underlying structure of a simplicial complex
constructed by gluing together triangles by a
non-equilibrium dynamics.

Every link is incident to

at most $\overline{k}$ triangles with $\overline{k}>1$.

Wu, Menichetti, Rahmede, Bianconi, Scientific Reports (2015)

Saturated and unsaturated links



We classify links $[r, s]$ as *unsaturated* and *saturated* depending on the value of the auxiliary variable ρ_{rs} defined as

$$\rho_{rs} = \begin{cases} 0 & \text{if} \quad k_{2,1}([r,s]) < \bar{k}, \\ 1 & \text{if} \quad k_{2,1}([r,s]) = \bar{k}. \end{cases} \tag{5.1}$$

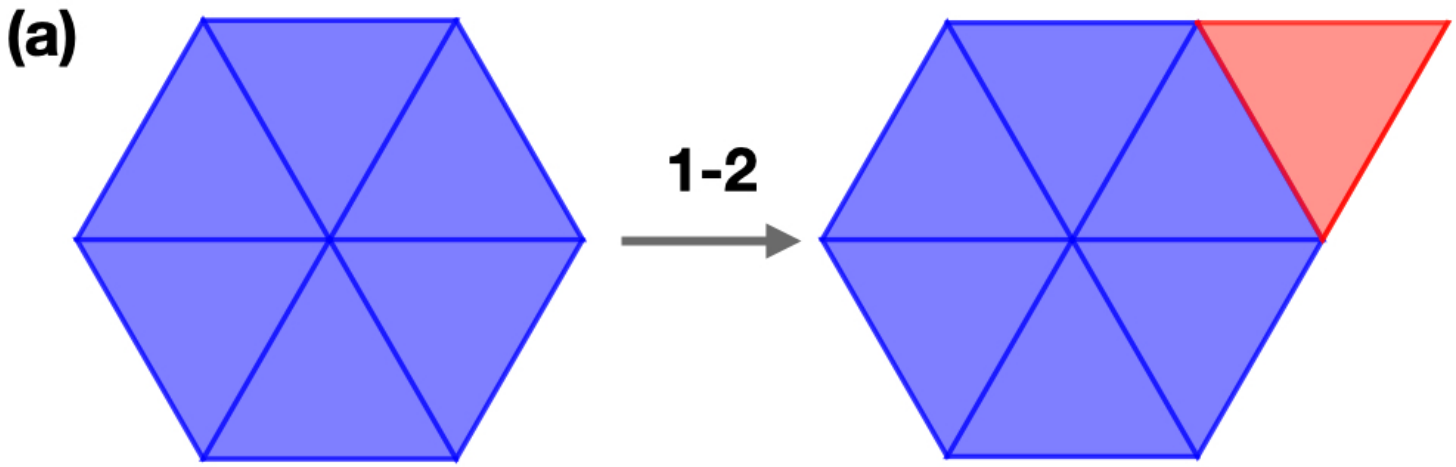
Therefore for each link $[r, s]$ there are two possibilities:

- if $\rho_{rs} = 0$ the link is *unsaturated* , i.e. less than $\bar{k}$ triangles are incident on it;
- if $\rho_{rs} = 1$ if the link is *saturated*, i.e. the number of incident triangles is given by $\bar{k}$.

Process (a)

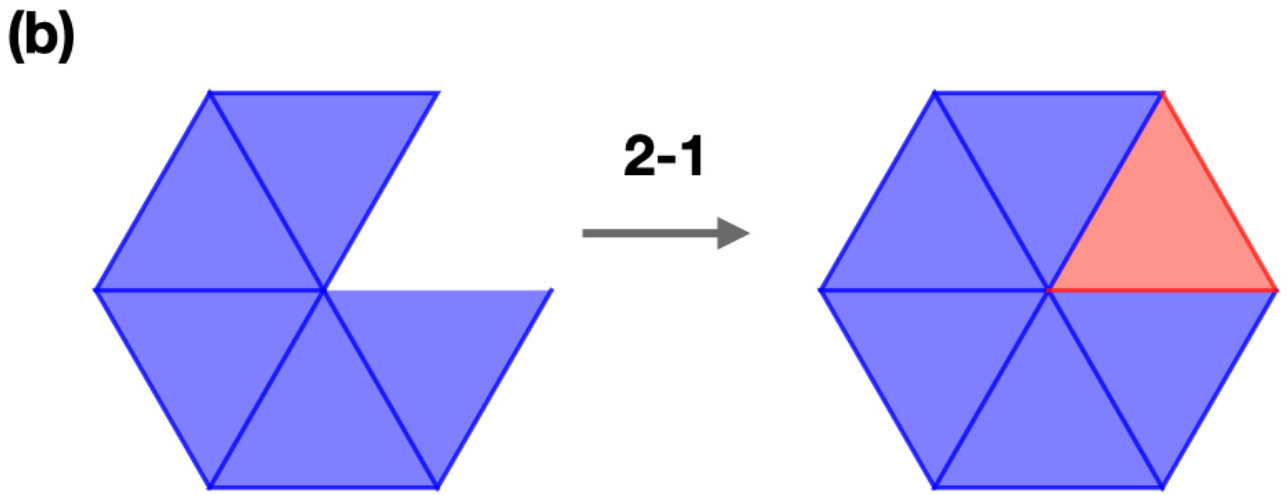
We choose a link (i,j) with probability and glue a new triangle the link

$$\Pi_{(i,j)} = \frac{\rho_{ij}}{\sum_{r,s} \rho_{rs}}$$



Process (b)

**We choose a two adjacent unsaturated links
and we add the link between the nodes at distance 2
and all triangles that this link closes
as long that this is allowed.**



The model

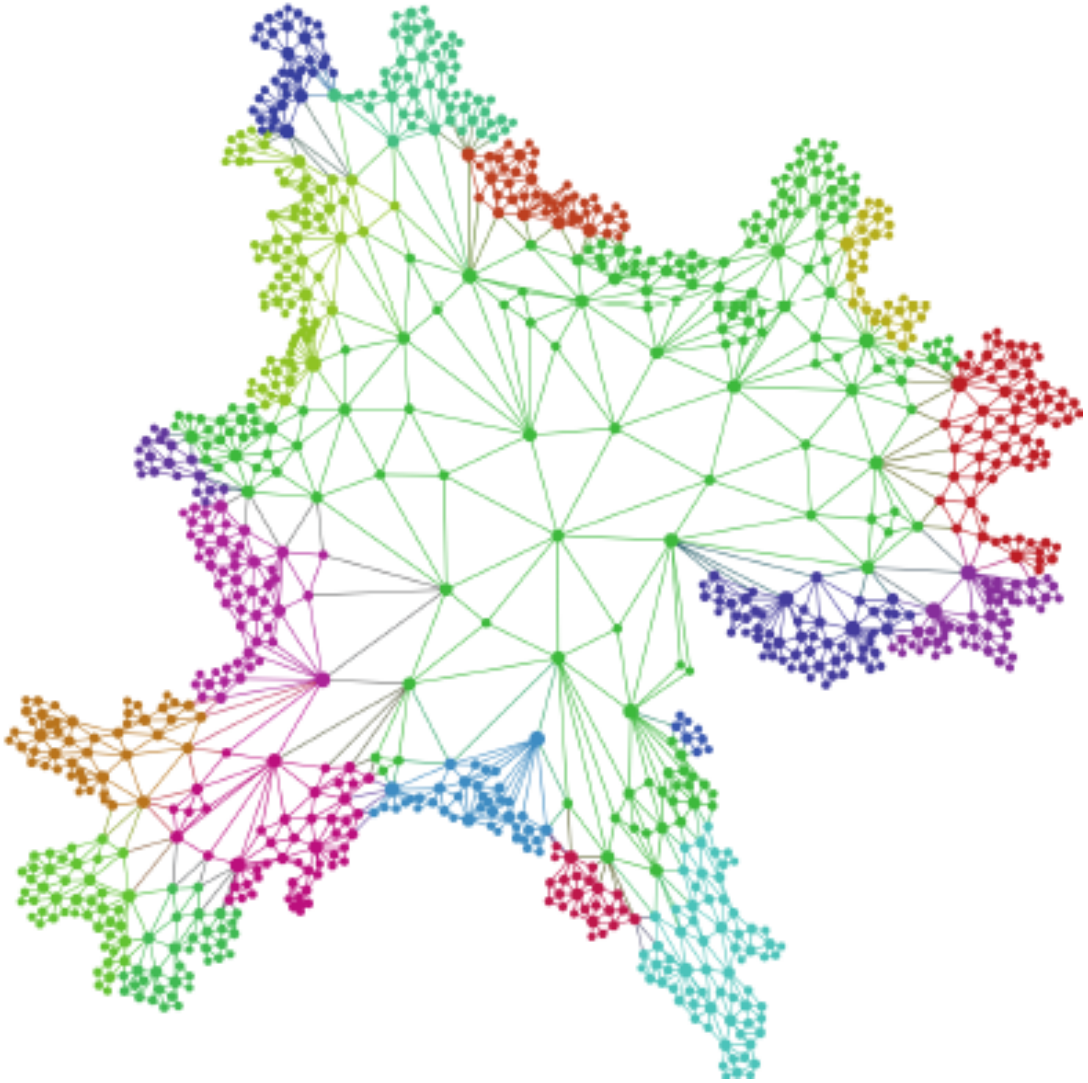
Starting from an initial triangle,
At each time

- **process (a) takes place**

and

- **process (b) takes place**
with probability p<1.

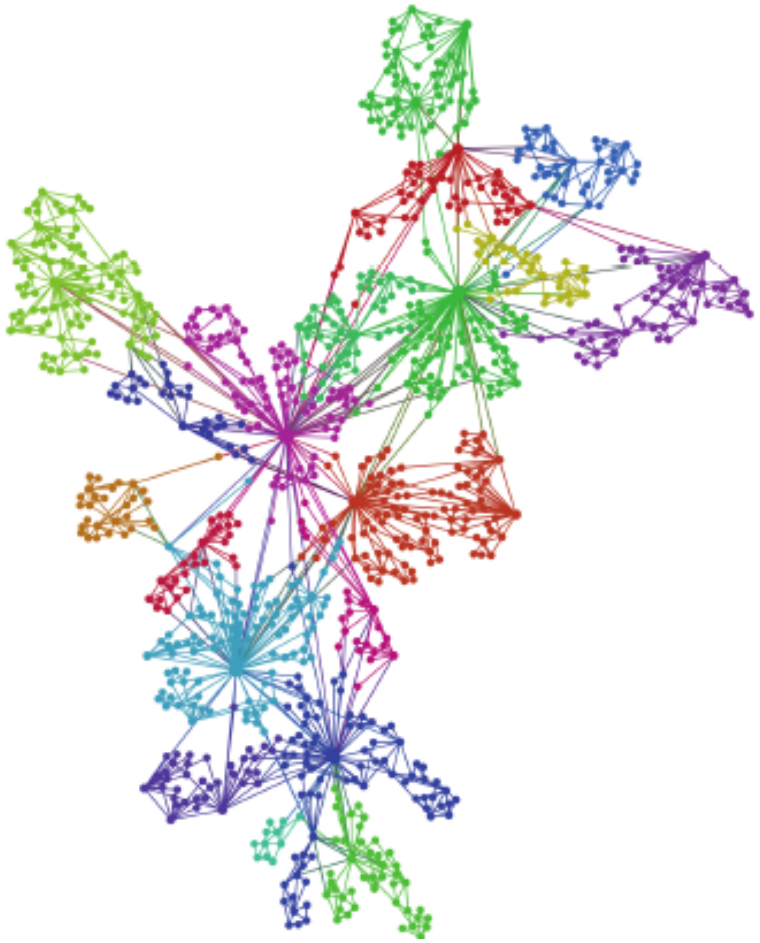
Discrete Manifolds



A discrete manifold of dimension d=2 is a simplicial complex formed by triangles such that every link is incident to at most two triangles.

Therefore the emergent network geometry for our model with $\bar{k} = 2$ is a discrete 2d manifold.

Scale-free networks



In the case $\bar{k} = \infty$
a scale-free network with high clustering, significant community structure, finite spectral dimension is generated.

Planar for p=0.

