Higher-order networks

An introduction to simplicial complexes
Lesson II

LTCC Module

20 February 2023

Ginestra Bianconi

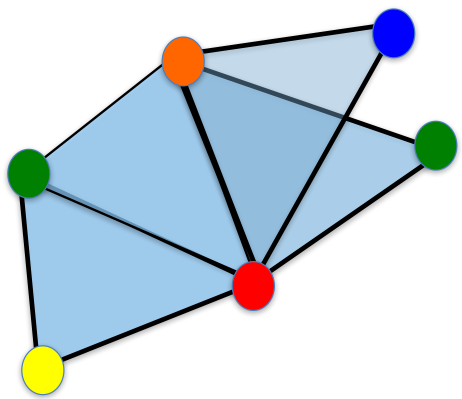
School of Mathematical Sciences, Queen Mary University of London
Alan Turing Institute

Queen Mary University of London

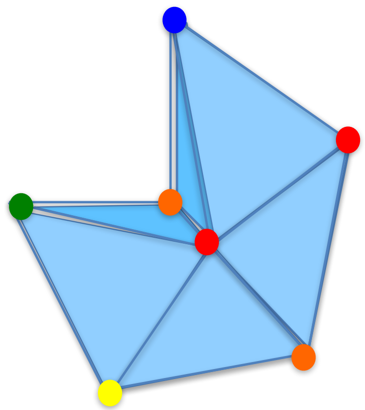
The Alan Turing Institute

Higher-order networks

Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



d=2 simplicial complex

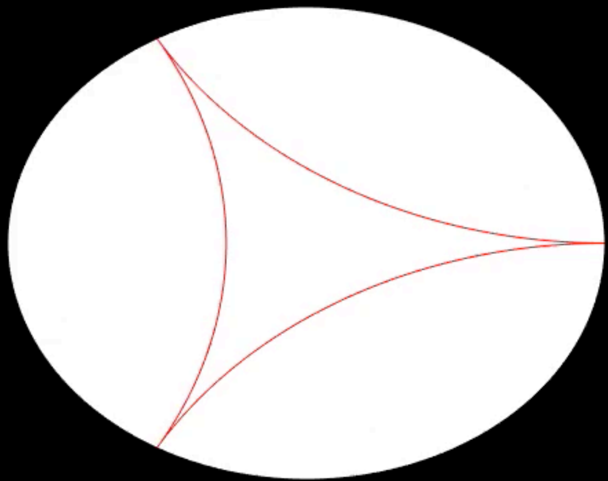


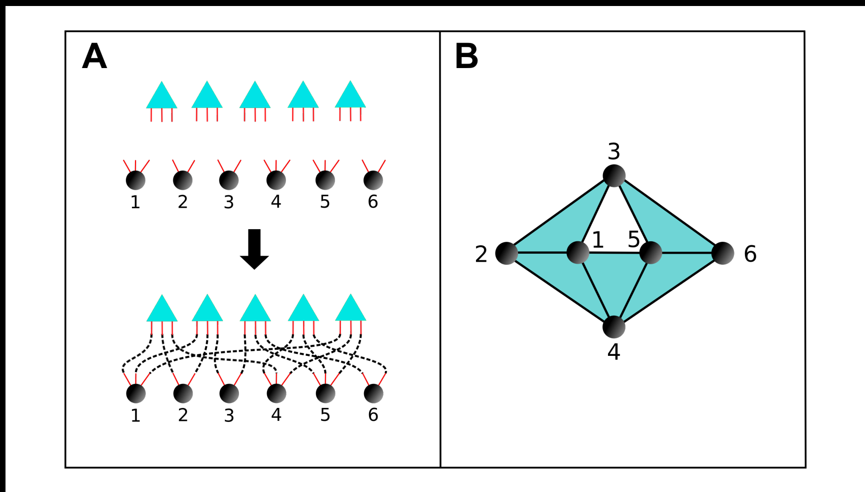
d=3 simplicial complex

Simplicial complex models

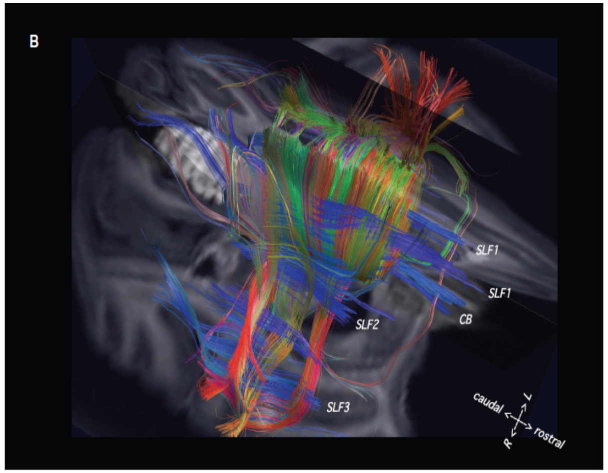
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

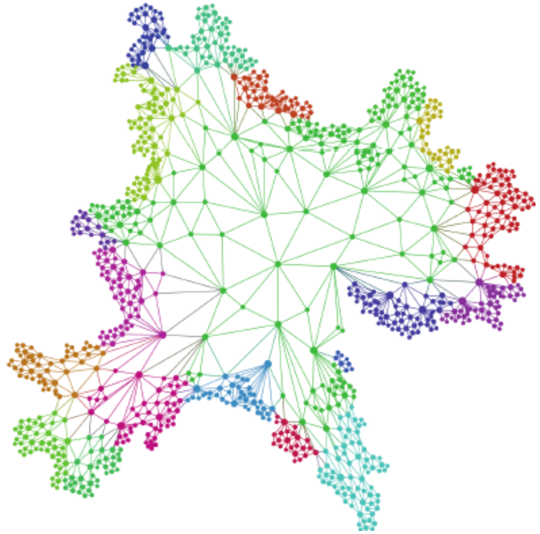
Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