Emergent preferential attachment

If we add triangles to link with uniform probability,

i.e. we add a new triangle to a link (i, j) with probability

$$\pi_{ij} = \frac{a_{ij}}{L},$$

the probability Π_i of adding a new link to a node i is given by

$$\Pi_i = 2\frac{k_i}{\sum_j k_j}$$

i.e. obeys preferential attachment

Emergent preferential attachment

If we add triangles to link with uniform probability,

i.e. we add a new triangle to a link (i, j) with probability

$$\pi_{ij} = \frac{a_{ij}}{L},$$

the probability Π_i of adding a new link to a node i is given by

$$\Pi_i = \sum_j \pi_{ij} = \sum_j \frac{a_{ij}}{L}$$

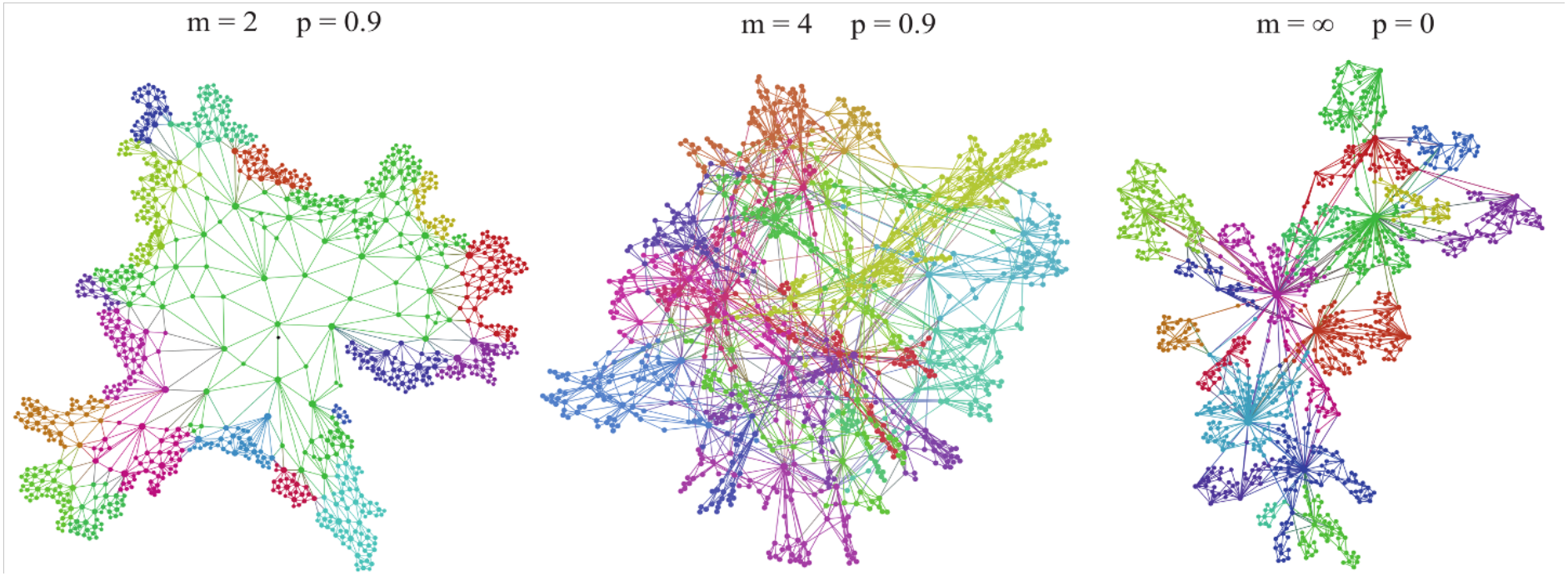
Using

$$k_i = \sum_j a_{ij} \quad L = \frac{1}{2} \sum_j k_j$$

we obtain

$$\Pi_i = 2 \frac{k_i}{\sum_j k_j}$$

Curvature distribution

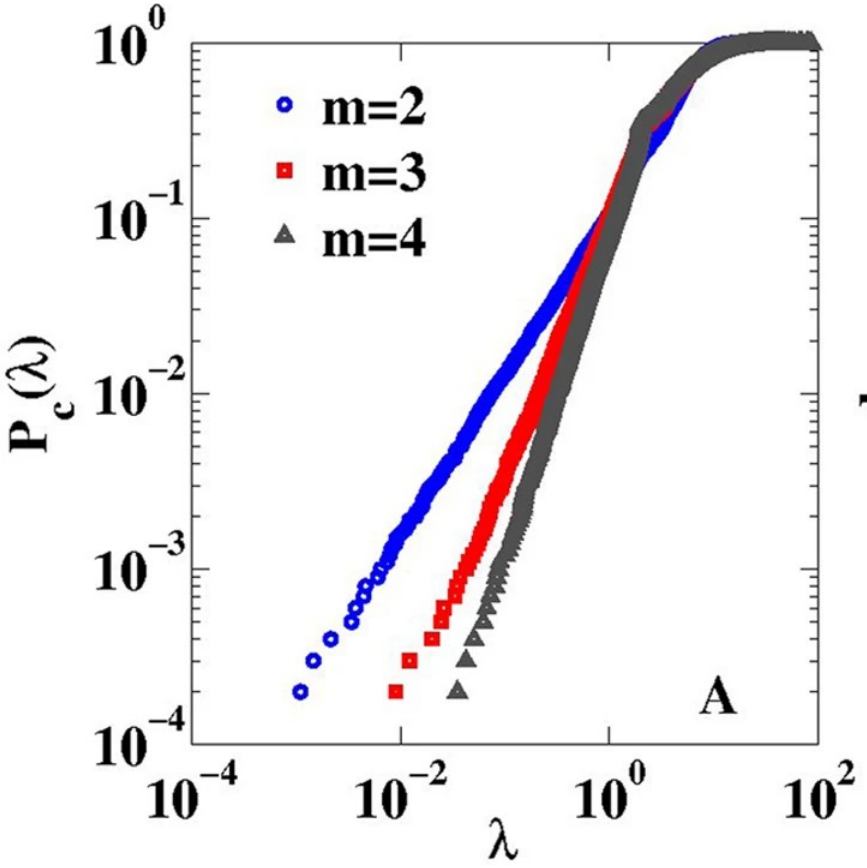


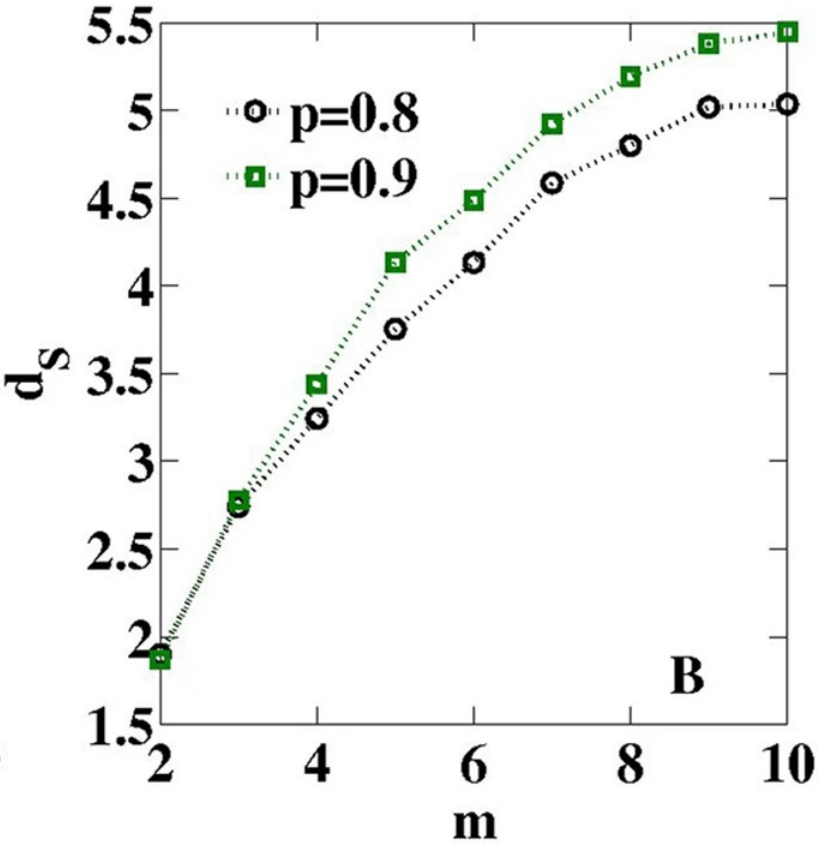
Planar $\langle R \rangle = 1$
Exponential degree distribution
Exponential negative tail of
Local curvatures
$\langle R^2 \rangle < \infty$ **as** $N \to \infty$

Non-planar
Broad
degree distribution

Planar $\langle R \rangle = 1$
Scale-free degree distribution
Power-law negative tail of
Local curvatures
$\langle R^2 \rangle \to \infty$ **as** $N \to \infty$

Spectral dimension of emergent geometry





$$\bar{k} = m$$

This emergent geometries display a finite spectral dimension

Properties of emergent network geometries

- **Small world**
- **Finite clustering**
- **High modularity**
- **Finite spectral dimension**

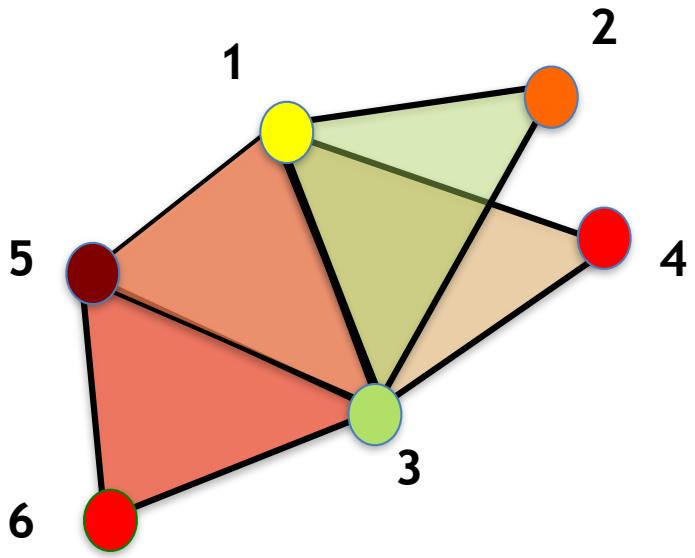
Which are properties of many real network datasets.

Properties of real datasets

Datasets	N	L	$\langle \ell \rangle$	C	M	d_S
1L8W (protein)	294	1608	5.09	0.52	0.643	1.95
1PHP (protein)	219	1095	4.31	0.54	0.638	2.02
1AOP chain A (protein)	265	1363	4.31	0.53	0.644	2.01
1AOP chain B (protein)	390	2100	4.94	0.54	0.685	2.03
Brain-(coactivation) [45]	638	18625	2.21	0.384	0.426	4.25
Internet [46]	22963	48436	3.8	0.35	0.652	5.083
Power-grid[38]	4941	6594	19	0.11	0.933	2.01
Add Health (school61)[47]	1743	4419	6	0.22	0.741	2.97

Network Geometry with Flavor

Network Geometry with Flavor



NETWORK GEOMETRY WITH FLAVOR (NEUTRAL MODEL) [29]

At time $t = 1$ the NGF is formed by a single d-dimensional simplex. At each time $t > 1$ the model evolves according to the following principles.

- GROWTH : At every timestep a new d-dimensional simplex formed by one new node and an existing $(d-1)$-face is added to the simplicial complex.
- ATTACHMENT: The probability that the new d-simplex is glued to a $(d-1)$-dimensional face α depends on the *flavor* $s \in \{-1, 0, 1\}$ and is given by

$$\Pi_{\alpha}^{[s]} = \frac{(1 + s n_{\alpha})}{\sum_{\alpha'}(1 + s n_{\alpha'})}. \tag{5.6}$$

Bianconi & Rahmede (2016)

Attachment probability

The attachment probability to (d-1)-dimensional faces is given by

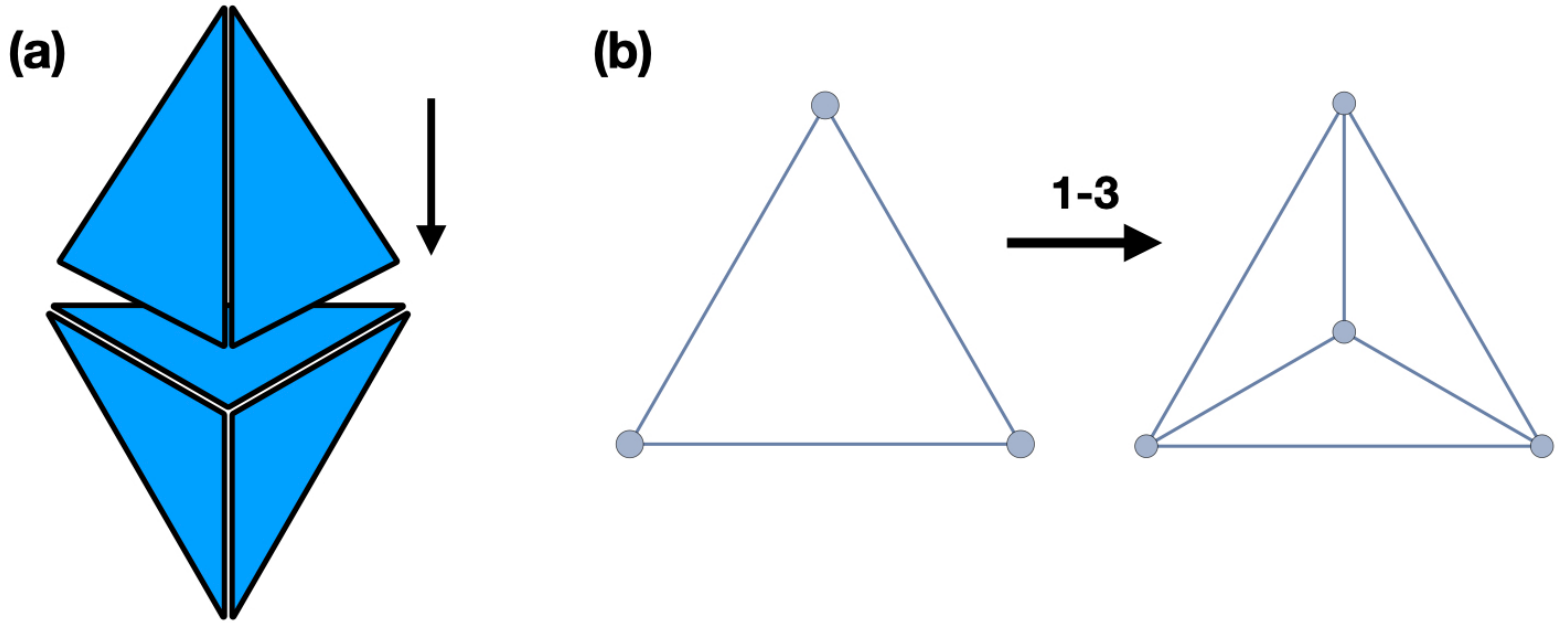
$$\Pi_{\alpha}^{[s]} = \frac{(1+sn_{\alpha})}{\sum_{\alpha'}(1+sn_{\alpha'})} \propto \begin{cases} 1-n_{\alpha} & \textbf{if } s=-1 \\ 1 & \textbf{if } s=0 \\ k_{d,d-1}(\alpha) & \textbf{if } s=1 \end{cases}$$

For s=-1 we obtain discrete manifolds $n_{\alpha} = 0,1$

For s=0 we have uniform attachment $n_{\alpha} = 0,1,2,3,4...$

For s=1 we have a generalised preferential attachment $n_{\alpha} = 0,1,2,3,4...$

Pachner move 1-d for NGF with s=-1



Emergence of preferential attachment

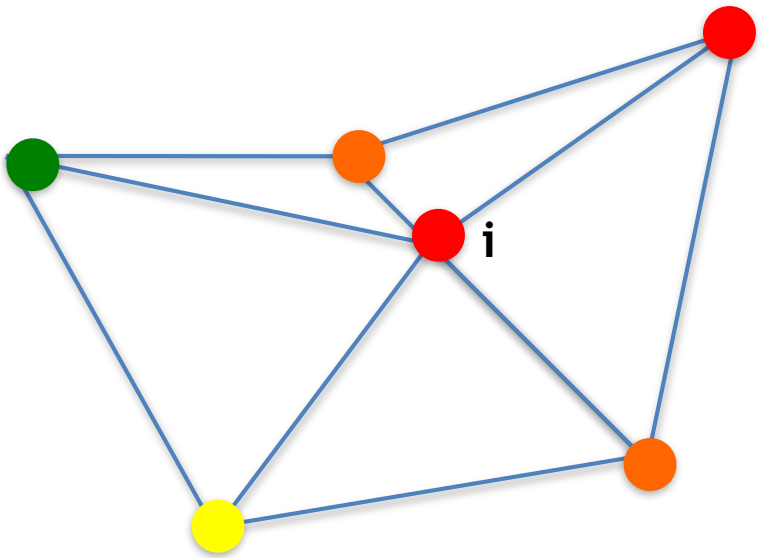
The probability of attaching a d-dimensional simplex to a δ-dimensional face is given by

$$\Pi_{d,\delta}(k) = \begin{cases} \dfrac{2-k}{(d-1)t} & \textbf{for } d+s-\delta-1=-1 \\ \dfrac{(d-\delta-1+s)k+1-s}{(d+s)t} & \textbf{for } d+s-\delta-1\geq 0 \end{cases}$$