Network Topology and Geometry



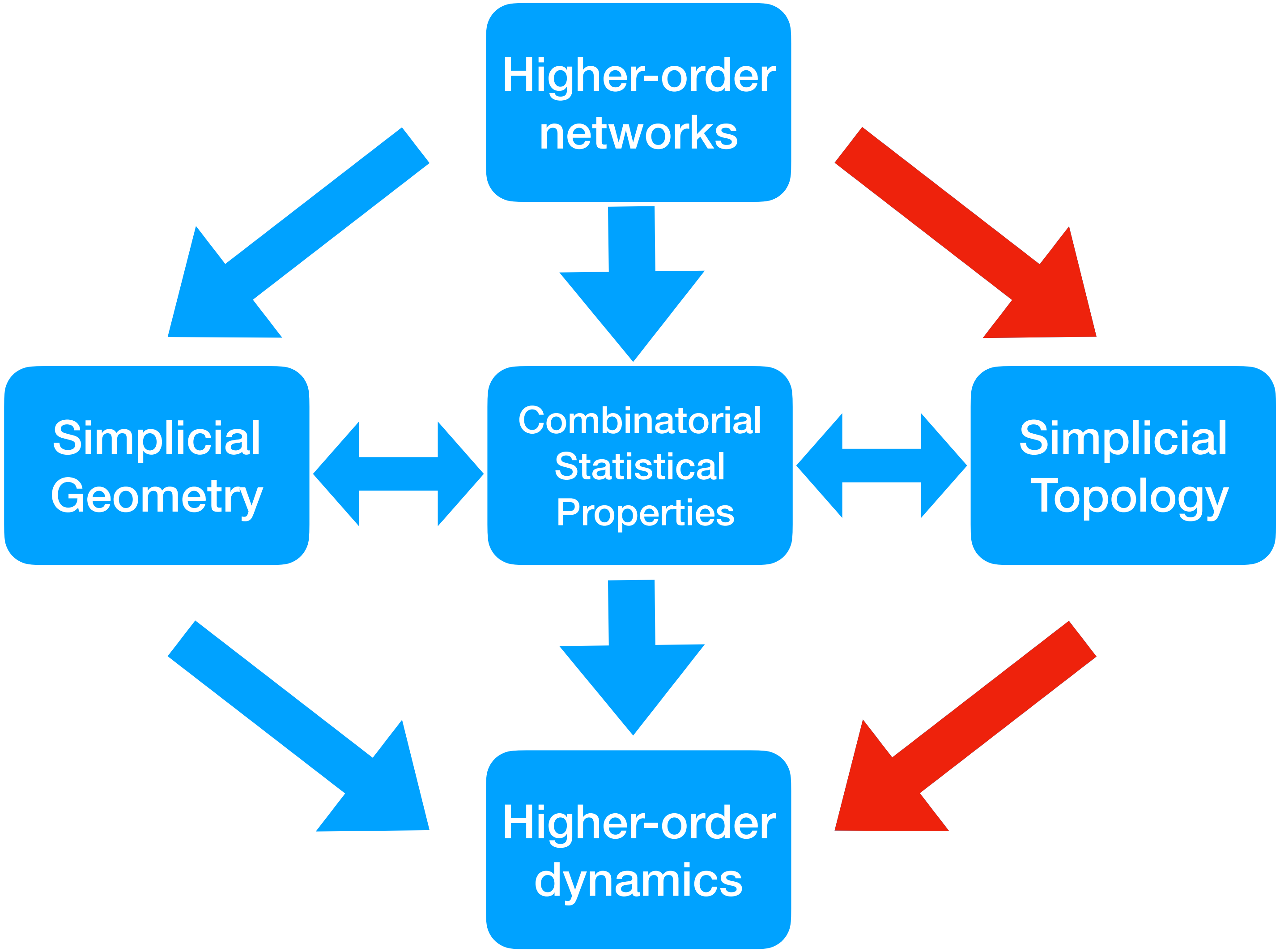


Are expected to have impact in a variety of applications,

ranging from

brain research to biological transportation networks

Higher-order structure and dynamics



Lesson II:
Introduction to Algebraic Topology

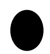
- **Introduction to algebraic topology**
- **Higher-order operators and their properties**

1. **Topological signals**
2. **The Hodge Laplacian and Hodge decomposition**
3. **Topology of weighed simplicial complexes**

Introduction to Algebraic Topology

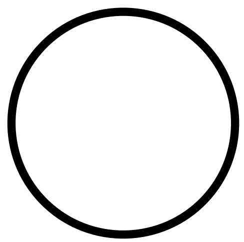
Betti numbers

Point



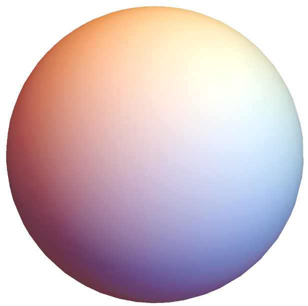
$$\beta_0 = 1 \quad \beta_1 = 0 \quad \beta_2 = 0$$

Circle



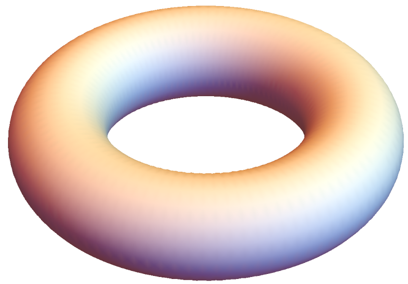
$$\beta_0 = 1 \quad \beta_1 = 1 \quad \beta_2 = 0$$

Sphere



$$\beta_0 = 1 \quad \beta_1 = 0 \quad \beta_2 = 1$$

Torus

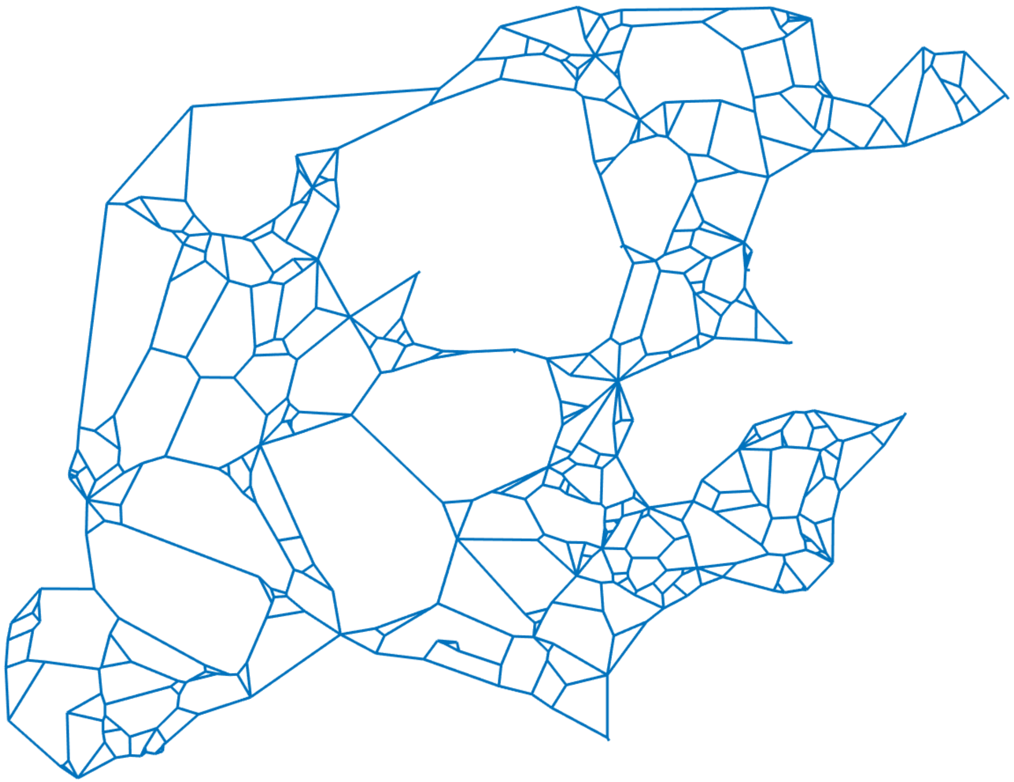


$$\beta_0 = 1 \quad \beta_1 = 2 \quad \beta_2 = 1$$

Euler characteristic

$$\chi = \sum_n (-1)^n \beta_n$$

Betti number 1



Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

Simplicial complex:notation

We consider a d-dimensional simplicial complex $\mathscr{K}$ having N_m

positively oriented simplices α_i^m

(or simply i) of dimension m.

We indicate the set of all the m positively oriented simplices of the simplicial complex

$Q_m(\mathscr{K})$

Orientation of a simplex

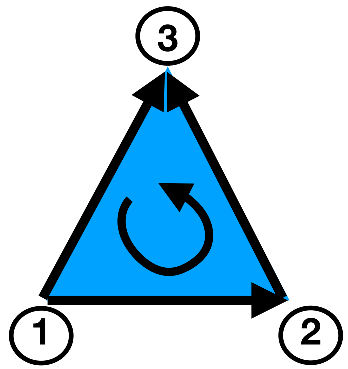
A m-dimensional *oriented simplex* α is a set of $m+1$ nodes

$$\alpha = [v_0, v_1, \ldots, v_m], \tag{3.1}$$

associated to an orientation wuch that

$$[v_0, v_1, \ldots, v_m] = (-1)^{\sigma(\pi)}[v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(m)}] \tag{3.2}$$

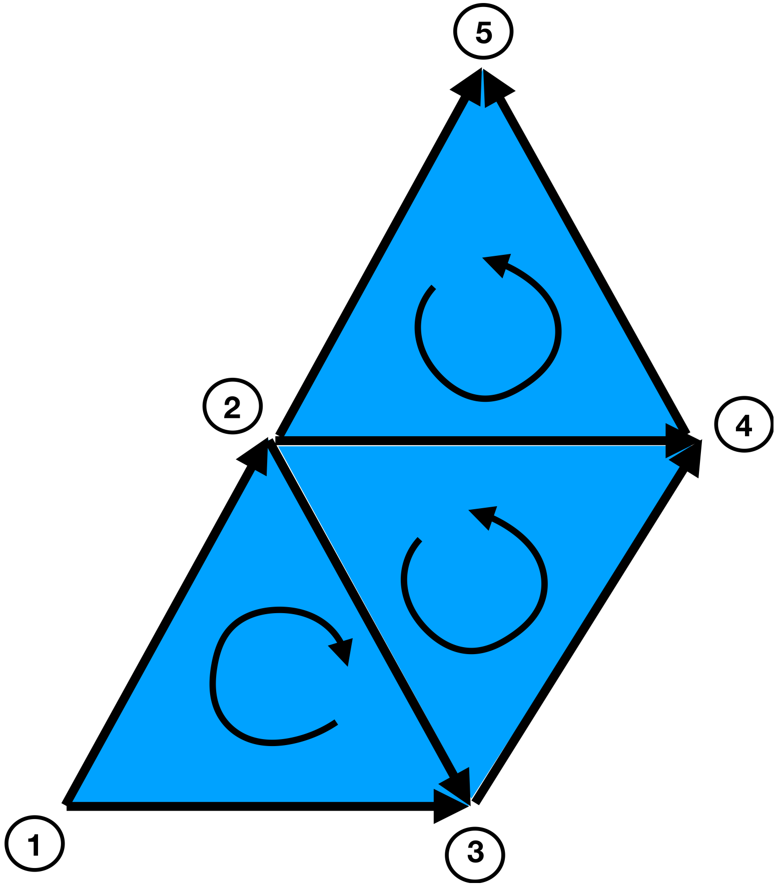
where $\sigma(\pi)$ indicates the parity of the permutation π.



$$[r, s] = -[s, r]$$

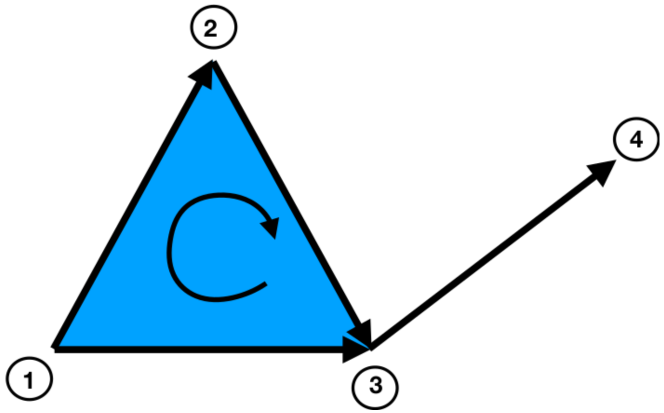
$$[r, s, q] = [s, q, r] = [q, r, s] = -[s, r, q] = -[q, s, r] = -[r, q, s]$$

Oriented simplicial complex



A typical choice of orientation
of a simplicial complex,
is to consider the orientation
induced by the node labels,
i.e. each simplex is oriented in an
increasing (or decreasing) order
of the node labels

Oriented simplicial complex



The set of positively oriented simplices on this simplicial complex are:

{[1,2,3], [1,2], [2,3], [1,3], [3,4], [1], [2], [3], [4]}

We adopt the convention that each 0-simplex is positively oriented

m-Chains

THE m-CHAINS

Given a simplicial complex, a m-chain C_m consists of the elements of a free abelian group with basis on the m-simplices of the simplicial complex.

Its elements can be represented as linear combinations of the of all oriented m-simplices

$$\alpha = [v_0, v_1, \ldots, v_m] \tag{3.6}$$

with coefficients in $\mathbb{Z}$.

m-chain $c_m \in C_m$

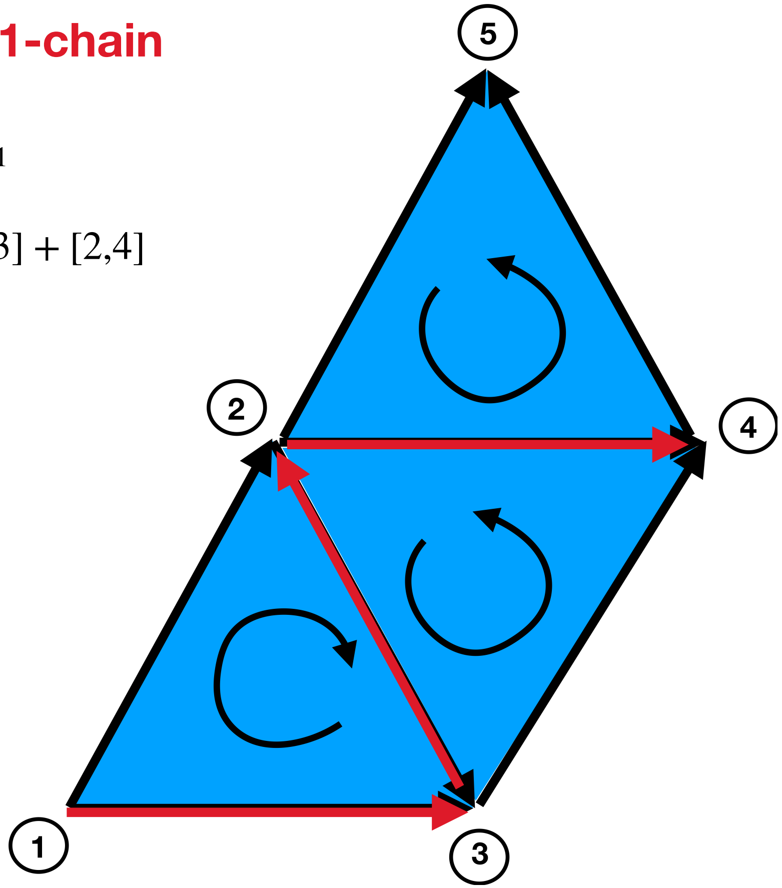
$$\boxed{c_m = \sum_{\alpha_i \in Q_m(\mathscr{K})} c_m^i \alpha_i^m, \ \textbf{with } c_m^i \in \mathbb{Z}}$$

Oriented simplicial complex and m-chains

Example of 1-chain

$$a \in \mathscr{C}_1$$

$$a = [1,3] - [2,3] + [2,4]$$



Boundary operator

THE BOUNDARY MAP

The boundary map ∂_m is a linear operator

$$\partial_m : C_m \to C_{m-1} \tag{3.8}$$

whose action is determined by the action on each m-simplex of the simplicial complex is given by

$$\partial_m[v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m]. \tag{3.9}$$

Boundary operator

The boundary map ∂_n is a linear operator

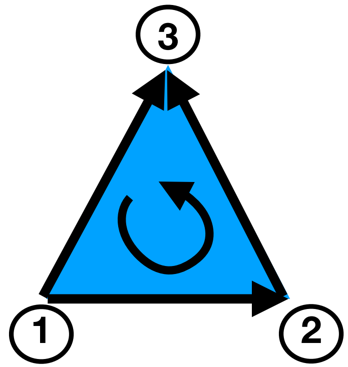
$$\partial_m : \mathscr{C}_m \rightarrow \mathscr{C}_{m-1}$$

whose action is determined by the action on each n-simplex of the simplicial complex

$$\boxed{\partial_m [v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m] \,.}$$

Therefore we have

$$\partial_1 [1,2] = [2] - [1] \,.$$



$$\partial_2 [1,2,3] = [2,3] - [1,3] + [1,2] \,.$$

Boundary operator

THE BOUNDARY MAP

The boundary map ∂_m is a linear operator

$$\partial_m : C_m \to C_{m-1} \tag{3.8}$$

whose action is determined by the action on each m-simplex of the simplicial complex is given by

$$\partial_m[v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m]. \tag{3.9}$$

From this definition it follows that the $\mathrm{im}(\partial_m)$ corresponds to the space of $(m-1)$ boundaries and the $\ker(\partial_m)$ is formed by the cyclic m-chains.