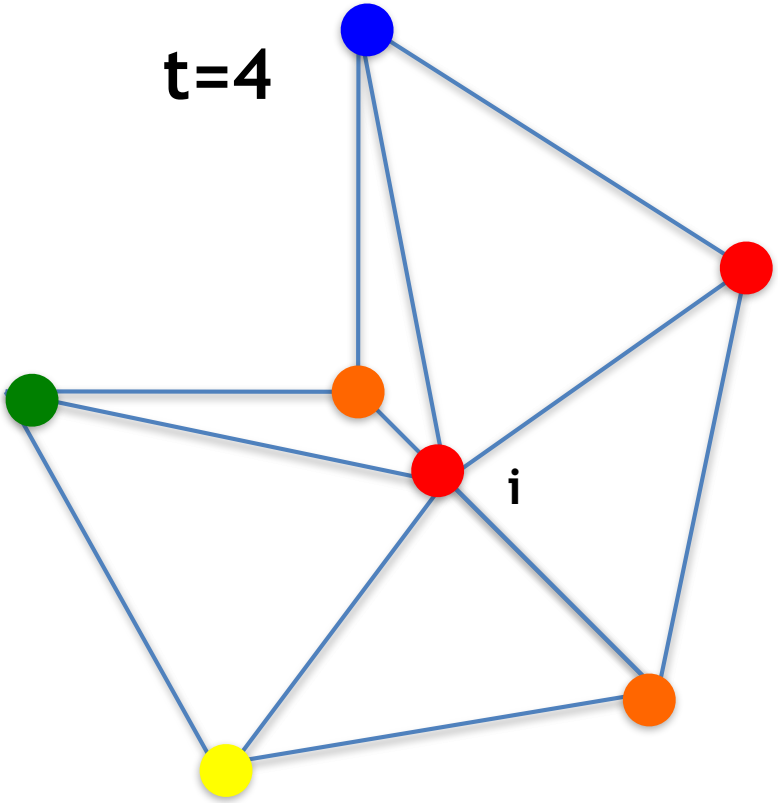
Therefore for $d-\delta > 1-s$ we observe a generalised preferential attachment as a consequence of the geometry and dimensionality of of the NGF

Effective preferential attachment in d=3 s=-1

t=3



t=4

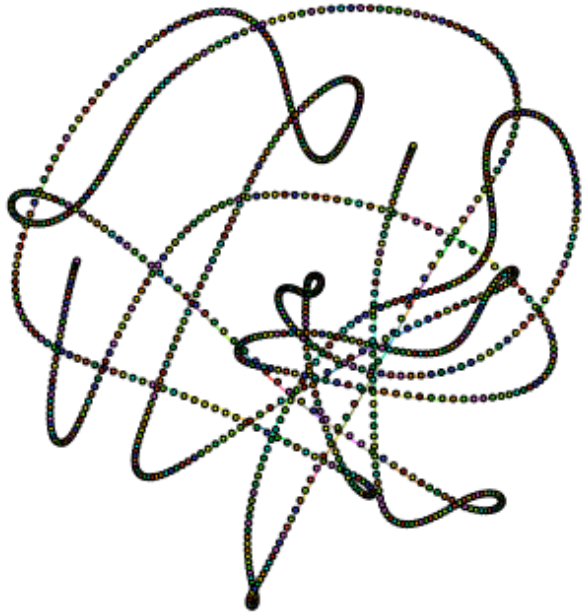


Node i has generalized degree 3
Node i is incident to 5 faces with n=0

Node i has generalized degree 4
Node i is incident to 6 faces with n=0

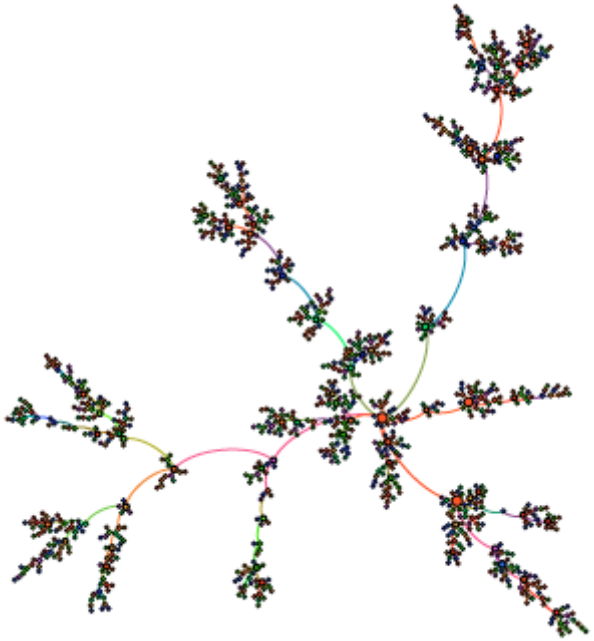
Dimension d=1

Manifold



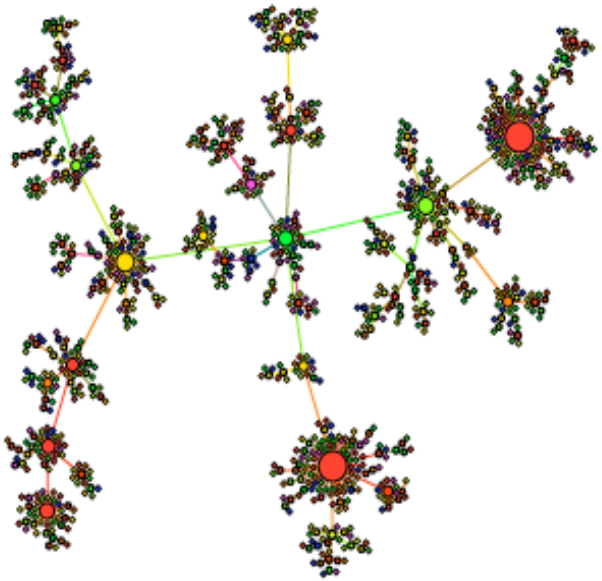
Chain

Uniform attachment



Exponential

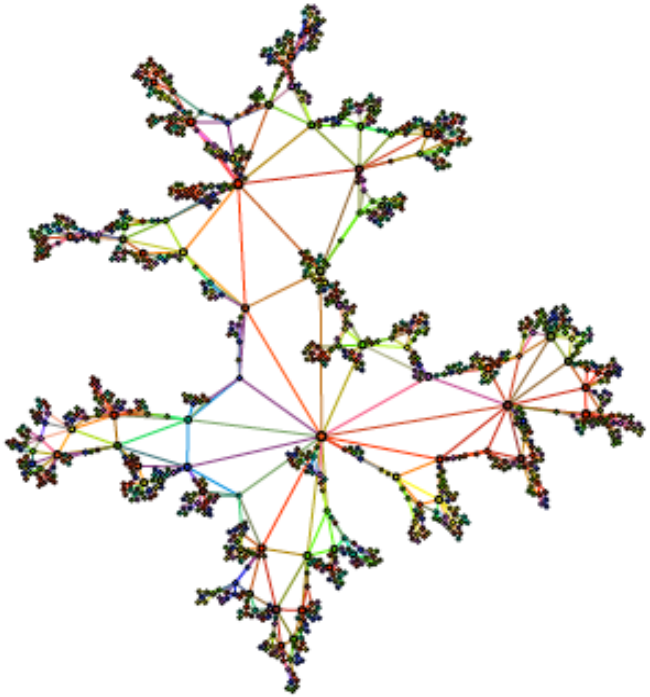
Preferential attachment



BA model

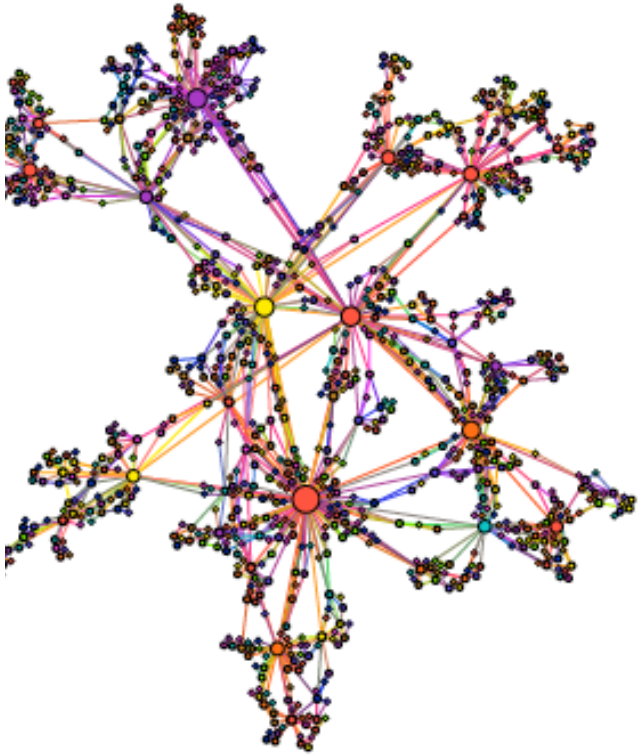
Dimension d=2

Manifold



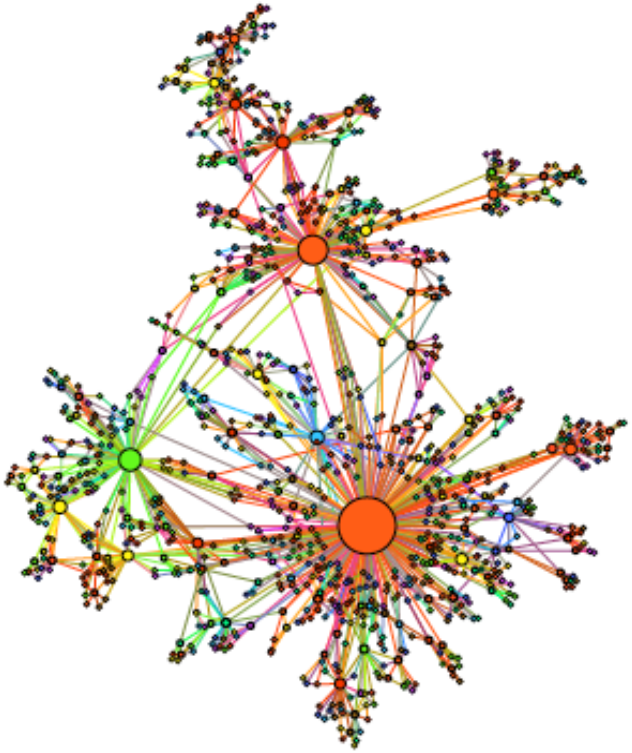
Exponential

Uniform attachment



Scale-free

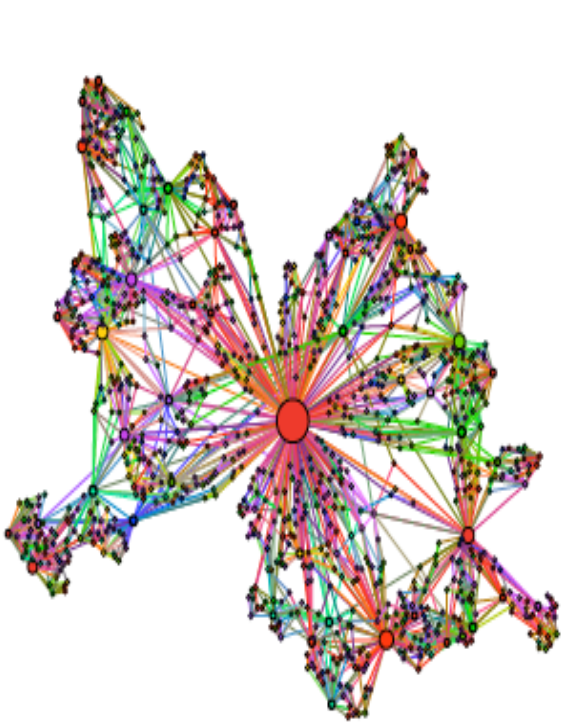
Preferential attachment



Scale-free

Dimension d=3

Manifold

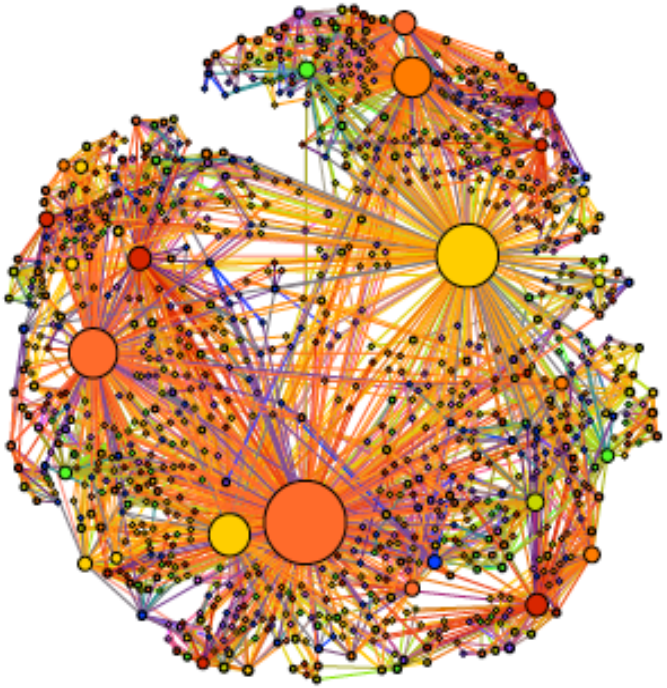


Scale-free

Uniform attachment

Scale-free

Preferential attachment



Scale-free

Degree distribution

For d+s=1

$$P_d^{[s]}(k) = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

For d+s>1

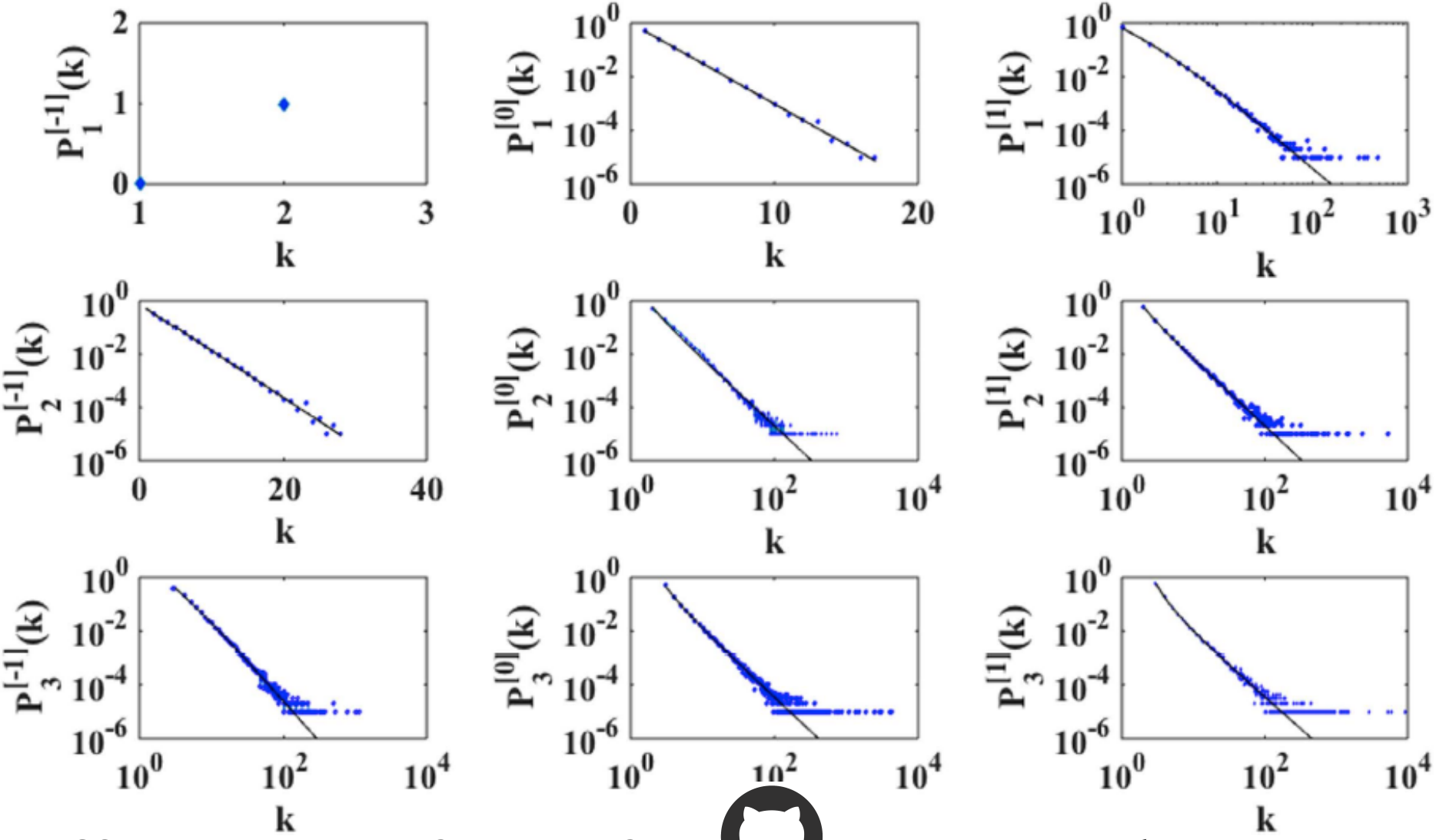
$$P_d^{[s]}(k) = \frac{d+s}{2d+s} \frac{\Gamma[1+(2d+s)/(d+s-1))]}{\Gamma[d/(d+s-1)]} \frac{\Gamma[k-d+d/(d+s-1)]}{\Gamma[k-d+1+(2d+s)/(d+s-1)]}$$