Special groups

Boundary group $\hat{B}_m = \mathbf{im}(\partial_{m+1})$
Cycle group $\hat{Z}_m = \mathbf{ker}(\partial_m)$

The boundary of a boundary is null

The boundary operator has the property

$$\boxed{\partial_m \partial_{m+1} = 0 \quad \forall m \geq 1}$$

Which is usually indicated by saying that the boundary of the boundary is null.

This property follows directly from the definition of the boundary, as an example we have

$$\partial_1 \partial_2 [r, s, q] = \partial_1([r, s] + [s, q] - [r, q]) = [s] - [r] + [q] - [s] - [q] + [r] = 0.$$

Proof

The boundary of the boundary is null.

Proof: Indicating with $\hat{v}_p$ the p^th missing vertex we have

$$\partial_{m-1}\partial_m[v_0, v_1, \ldots, v_m] = \sum_{p=0}^{m} (-1)^p \partial_m[v_0, v_1, \ldots \hat{v}_p \ldots v_m]$$

$$= \sum_{p=0}^{m} (-1)^p \sum_{p'=0}^{p-1} (-1)^{p'} [v_0, v_1, \ldots \hat{v}_{p'} \ldots \hat{v}_p \ldots v_m]$$

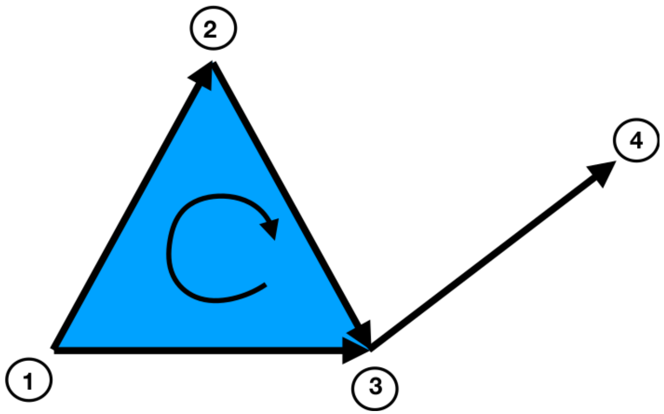
$$+ \sum_{p=0}^{m} (-1)^p \sum_{p'=p+1}^{m} (-1)^{p'-1} [v_0, v_1, \ldots \hat{v}_p \ldots \hat{v}_{p'} \ldots v_m] = 0$$

Incidence matrices

Given a basis for the m simplices and m-1 simplices
the m-boundary operator

$$\partial_m[v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m]\,.$$

is captured by the $N_{m-1} \times N_m$ **incidence (or boundary) matrix** $\mathbf{B}_{[m]}$



$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array},$$

$$\mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

Boundary of the boundary is null

In terms of the incidence matrices the relation

$$\partial_m \partial_{m+1} = 0 \quad \forall m \geq 1$$

Can be expressed as

$$\boxed{\mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0} \quad \forall m \geq 1 \qquad \mathbf{B}_{[m+1]}^\top \mathbf{B}_{[m]}^\top = \mathbf{0} \quad \forall m \geq 1}$$

Homology groups

THE HOMOLOGY GROUPS

The homology group $\mathcal{H}_m$ is the quotient space

$$\mathcal{H}_m = \frac{\ker(\partial_m)}{\mathrm{im}(\partial_{m+1})}, \tag{3.14}$$

denoting homology classes of m-cyclic chains that are in the $\ker(\partial_m)$ and they do differ by cyclic chains that are not boundaries of $(m+1)$-chains, i.e. they are in $\mathrm{im}(\partial_{m+1})$.

It follows that $a \in \ker(\partial_m)$ is in the same homology class than $a + b \in \ker(\partial_m)$ with $b \in \mathrm{im}(\partial_{m+1})$

Betti numbers

BETTI NUMBERS

The Betti number β_m indicates the number of m-dimensional cavities of a simplicial complex and is given by the rank of the homology group $\mathcal{H}_m$, i.e.

$$\beta_m = \mathrm{rank}(\mathcal{H}_m) = \mathrm{rank}(\ker(\partial_m)) - \mathrm{rank}(\mathrm{im}(\partial_{m+1})). \tag{3.15}$$

Euler characteristic

THE EULER CHARACTERISTIC AND THE EULER-POINCARÉ FORMULA

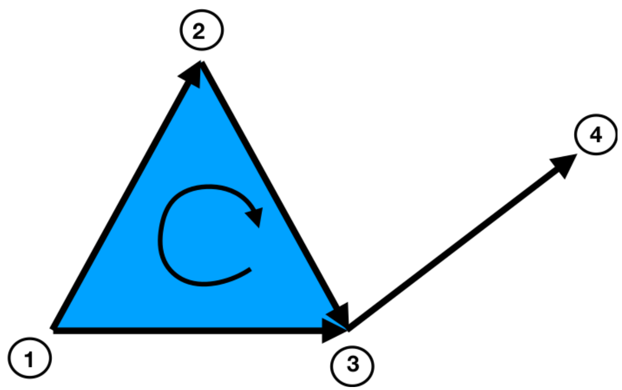
The Euler characterisic χ is defined as the alternating sum of the number of m-dimensional simplices, i.e.

$$\chi = \sum_{m \geq 0} s_m, \tag{3.16}$$

where s_m is the number of m-dimensional simplices in the simplicial complex. According to the Euler-Poincaré formula, the Euler characteristic χ of a simplicial complex can be expressed in terms of the Betti numbers as

$$\chi = \sum_{m \geq 0} (-1)^m \beta_m. \tag{3.17}$$

Boundary Operators



Boundary operators

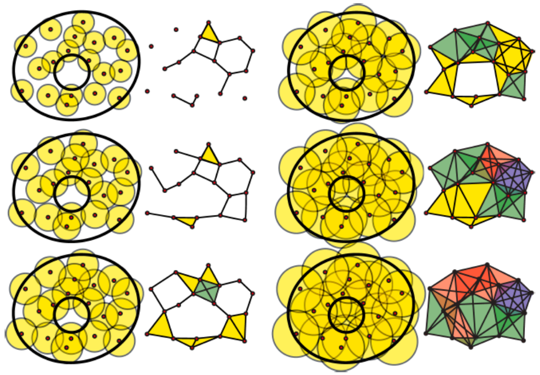
$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

The boundary of the boundary is null

$$\boxed{\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}}$$

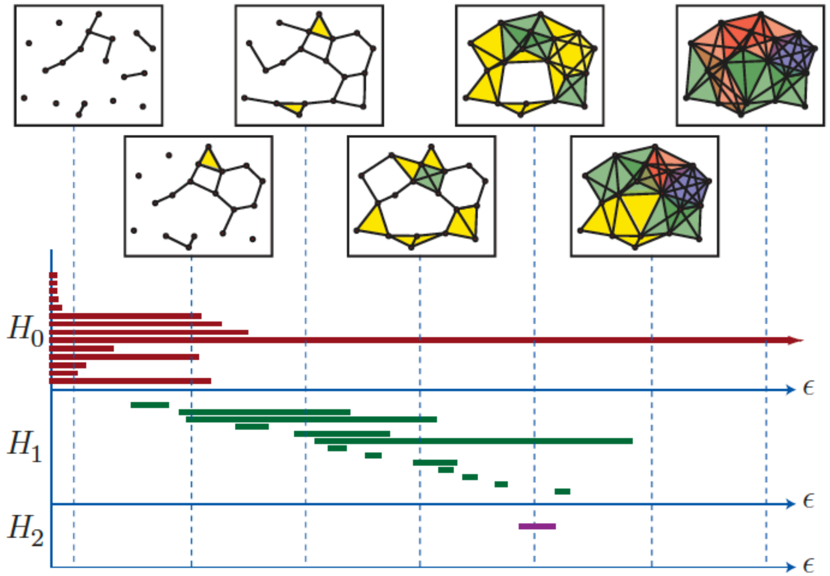
Persistent homology

Filtration: distance/weights



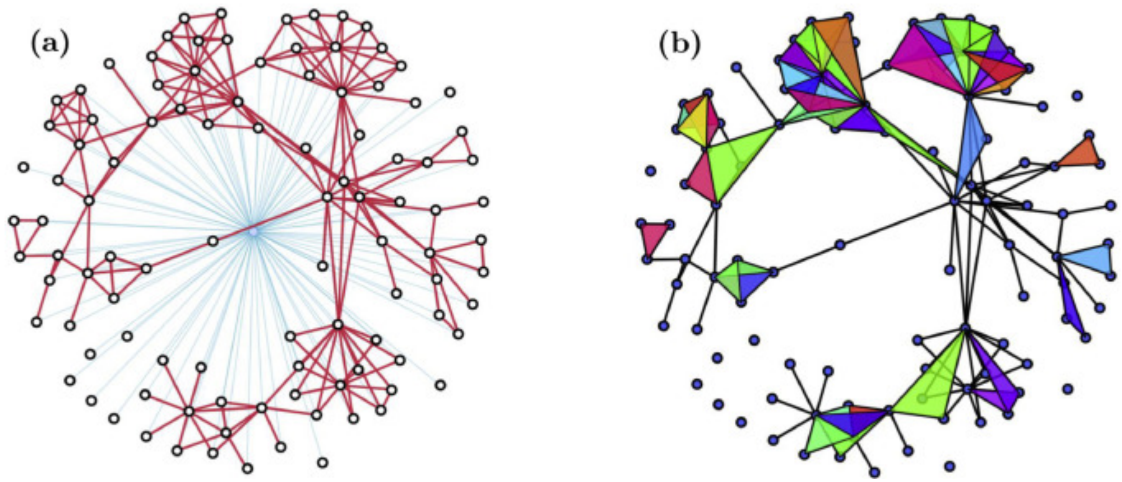
Ghrist 2008

Persistent homology Barcode



Topological clustering

The node neighbourhood is the clique simplicial complex formed by the set of all the neighbours of a node and their connections



Properties of the node

The degree k_i

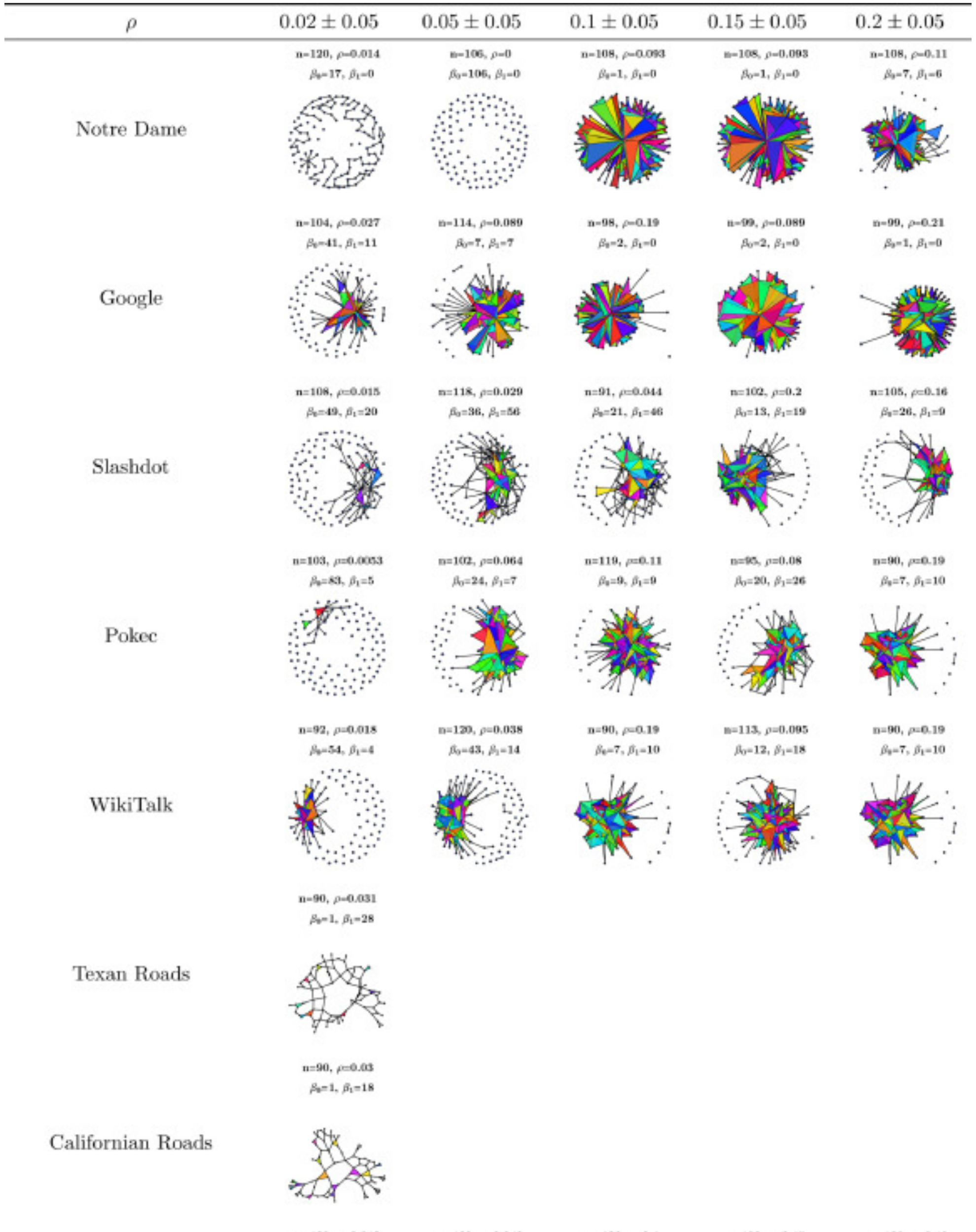
The local clustering coefficient C_i

AP Kartun-Giles et al. (2019)