NGF are always scale-free for d>1-s

- **For s=1 NGF are always scale free**
- **For s=0 and d>1 the NGF are scale-free**
- **For s=-1 and d>2 the NGF are scale-free**

[Bianconi & Rahmede (2016)]

Degree distribution of NGF



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Generalized degree distributions

The generalized degree distribution depends on both d and m

and can be calculated with the master equation approach

getting the following exact asymptotic results

$$P^{[s]}_{d,m}(k) = \begin{cases} (d-1)/d & \textbf{for} \quad k = 1, \\ 1/d & \textbf{for} \quad k = 2. \end{cases}$$

For m+d+s=0

$$P^{[s]}_{d,m}(k) = \left(\frac{d-m}{d+1}\right)^k \frac{m+1}{d-m}.$$

For m+d+s=1

$$P^{[s]}_{d,m}(k) = \mathscr{C} \frac{\Gamma[k + (1-s)/(d-m+s-1)]}{\Gamma[k+1+(d+1)/(d-m+s-1)]},$$

For m+d+s>1

[Bianconi & Rahmede (2016)]

Generalized degree distribution

Simplicial complexes can have generalised degree distribution following different statistics depending on the dimension of the faces considered

Flavor	$s = -1$	$s = 0$	$s = 1$
$m = d - 1$	Bimodal	Exponential	Power-law
$m = d - 2$	Exponential	Power-law	Power-law
$m \leq d - 3$	Power-law	Power-law	Power-law

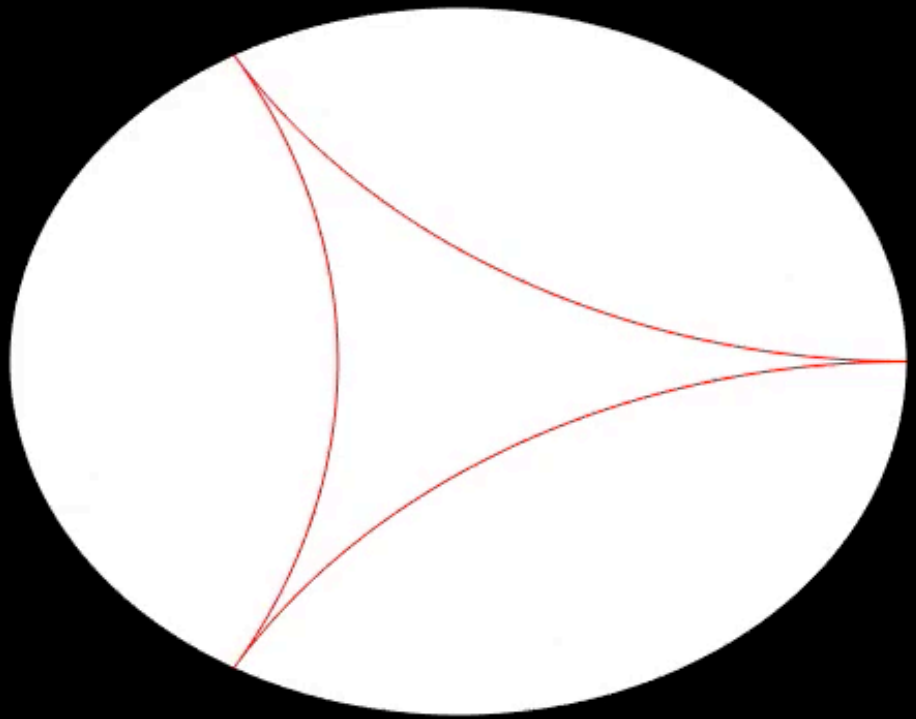
The generalized degree distribution depends on the flavor s and on the dimension m of the faces

[Bianconi & Rahmede (2016)]

Emergent Hyperbolic geometry

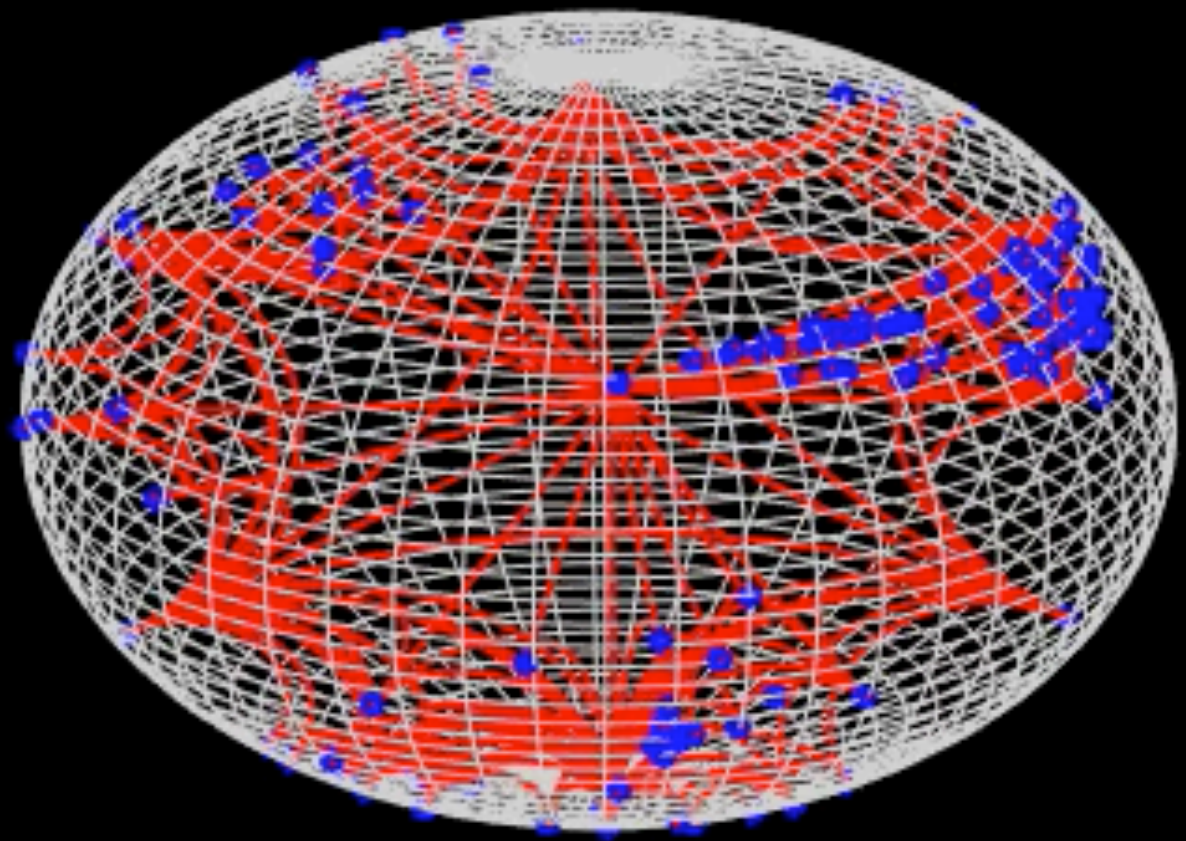
The emergent hidden geometry is the hyperbolic H^d space

Here all the links have equal length



d=2

Emergent hyperbolic geometry



d=3

NGF an hyperbolic network geometry

NGF for flavor s=-1 are discrete hyperbolic manifolds

NGF of any flavor and any dimension are δ-hyperbolic networks

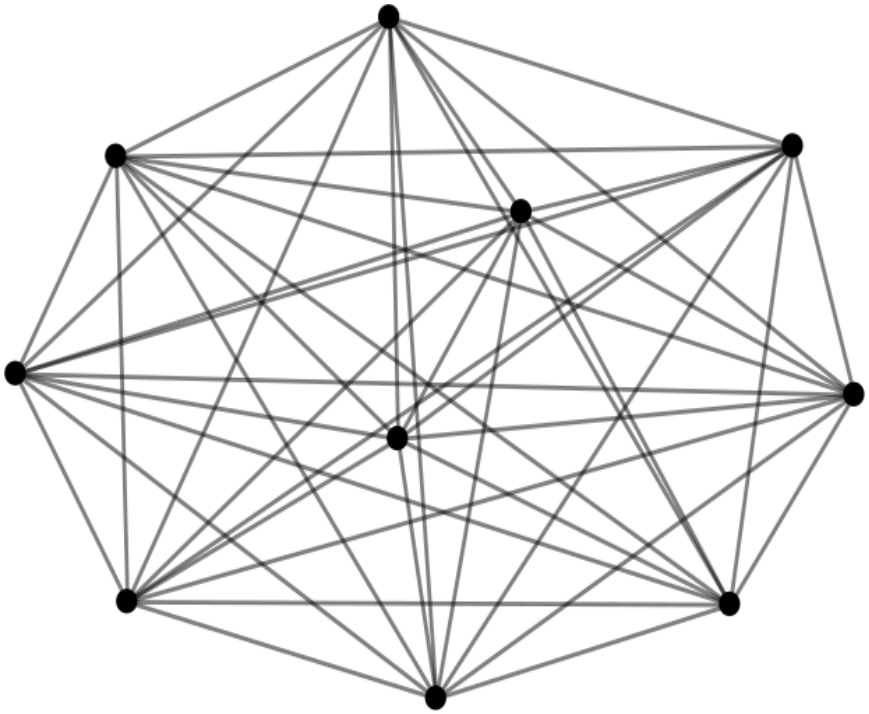
[with δ=1 in the case of simplicial complexes]

What is a “natural” random geometry?

Randomness and order
Random graph

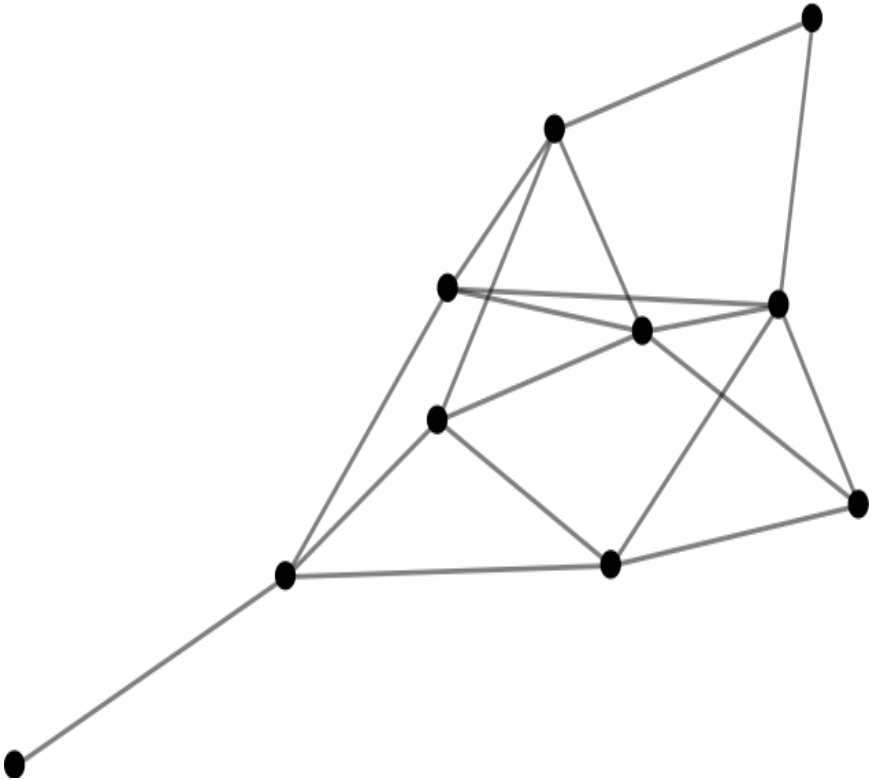
A fully connected network -trivial/no geometry- where some random links are selected

p=1



Complete graph

p=0.4



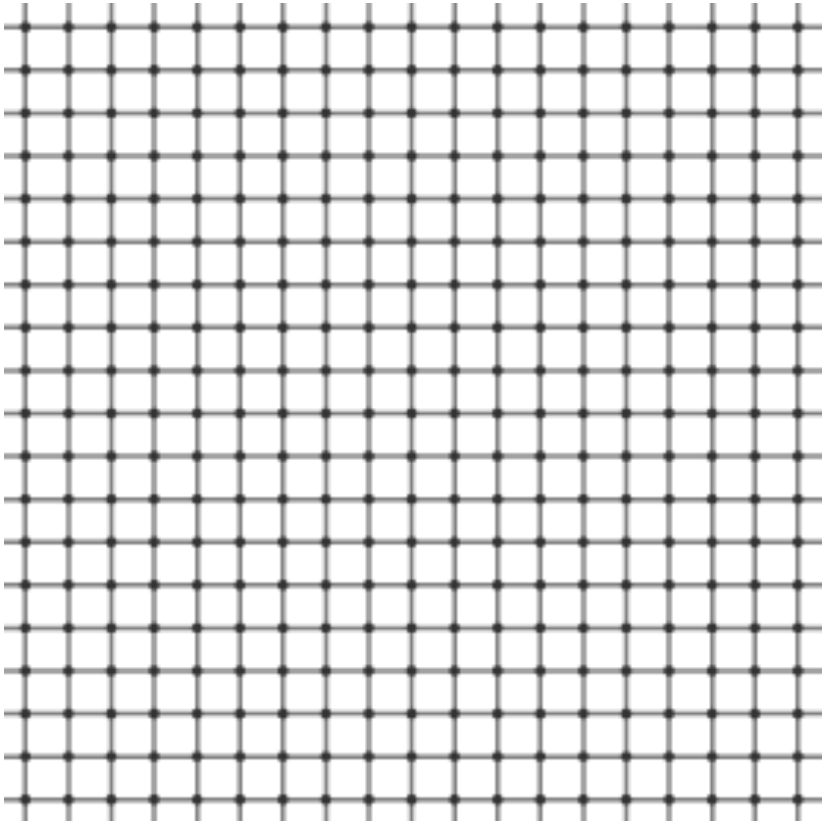
Random graph

Randomness and order
Percolation

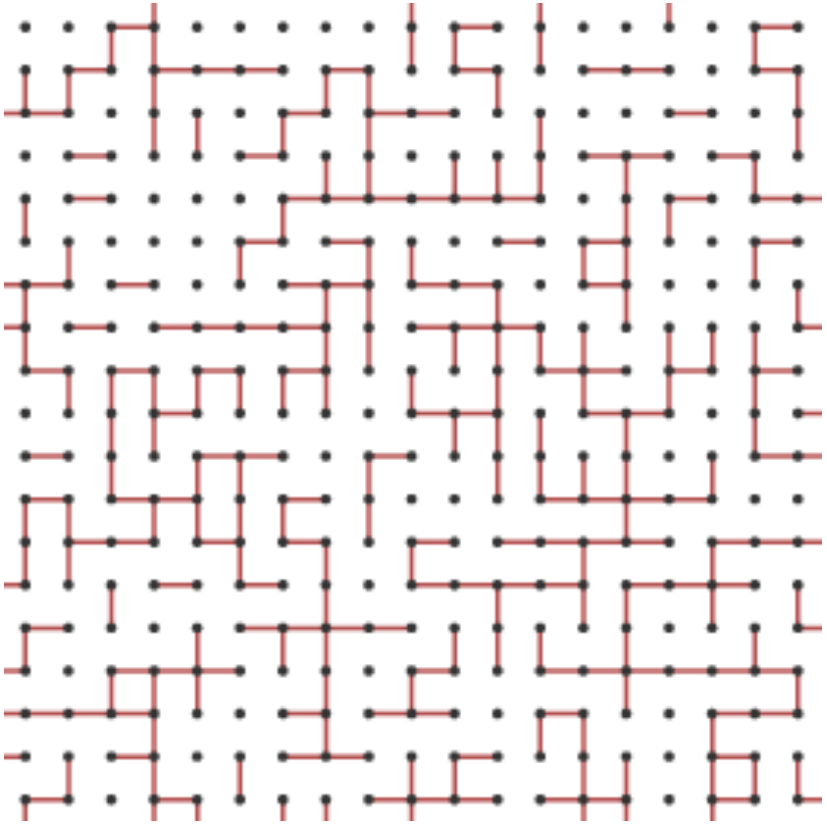
A square lattice -known, given geometry-

where only few links are preserved

p=1

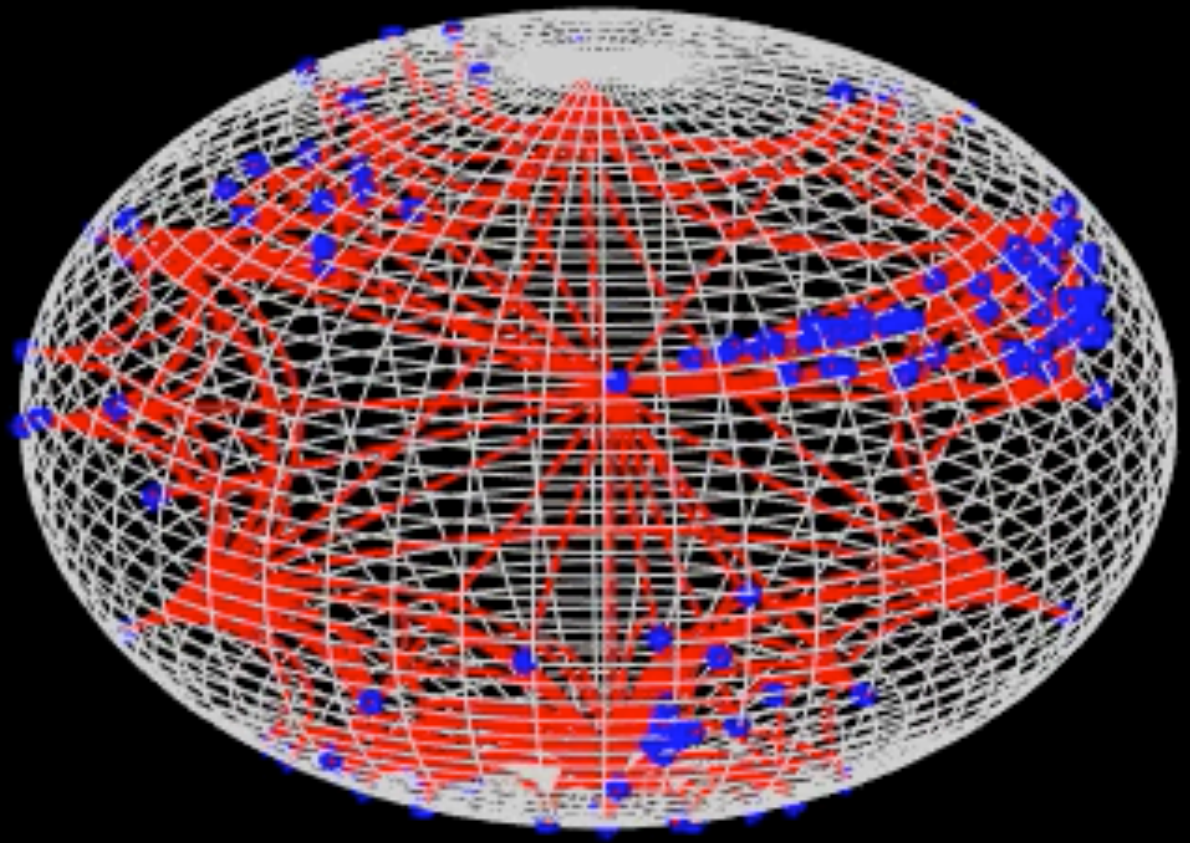


p=0.4



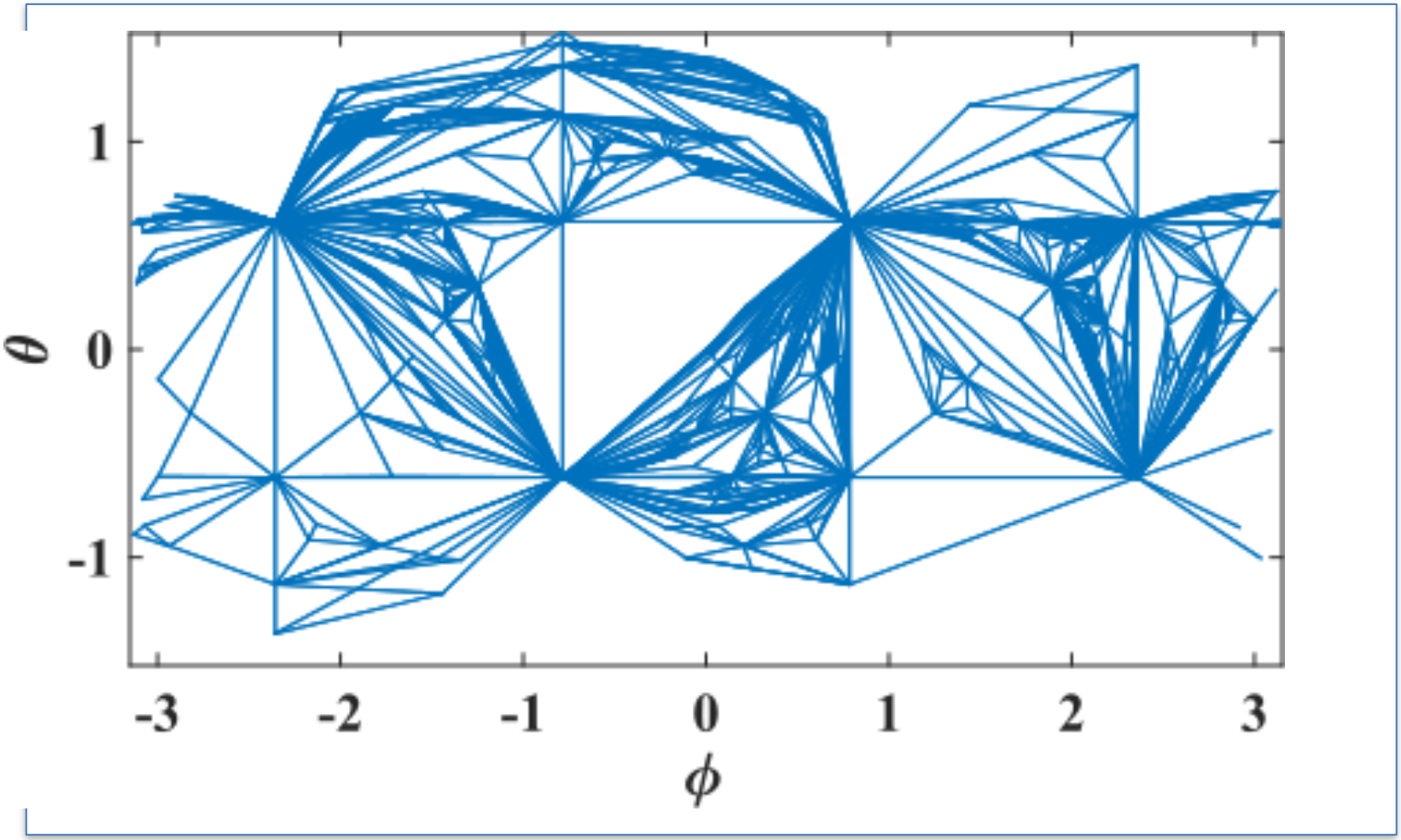
Emergent hyperbolic geometry

A growing cluster on -emergent- hyperbolic lattice

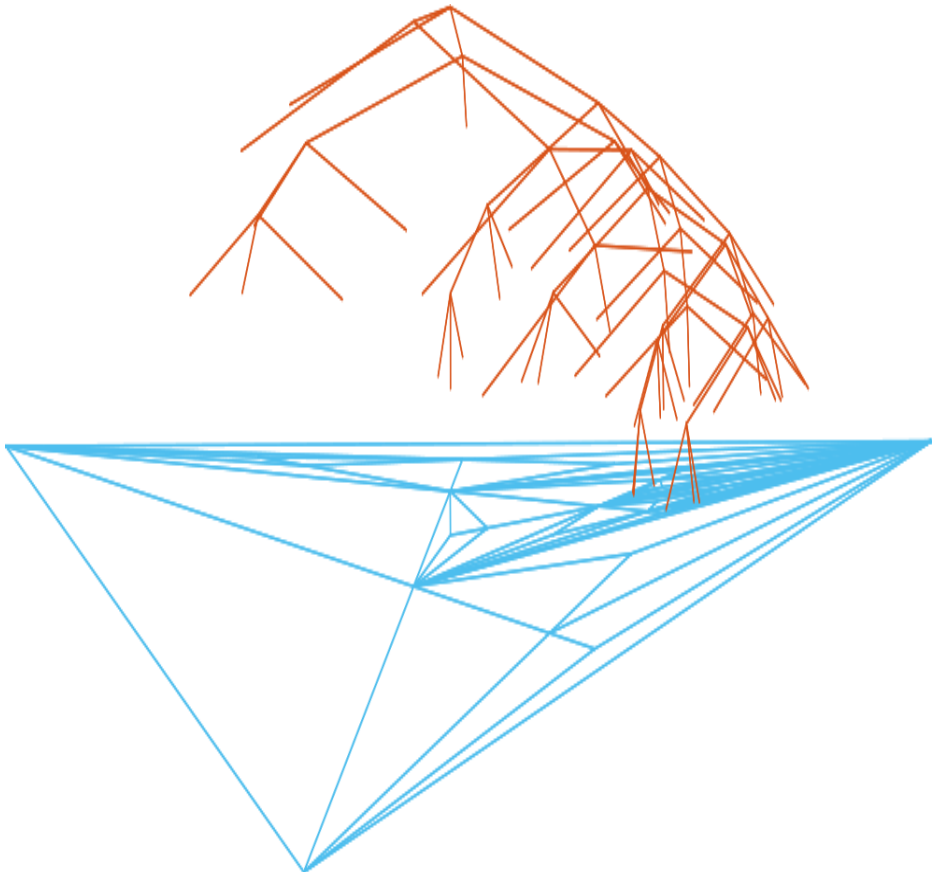


d=3

Planar projection of the d=3 NGF with s=-1

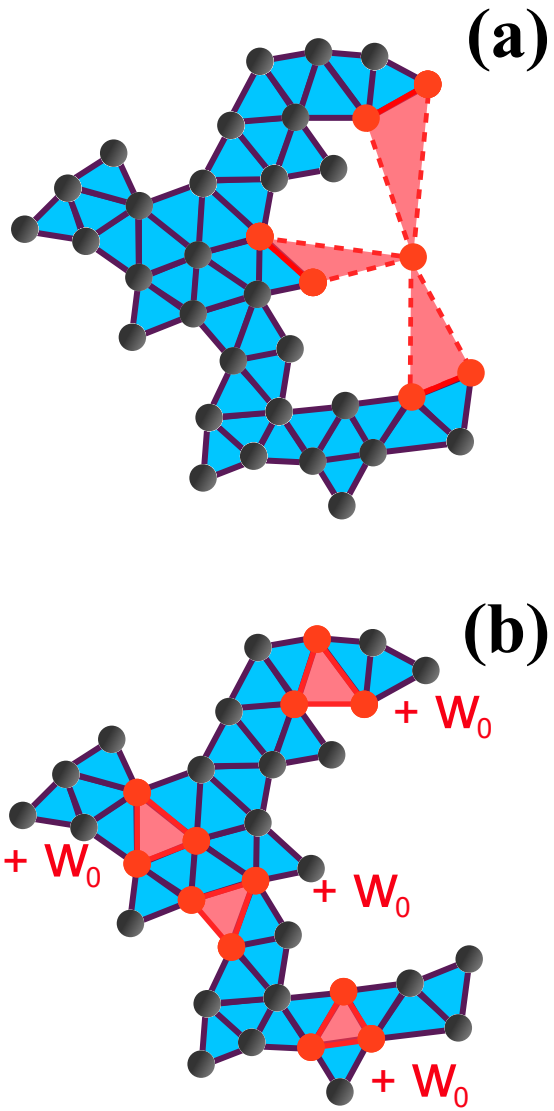


The relation to Trees



Line graph of the NGF

Growing weighted simplicial complex



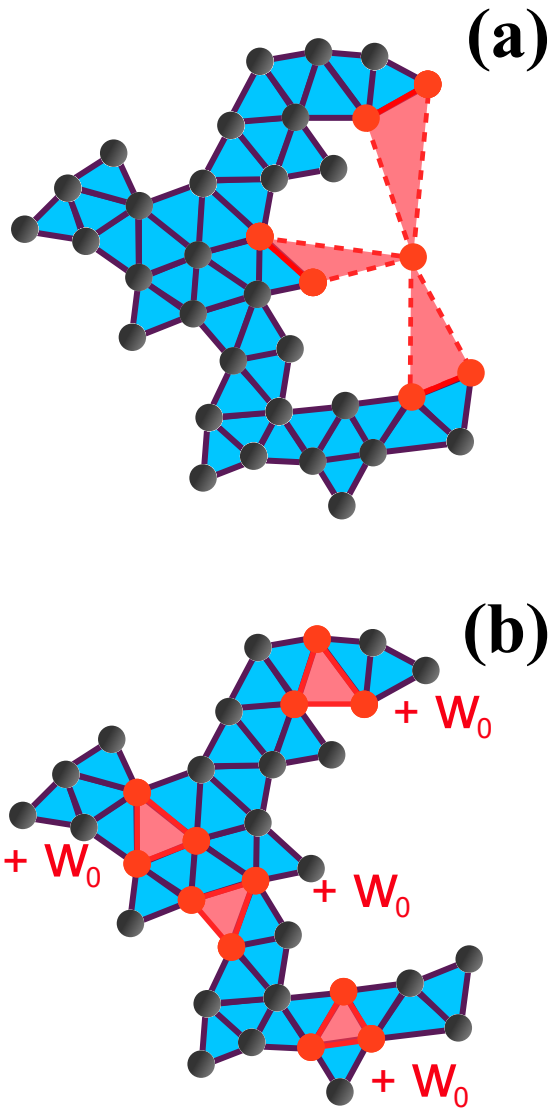
We considered a weighted network model in which we assume:

- that each new node can attach m simplices to the rest of network
- that simplices can increase their weight in time

We found deep correlations between the weights of the simplices and the network topology.