Properties of the node neighbourhood

Number of nodes n

Density of the links ρ



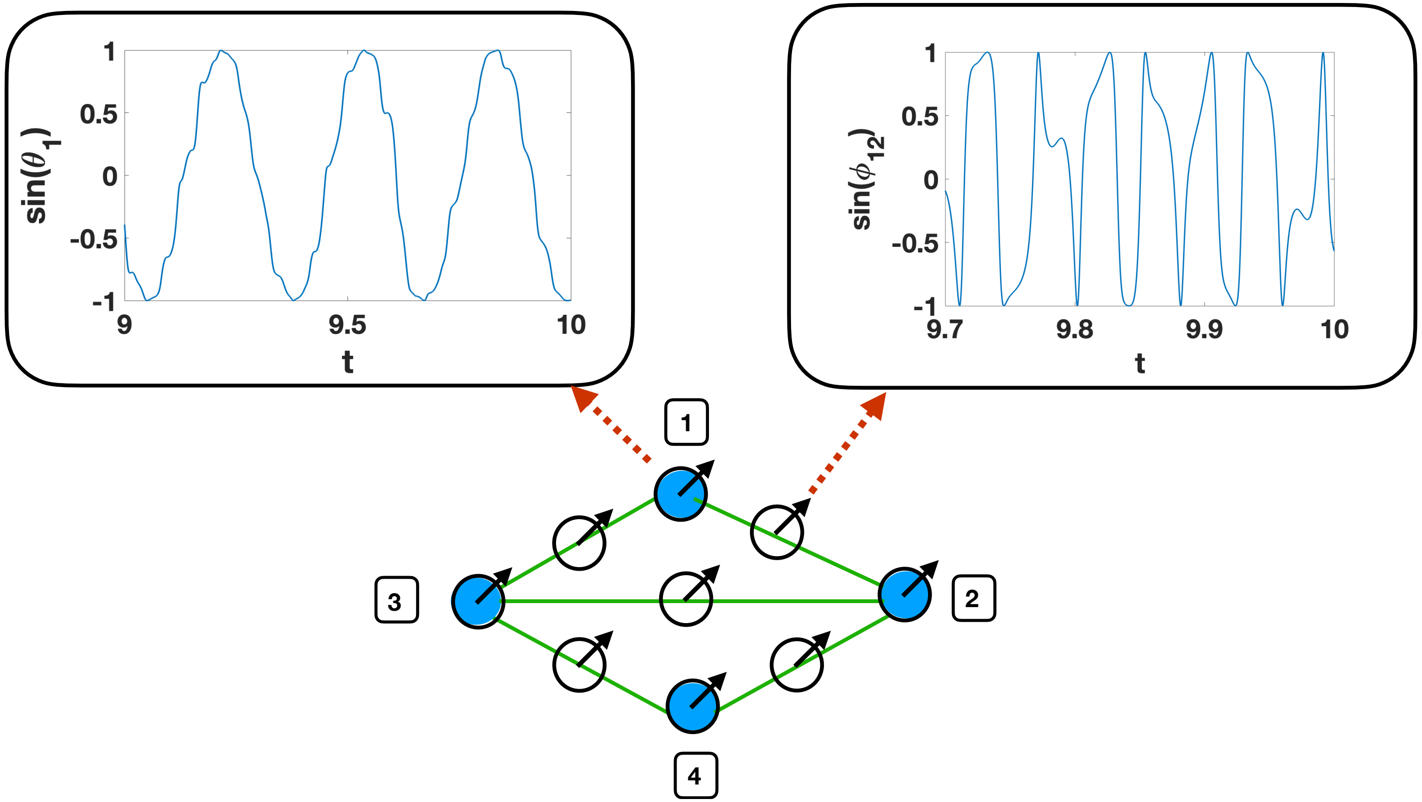
Node neighbourhoods with the same number of nodes and the same density of links can have very different topology

AP Kartun-Giles et al. (2019)

Topological signals,
coboundary operators

Topological signals

Simplicial complexes and networks can sustain dynamical variables (signals) not only defined on nodes but also defined on higher order simplices these signals are called *topological signals*

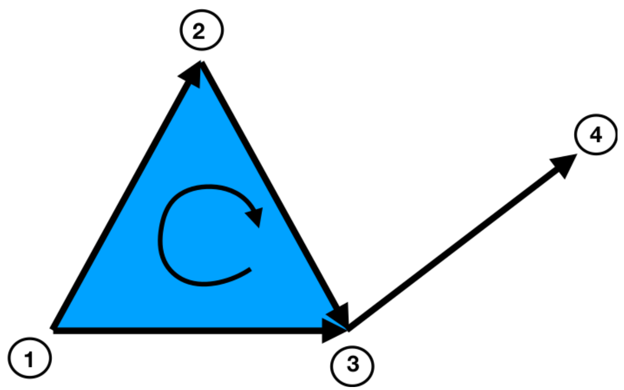


Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are cochains or vector fields

Boundary Operators



Boundary operators

$$
\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.
$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}, \quad \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n-1]}^{\top} = \mathbf{0}$$

Cochains

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \to \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

m-cochain $f \in C^m$

Given the m-chain
$c_m \in C_m$

$$c_m = \sum_{i \in Q_m(\mathcal{K})} c_m^i \alpha_i^m, \text{ with } c_m^i \in \mathbb{Z}$$

$$\boxed{f(c_m) = \sum_{i \in Q_m(\mathcal{K})} c_m^i f([\alpha_i^m]), \text{ with } c_m^i \in \mathbb{Z}}$$

Oriented simplicial complex and m-chains

Example of 1-chain

$$a \in \mathscr{C}_1$$

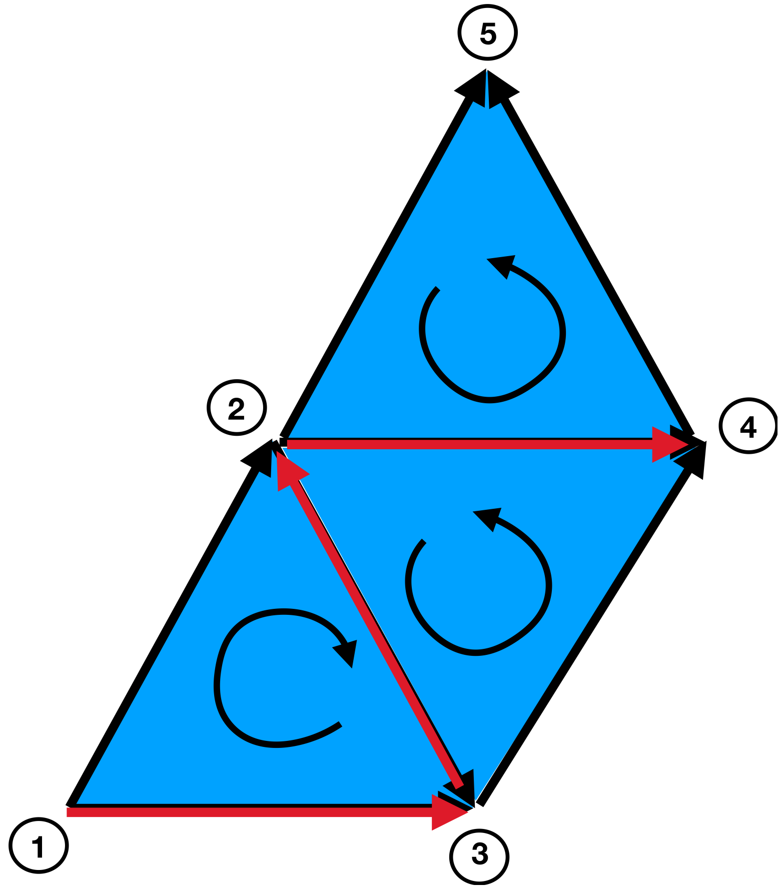
$$a = [1,3] - [2,3] + [2,4]$$

Example

Given $f \in C^1$

then

$$f(a) = f([1,3]) - f([2,3]) + f([2,4])$$



Cochains:properties

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \rightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