Courtney Bianconi (2017)

Growing weighted simplicial complex



- As long as $m = 1$ these simplicial complexes have a finite spectral dimension.
- For $m > 1$ these simplicial complexes acquire a spectral gap, i.e. a finite Fiedler eigenvalue in the limit $N \to \infty$
- This reveals the mean-field nature of these simplicial complexes with $m > 1$ (characterised by the loss of the local attachment of new simplices)

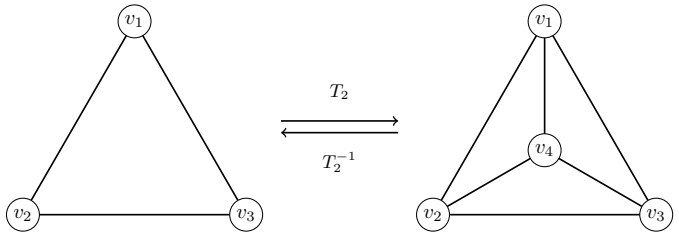
Triangulated Maximally Filtered Graph

Triangulated Maximally Filtered Graphs can be used represent the backbone of correlation matrices

As NGF they are constructed by adding simplices attached to faces but they are following a deterministic construction and the new simplices are attached in order to maximise a gain function based on data

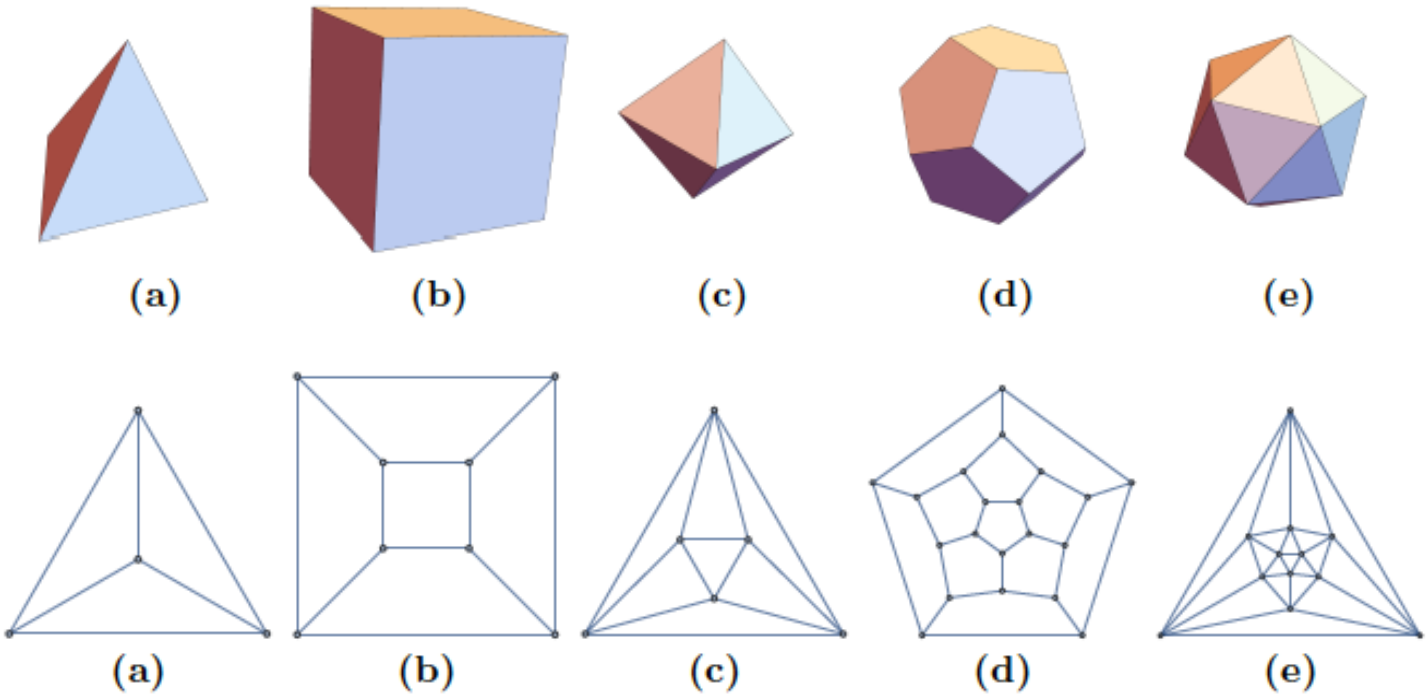
Algorithm 1: TMFG algorithm

input : A dense matrix W with positive weights, e.g. a matrix of squared correlation coefficients
output: A sparse matrix, TMFG, a filtered version of W fulfilling the planarity constraint
// Initialise a tetrahedron th_1 e.g. by using the highest edge-weights, this gives four triangles t_1, t_2, t_3, t_4 as in Fig. 1
$th_1 \leftarrow$ MaxTetrahedron(W) // Tetrahedron with highest overall total gain function
$[t_1, t_2, t_3, t_4] \leftarrow$ Triangles in th_1;
Triangles $\leftarrow [t_1, t_2, t_3, t_4]$;
VertexList $\leftarrow$ List of vertices of W not belonging to th_1;
MaxGain $\leftarrow$ Vector indexed by triangles as in Eq.5;
BestVertex $\leftarrow$ Vector indexed by triangles as in Eq.6;
MaxGain $\leftarrow$ UpdateMaxGain(VertexList, $[t_1, t_2, t_3, t_4]$);
BestVertex $\leftarrow$ UpdateBestVertex(VertexList, $[t_1, t_2, t_3, t_4]$);
$p \leftarrow$ number of vertices in VertexList;
// Insert $p-4$ vertices via T_2
for $i = 1$ *to* $p-4$ **do**
  // Get the triangle with the highest score ...
  $t_{abc} = \underset{t_{xyz} \in \text{Triangles}}{\text{argmax}} \{\text{MaxGain}(t_{xyz})\}$;
  // ...and the corresponding vertex
  $v_i = \text{BestVertex}(t_{abc})$;
  $[t_{a_1}, t_{a_2}, t_{a_3}] \leftarrow$ triangles created by the insertion of v_i into t_{abc};
  VertexList $\leftarrow$ VertexList $\setminus v_i$;
  Triangles $\leftarrow$ Triangles $\cup \{t_{a_1}, t_{a_2}, t_{a_3}\} \setminus t_{abc}$;
  Separators $\leftarrow$ Separators $\cup t_{abc}$;
  MaxGain $\leftarrow$ UpdateMaxGain(VertexList, Triangles);
  BestVertex $\leftarrow$ UpdateBestVertex(VertexList, Triangles);
end
return TMFG;



Massara, Di Matteo, Aste 2017

Cell complexes



A cell complex $\mathcal{K}$ has the following two properties:

(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

Network Geometry with Flavor

Consider pure cell complexes formed by gluing identical regular polytopes along d-1 faces

- Starting from a single d-dimensional regular polytope

(1) GROWTH :

At every timestep we add a new d-dimension polytope glued to an existing (d-1)-face).

(2) ATTACHMENT:

The probability that the new polytope will be connected to a face α depends on the flavor s=-1,0,1 and is given by

$$\Pi_{\alpha}^{[s]} = \frac{(1 + s n_{\alpha})}{\sum_{\alpha'} (1 + s n_{\alpha'})}$$

Power-law exponent γ

depends also on the nature of the regular polytope that constitute the building block of the cell complex

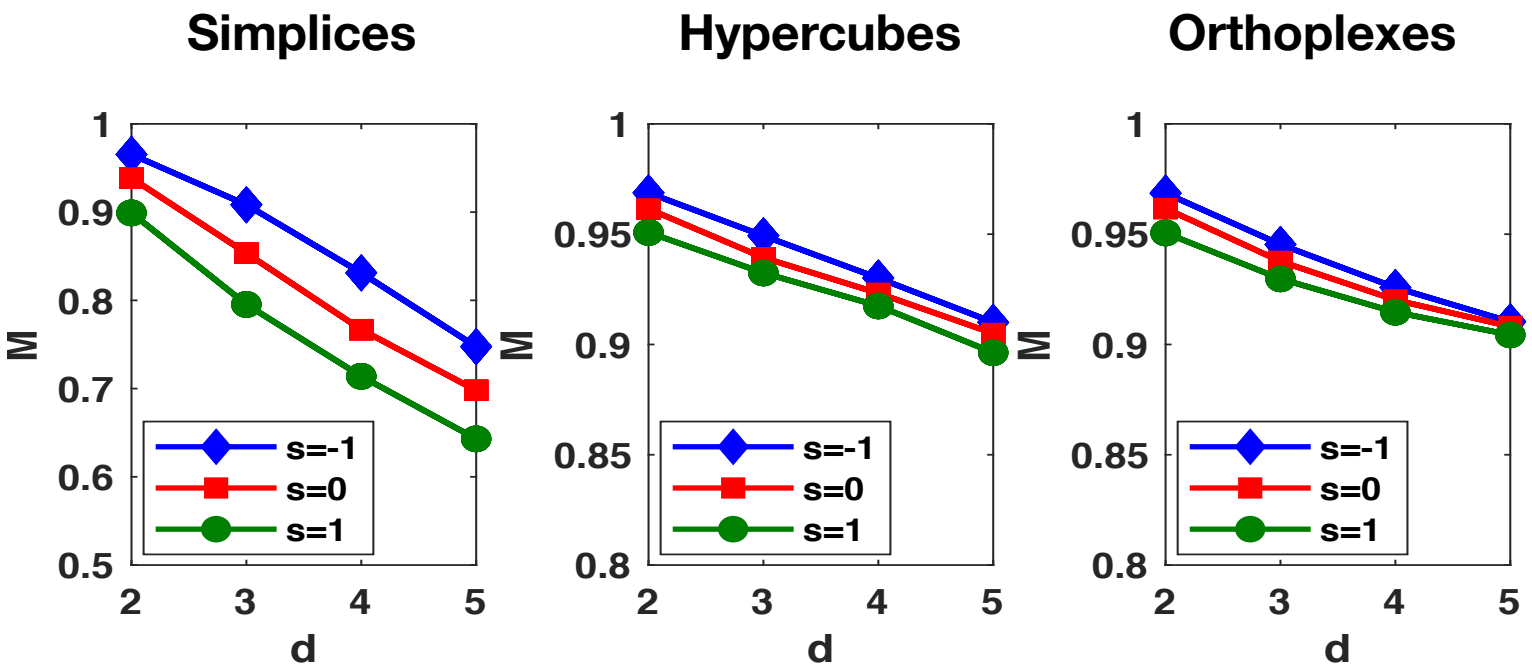
- *Simplicial complexes with power-law degree distribution are always scale free*
- *Other cell complexes are scale-free only if they have flavor s=1 (preferential attachment)*
- *Some cell complexes in d=2,3,4 are not even scale-free for flavor s=1*

γ	$s=-1$	$s=0$	$s=1$
$d=1$			
link	N/A	N/A	3
$d=2$			
p-polygon	N/A	p	$1+\frac{p}{2}$
$d=3$			
tetrahedron	3	$2\frac{1}{2}$	$2\frac{1}{3}$
cube	5	$3\frac{1}{3}$	3
octahedron	4	$3\frac{1}{3}$	3
dodecahedron	11	$6\frac{1}{2}$	5
icosahedron	7	$5\frac{3}{4}$	5
$d=4$			
pentachoron	$2\frac{1}{2}$	$2\frac{1}{3}$	$2\frac{1}{4}$
tesseract	4	$3\frac{1}{3}$	3
hexadecachoron	$3\frac{1}{3}$	$3\frac{1}{7}$	3
24-cell	$6\frac{1}{2}$	$5\frac{3}{5}$	5
120-cell	60	$40\frac{2}{3}$	31
600-cell	$34\frac{2}{9}$	$32\frac{10}{19}$	31
$d>4$			
simplex	$2+\frac{1}{d-2}$	$2+\frac{1}{d-1}$	$2+\frac{1}{d}$
cube	$3+\frac{2}{d-2}$	$3+\frac{1}{d-1}$	3
orthoplex	$3+\frac{1}{2^{(d-2)}-1}$	$3+\frac{1}{2^{d-1}-1}$	3

Modularity of NGFs

Network Geometry with Flavor

Displays emergent community structure



Hausdorff dimension

NGFs as well as Apollonian and pseudo-fractal simplicial complexes

have an infinite Hausforff dimension

$$d_H = \infty$$

Therefore the spectral dimension satisfies

$$d_S \geq 2$$

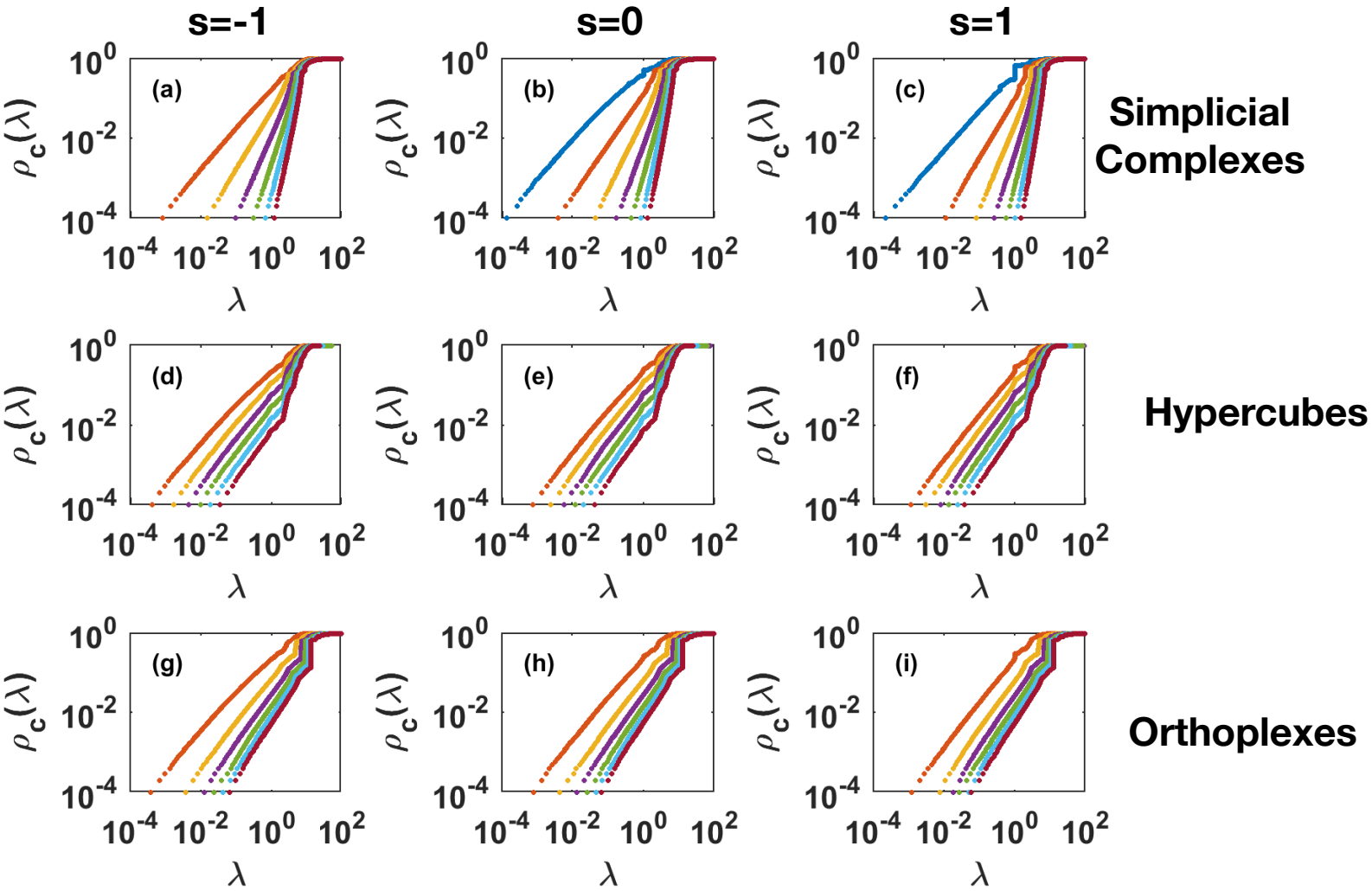
The Area of these simplicial complexes

(number of (d-1)-faces with zero incidence number)

is

$$A \propto N$$

Laplacian spectrum of NGFs

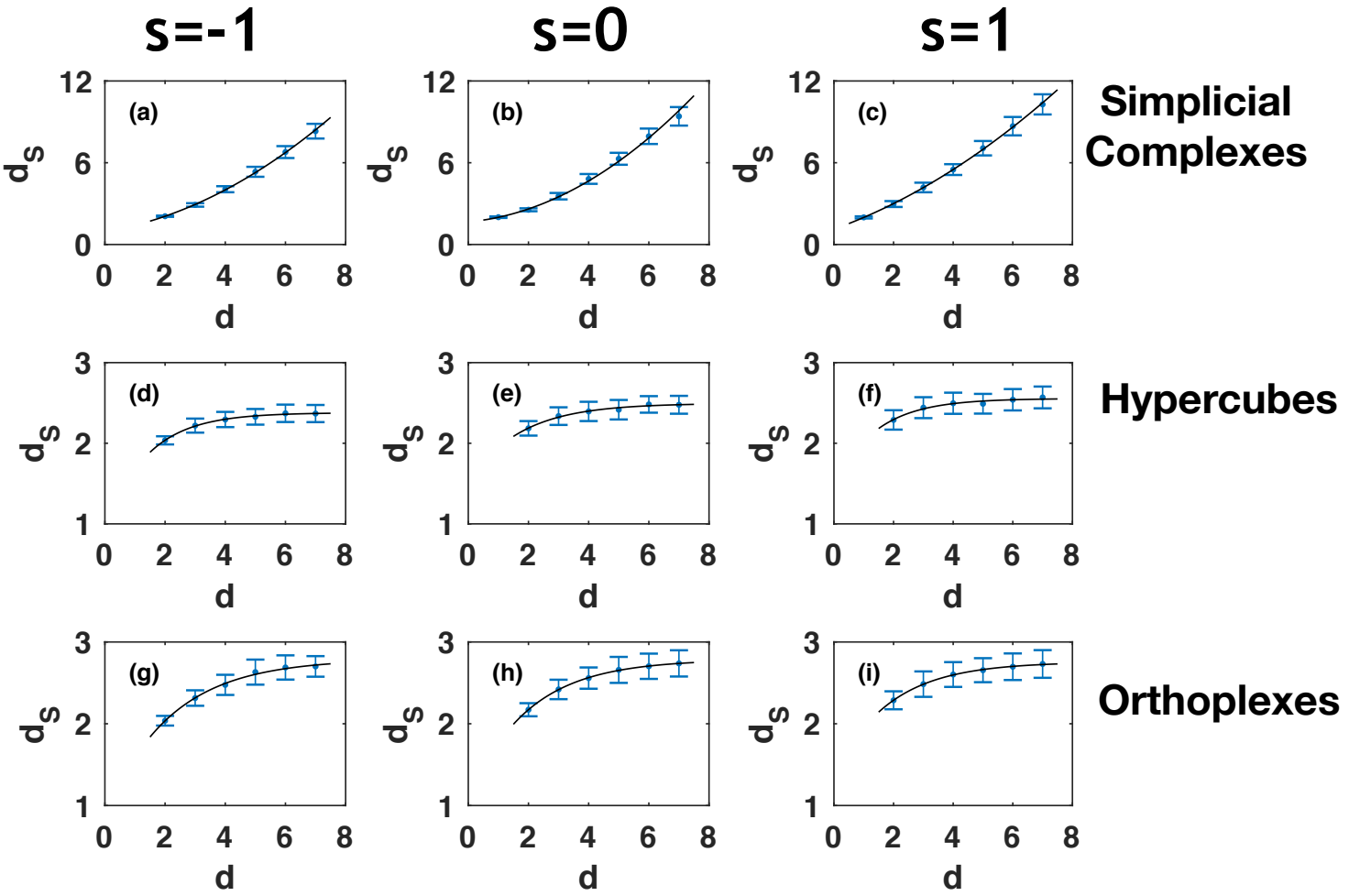


The spectral dimension depends

on d, on the flavor s and on the building block of the cell complex

Different colors indicate different dimensions d

Spectral dimension of NGF



Higher-order spectral dimension of NGF

$$\mathbf{d_S} = (d_S^{[0]}, d_S^{[1]}, \ldots, d_S^{[d-2]})$$

Different Higher-order

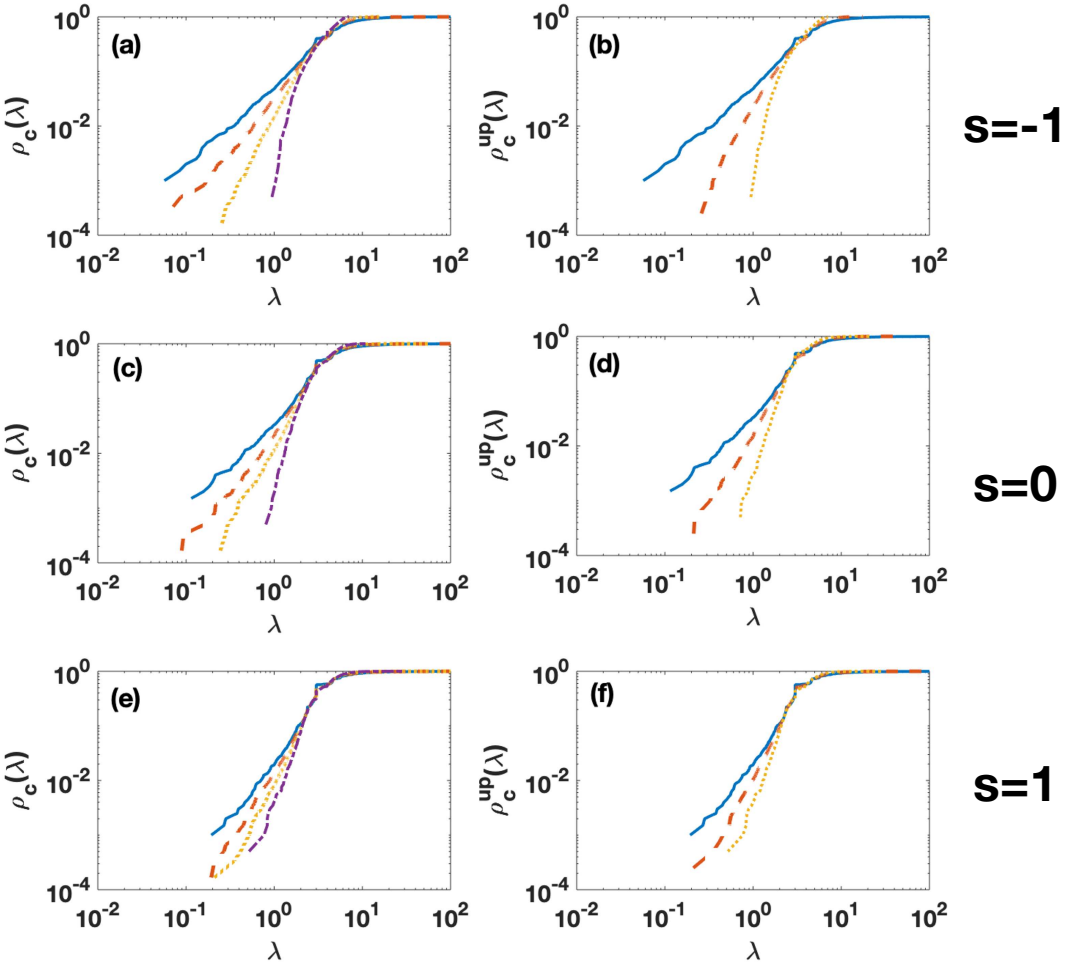
Spectral dimensions

coexist for the same

d=3 dimensional NGF

Corresponding to orders

0(blue), 1(red) 2 (yellow)



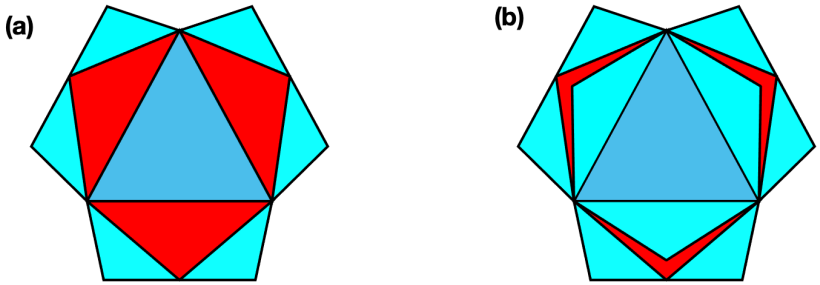
NGF and non-universal spectral dimension

The spectral dimension of NGF can change depending on:

- The dimension d of the simplicial complex
- The nature of the cell constituting the building block of NGF (simplex, hypercube, orthoplex)
- The order of the Laplacian

Renormalization group predictions of the spectral dimension on Apollonian and pseudo-fractal simplicial complexes

Apollonian and pseudo-fractal simplicial complexes

- We start at 

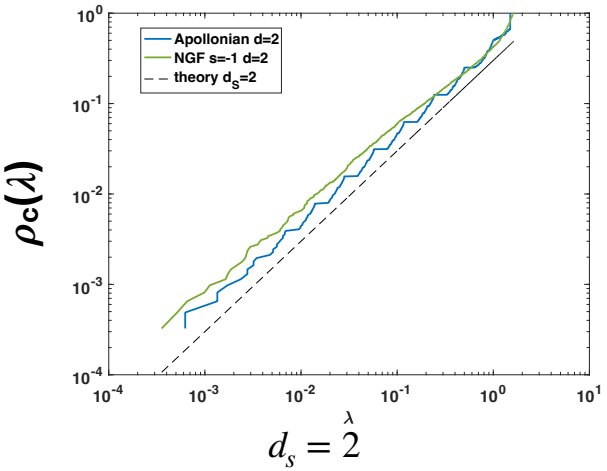
- At each time t>1, we glue a d-simplex

    A. to every (d-1)-face added at the previous time (Apollonian simplicial complexes)

    B. to every (d-1)-face of the simplicial complex (pseudo-fractal simplicial complexes)

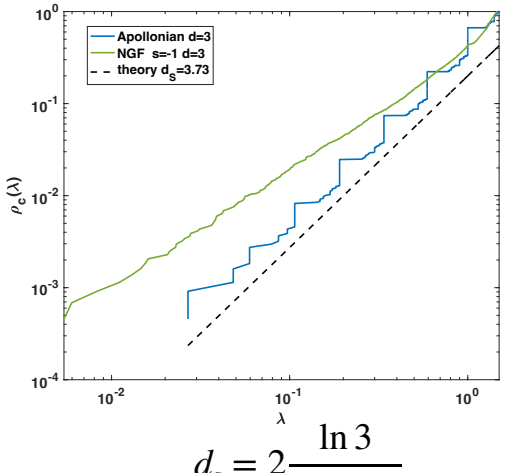
Spectral dimension of Apollonian networks and effect of randomness

d=2



$$d_s = 2$$

d=3



$$d_s = 2\frac{\ln 3}{\ln(9/5)}$$

Asymptotic expression for $d \gg 1$

$$d_s = -2\frac{\ln d}{\ln[1-1/d-1/d^2)]} \simeq 2\ln(d)\left[d - \frac{3}{2} + O(1/d)\right].$$

[G. Bianconi and S.N. Dorogovstev (2020)]

Higher-order spectral dimension

Apollonian simplicial complexes

d/m	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$m=d-3$	—	3.738 13	4.5742	5.199 79	5.700 72	6.119 32	6.479 49	6.795 96
$m=d-4$	—	—	7.399 62	8.482 12	9.356 64	10.0913	10.7253	11.2833
$m=d-5$	—	—	—	11.729	12.9719	14.0179	14.9217	15.7178
$m=d-6$	—	—	—	—	16.5732	17.9293	19.1017	20.1346
$m=d-7$	—	—	—	—	—	21.8337	23.2741	24.5434
$m=d-8$	—	—	—	—	—	—	27.4423	28.9478
$m=d-9$	—	—	—	—	—	—	—	33.3496