Upon a change of orientation of a simplex the value of the cochain associated to a simplex changes sign

$$\boxed{f([\alpha_i^m]) = -f(-[\alpha_i^m]) \;\forall \alpha_i^m \in Q_m(\mathcal{K})}$$

For $m = 1$ a cochain can for instance express a flux along the links of the simplicial complex

Cochains:properties

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \longrightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

Given a basis for the m-simplices of the simplicial complex,
A m-cochain can be expressed as a vector $\mathbf{f}$ of elements

$$\boxed{f_i = f([\alpha_i^m])\ \forall \alpha_i^m \in Q_m(\mathscr{K})}$$

L^2 norm between cochains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{i \in Q_m(\mathscr{K})} f_i^2$$

This scalar product can be generalised by introducing metric matrices (see next)

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \longrightarrow C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{m+1}^\top \mathbf{f} \equiv \bar{\mathbf{B}}_{m+1}\mathbf{f}$

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

if follows that

$$\boxed{\delta_{m+1} \circ \delta_m = 0 \ \forall m \geq 1 \text{ hence } \mathbf{B}^\top_{[m+1]} \mathbf{B}^\top_{[m]} = \mathbf{0}}$$

Discrete Gradient

If $f \in C^0$, then $g = \delta_1 f \in C^1$ indicates its discrete gradient

Indeed we have

$$\mathbf{g} = \mathbf{B}_{[1]}^{\top}\mathbf{f}$$

which implies

$$g_{[i,j]} = f_j - f_i$$

Discrete Curl

If $f \in C^1$, then $g = \delta_1^* f \in C^2$ indicates its discrete curl

Indeed we have

$$\mathbf{g} = \mathbf{B}_{[2]}^\top \mathbf{f}$$

which implies

$$g_{[r,s,t]} = f_{[r,s]} - f_{[s,t]} + f_{[r,t]}$$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \to C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

for any $f \in C^m$ and $g \in C^{m+1}$.

It follows that if $f' = \delta_m^* g$ then $\mathbf{f}' = \mathbf{B}_{[m+1]}\mathbf{g}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \longrightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

If follows that if $f' \in C^m$ is given by $f' = \delta_m^* g$.

Then $\mathbf{f}' = \bar{\mathbf{B}}_{[m+1]}^\top \mathbf{g} = \mathbf{B}_{[m+1]} \mathbf{g}$

Discrete Divergence

If $g \in C^1$, then $f = \delta_0^* g \in C^0$ indicates its discrete divergence

Indeed we have

$$\mathbf{f} = \mathbf{B}_{[1]}\mathbf{g}$$

which implies

$$f_i = \sum_j g_{[ji]} - \sum_j g_{[ij]}$$

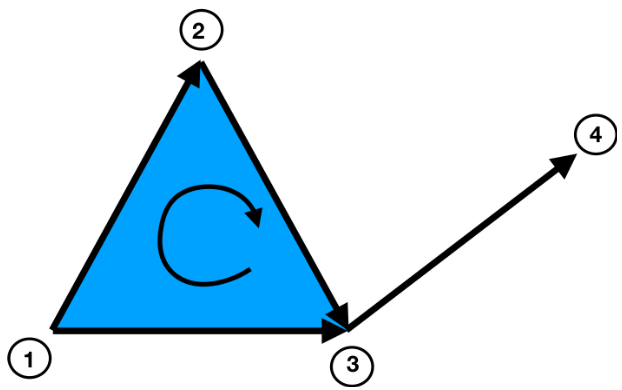
Coboundary action

In summary, the coboundary operator and its adjoint act on the cochains according to the following diagram

$$C^{m+1} \xleftarrow{\delta_m} C^m \xleftarrow{\delta_{m-1}} C^{m-1}$$

$$C^{m+1} \xrightarrow{\delta_m^*} C^m \xrightarrow{\delta_{m-1}^*} C^{m-1}$$

Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

Hodge Laplacians

Hodge Laplacian

The Hodge-Laplacians

The m-dimensional Hodge-Laplacian L_m is defined as

$$L_m = L_m^{up} + L_m^{down}$$

where up and down m-dimensional Hodge Laplacians are given by

$$\begin{aligned} L_m^{up} &= \delta_m^* \delta_m, \\ L_m^{down} &= \delta_{m-1} \delta_{m-1}^*. \end{aligned}$$

Hodge Laplacians

The Hodge Laplacians describe diffusion

from n-simplices to m-simplices through (m-1) and (m+1)

simplices

$$\mathbf{L}_{[m]} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]} + \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

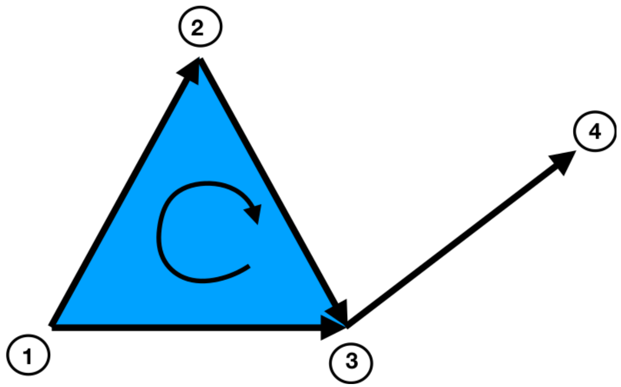
The higher order Hodge Laplacian can be decomposed as

$$\mathbf{L}_{[m]} = \mathbf{L}_{[m]}^{down} + \mathbf{L}_{[m]}^{up},$$

with

$$\mathbf{L}_{[m]}^{down} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]},$$
$$\mathbf{L}_{[m]}^{up} = \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

Simplicial complexes and Hodge Laplacians



Hodge Laplacians

The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices

For a 2-dimensional simplicial complex we have

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^\top \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^\top\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^\top \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^\top\mathbf{B}_{[2]}$$

Properties of the Hodge Laplacians

- The Hodge Laplacians $L_m, L_m^{up}, L_m^{down}$ are semidefinite positive.
- They obey Hodge decomposition
- The dimension of the kernel of the Hodge Laplacian L_m is the m-Betti number β_m

The Hodge-Laplacians are semi-definitive positive

The Hodge Laplacians $L_m, L_m^{up}, L_m^{down}$

are semidefinite positive.