Pseudo-fractal simplicial complexes

d/m	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$m=d-2$	3.169 93	4.0	4.643 86	5.169 93	5.614 71	6.0	6.339 85	6.643 86
$m=d-3$	—	5.315 62	5.869 24	6.280 83	6.605 35	6.871 91	7.0975	7.292 81
$m=d-4$	—	—	8.376 10	8.997 32	9.497 05	9.915 47	10.276	10.5934
$m=d-5$	—	—	—	12.7140	13.7232	14.4689	15.057	15.5463
$m=d-6$	—	—	—	—	17.3048	18.5860	19.5562	20.3283
$m=d-7$	—	—	—	—	—	22.2618	23.7403	24.897
$m=d-8$	—	—	—	—	—	—	27.5667	29.1935
$m=d-9$	—	—	—	—	—	—	—	33.1841

M. Reitz, G. Bianconi (2020)]

Network Geometry with Flavor

With fitness of the faces

Energy of the m-faces

Energy and Fitness of the faces of the NGF simplicial complexes [29]

The energy ε_α of the m-dimensional face α indicates its intrinsic (non-topological) properties. The energy $\varepsilon_{[r]}$ of a node r is a non negative number drawn from a given distribution $g(\varepsilon)$. The energy of a face α of dimension $m > 0$ is the sum of the energies of the nodes belonging to it, i.e.

$$\varepsilon_\alpha = \sum_{r \subset \alpha} \varepsilon_{[r]}. \tag{5.14}$$

The *fitness* associated to a m-dimensional face α describes the rate at which the face increases its generalized degree and is given by

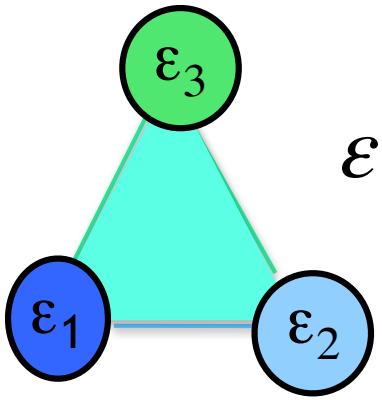
$$\eta_\alpha = e^{-\beta \varepsilon_\alpha} \tag{5.15}$$

where $\beta > 0$ is a parameter called *inverse temperature*. For $\beta = 0$ all the fitnesses are the same, and equal to one, while for $\beta \gg 1$ the small difference in energy leads to big differences in the fitnesses of the faces.

Energy of a link

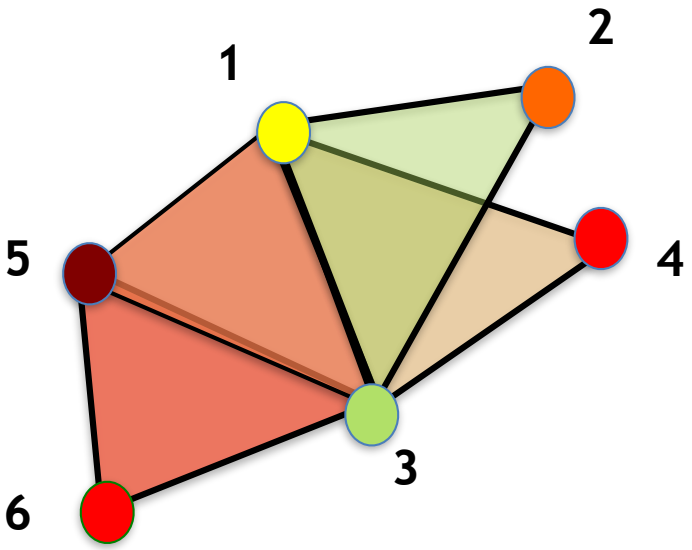
$$\varepsilon_1 + \varepsilon_2$$

Energy of a triangle



$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

Network Geometry with Flavor



$$\Pi_\alpha^{[s]} = \frac{e^{-\beta\varepsilon_\alpha}\,(1+sn_\alpha)}{\sum_{\alpha'} e^{-\beta\varepsilon_{\alpha'}}\,(1+sn_{\alpha'})}$$

NETWORK GEOMETRY WITH FLAVOR (WITH FITNESS) [29]

At time $t = 1$ the simplicial complex is formed by a single d-dimensional simplex. Each node r of this simplex has energy $\varepsilon_{[r]}$ drawn from a $g(\varepsilon)$ distribution. The energies of the higher-dimensional faces are calculated according to Eq. (5.14).

- GROWTH : At every timestep a new d-dimensional simplex formed by one new node and an existing $(d-1)$-face is added to the simplicial complex. Each new node r has energy $\varepsilon_{[r]}$ drawn from a $g(\varepsilon)$ distribution. The energies of the new higher-dimensional faces are calculated according to Eq. (5.14).
- ATTACHMENT: At every timestep the probability that the new d-simplex is connected to the existing $(d-1)$-dimensional face α depends on the *flavor* $s \in \{-1, 0, 1\}$ and on the *inverse temperature* $\beta > 0$ and is given by

$$\Pi_\alpha^{[s]} = \frac{e^{-\beta\varepsilon_\alpha}(1+sn_\alpha)}{\sum_{\alpha'} e^{-\beta\varepsilon_{\alpha'}}(1+sn_{\alpha'})}. \qquad (5.16)$$

For $\beta = 0$ the NGF (with fitness of the m-faces) reduces to the neutral NGF model, i.e. $\Pi_\alpha^{[s]}$ reduces to Eq. (5.6).

Bianconi & Rahmede (2016)

The average of the generalized degree of the NGF over δ-faces of energy ε

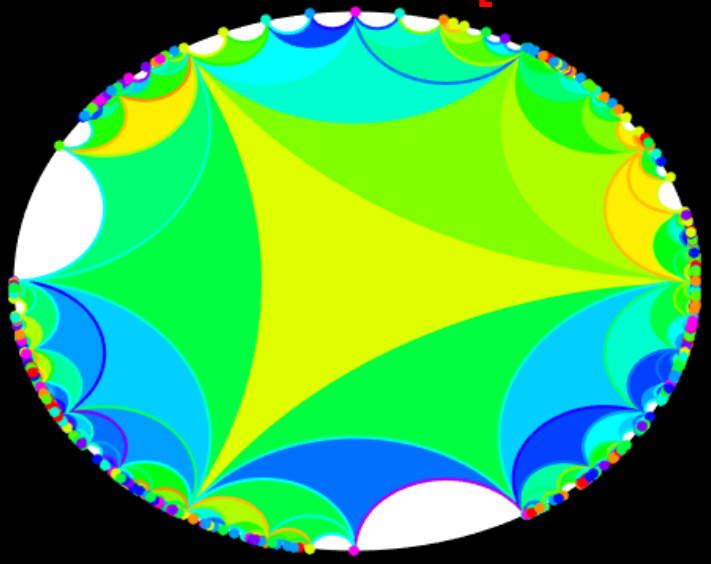
$$\left\langle [k_{d,m}(\alpha) - 1] \,|\, \varepsilon_\alpha = \varepsilon \right\rangle$$

follows a regular pattern

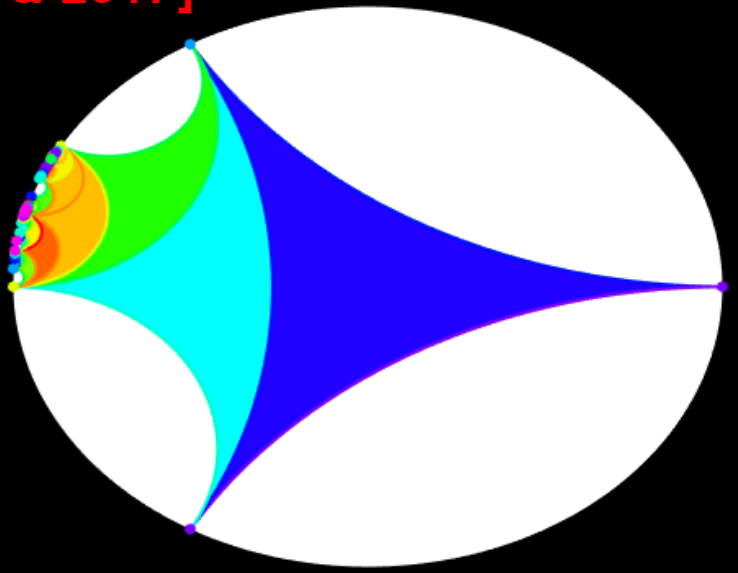
Flavor	$s = -1$	$s = 0$	$s = 1$
$m = d - 1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$m = d - 2$	Boltzmann	Bose-Einstein	Bose-Einstein
$m \leq d - 3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

Topological transitions

Emergent Hyperbolic Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]



β=0.01

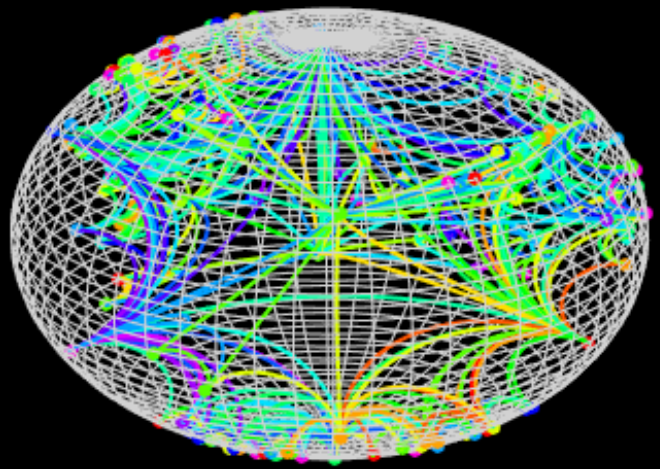


β=5

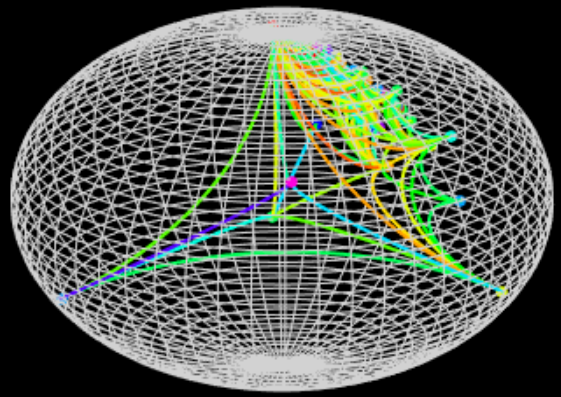
CODES AVAILABLE AT GITHUB **ginestrab**

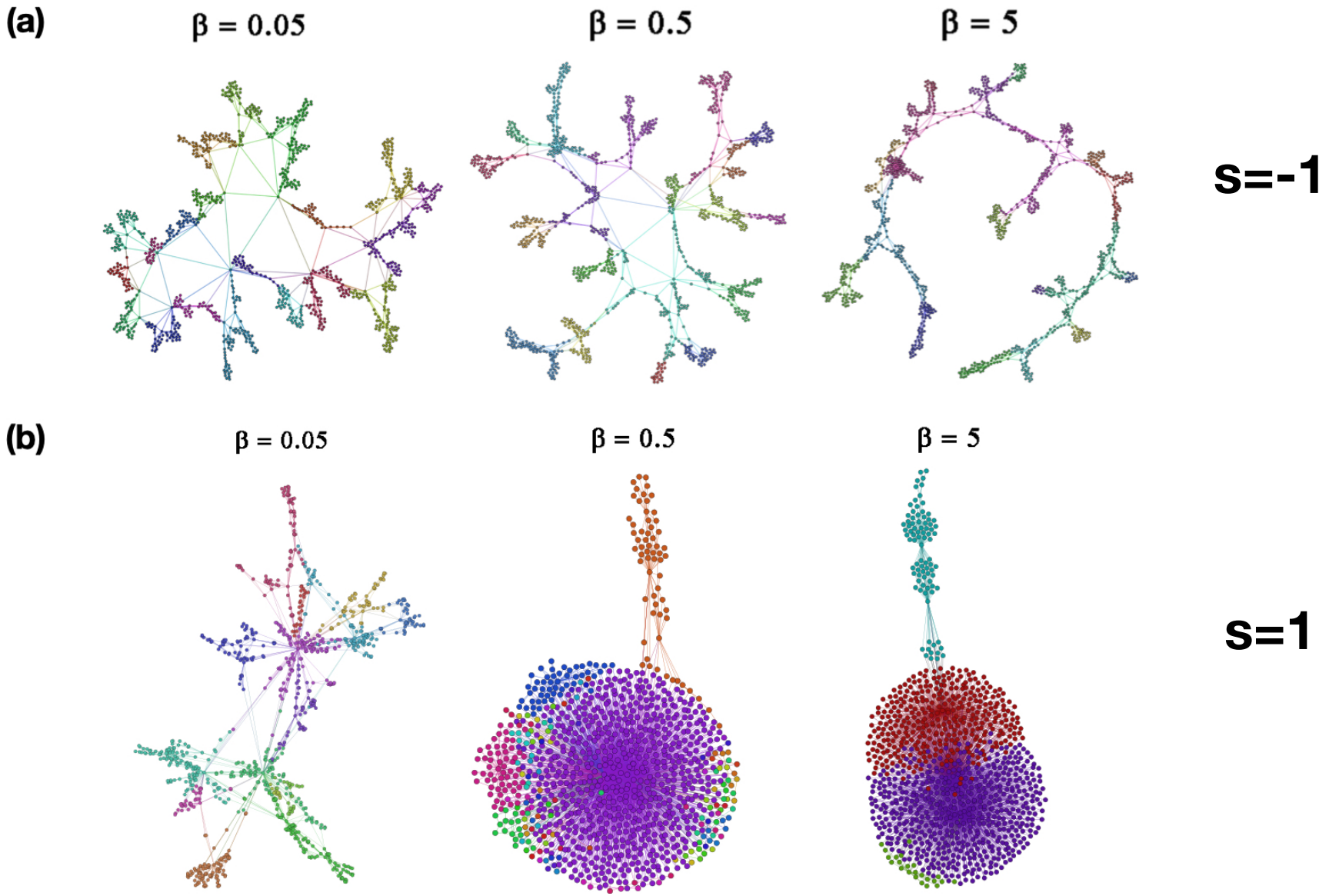
Emergent geometry at high temperature

s=-1
d=3
β=0.01



s=-1
d=3
β=5




(a)
β = 0.05
β = 0.5
β = 5
s=-1
(b)
β = 0.05
β = 0.5
β = 5
s=1

Conclusions

- **Non-equilibrium models of simplicial complex are a fundamental framework to address the problem of emergent geometry and emergent community structure**
- **NGF display statistical properties depending on the dimension of the faces that are considered**
- **NGF display a dependence of their spectral dimension with the nature topological dimension, the dimension of the building block from which they are formed, and the order at which the diffusion is studied**
- **NGF with fitness of the faces display emergent quantum statistics**

Higher-order structure and dynamics