Indeed we have:

$$\langle f, L_m^{up} f\rangle = \langle f, \delta_m^* \delta_m f\rangle = \langle \delta_m f, \delta_m f\rangle \geq 0$$

$$\langle f, L_m^{down} f\rangle = \langle f, \delta_{m-1} \delta_{m-1}^* f\rangle = \langle \delta_{m-1}^* f, \delta_{m-1}^* f\rangle \geq 0$$

$$\langle f, L_m f\rangle = \langle f, L_m^{up} f\rangle + \langle f, L_m^{down} f\rangle \geq 0$$

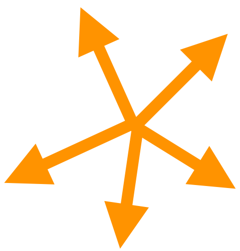
Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

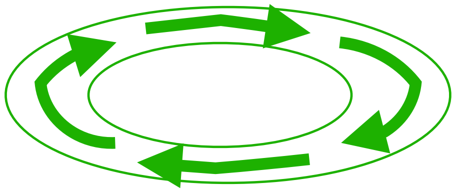
in a irrotational, harmonic and solenoidal components

$$\boxed{C^m = \mathbf{im}(\mathbf{B}_{[m]}^\top) \oplus \mathbf{ker}(\mathbf{L}_{[m]}) \oplus \mathbf{im}(\mathbf{B}_{[m+1]})}$$

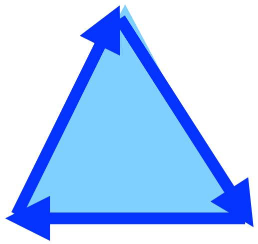
which in the case of topological signals of the links can be sketched as



Irrotational component
Gradient Flow



Harmonic component



Solenoidal component
Curl Flow

Hodge-decomposition

Given that $\mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0} \quad \mathbf{B}^{\top}_{[m-1]}\mathbf{B}^{\top}_{[m]} = \mathbf{0}$

and that $\mathbf{L}^{up}_{[m]} = \mathbf{B}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}, \quad \mathbf{L}^{down}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{B}_{[m]}$

We have:

$$\mathbf{L}^{down}_{[m]}\mathbf{L}^{up}_{[m]} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}^{up}_{[m]} \subseteq \mathrm{ker}\mathbf{L}^{down}_{[m]}$$

$$\mathbf{L}^{up}_{[m]}\mathbf{L}^{down}_{[m]} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}^{down}_{[m]} \subseteq \mathrm{ker}\mathbf{L}^{up}_{[m]}$$

Hodge decomposition

The Hodge decomposition can be summarised as

$$C^m = \textbf{im}(\mathbf{B}^\top_{[m]}) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})$$

This means that $\mathbf{L}_{[m]}, \mathbf{L}^{up}_{[m]}, \mathbf{L}^{down}_{[m]}$ are commuting and can be diagonalised simultaneously. In this basis these matrices have the block structure

$$\mathbf{U}^{-1}\mathbf{L}_{[m]}\mathbf{U} = \begin{pmatrix} \mathbf{D}^{down}_{[m]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^{up}_{[m]} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}^{down}_{[m]}\mathbf{U} = \begin{pmatrix} \mathbf{D}^{down}_{[m]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}^{up}_{[m]}\mathbf{U} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^{up}_{[m]} \end{pmatrix}$$

- **Therefore an eigenvector in the ker of $\mathbf{L}_{[m]}$ is also in the ker of both $\mathbf{L}^{up}_{[m]}, \mathbf{L}^{down}_{[m]}$**
- **An eigenvector corresponding to an non-zero eigenvalue of $\mathbf{L}_{[m]}$ is either a non-zero eigenvector of $\mathbf{L}^{up}_{[m]}$ or a non-zero eigenvector of $\mathbf{L}^{down}_{[m]}$**

Betti numbers

The dimension of the kernel of the Hodge Laplacian L_m is the m-Betti number β_m

Indeed, thanks to Hodge decomposition

$$
\begin{aligned}
\dim \ker \mathbf{L}_{[m]} &= \dim(\ker \mathbf{L}_{[m]}^{up} \cap \ker \mathbf{L}_{[m]}^{down}) \\
&= \dim(\ker \mathbf{L}_{[m]}^{up}) - \dim(\mathrm{im}\, \mathbf{L}_{[m]}^{down}) \\
&= \dim(\ker \mathbf{L}_{[m]}^{down}) - \dim(\mathrm{im}\, \mathbf{L}_{[m]}^{up}) \\
&= \mathrm{rank} \mathscr{H}_m = \beta_m
\end{aligned}
$$

Graph Laplacian in terms of the incidence matrix

The graph Laplacian of elements

$$\left(L_{[0]}\right)_{ij} = \delta_{ij}k_i - a_{ij}$$

Can be expressed in terms of the 1-incidence matrix

as

$$\boxed{\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}\,.}$$

Expression of the matrix elements of the Hodge Laplacians

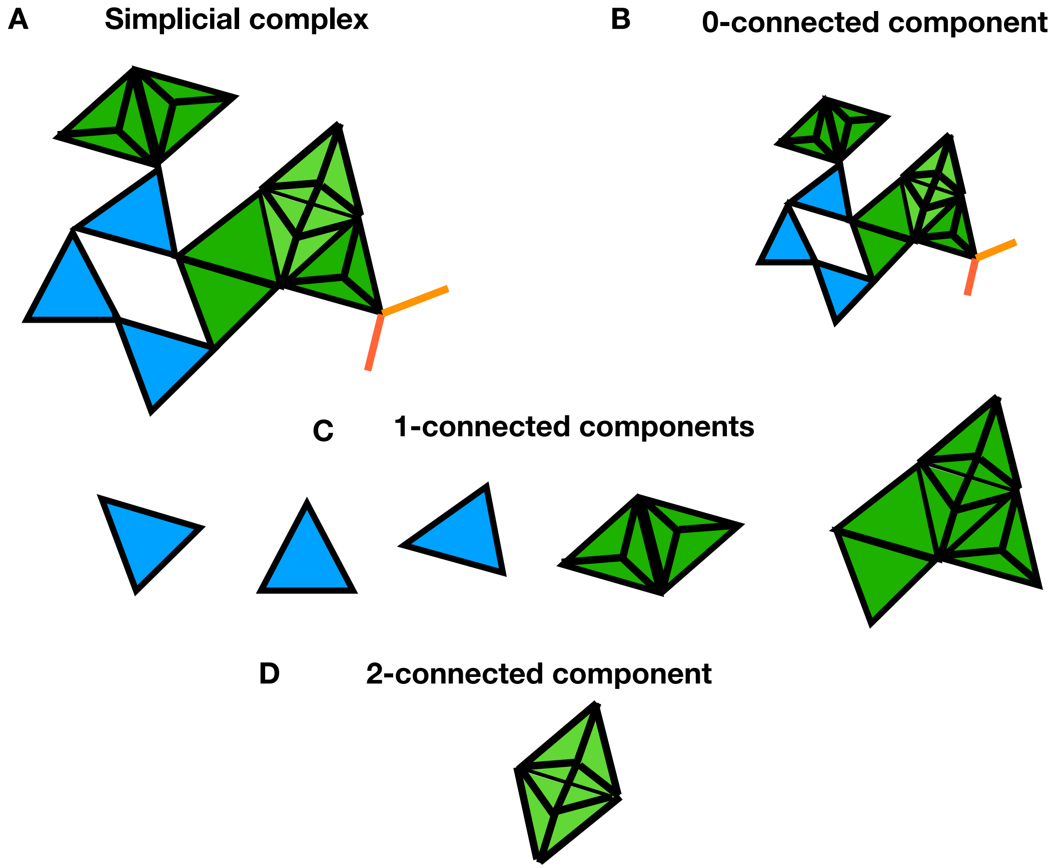
$$\mathbf{L}_m^{\mathrm{up}}(i,j) = \begin{cases} k_{m+1,m}(\alpha_i^m), & i = j. \\ -1, & i \neq j, \alpha_i^m \frown \alpha_j^m, \alpha_i^m \sim \alpha_j^m. \\ 1, & i \neq j, \alpha_i^m \frown \alpha_j^m, \alpha_i^m \nsim \alpha_j^m. \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{L}_m^{\mathrm{down}}(i,j) = \begin{cases} m+1, & i = j. \\ 1, & i \neq j, \alpha_i^p \smile \alpha_j^m, \alpha_i^m \sim \alpha_j^m. \\ -1, & i \neq j, \alpha_i^p \smile \alpha_j^m, \alpha_i^p \nsim \alpha_j^m. \\ 0, & \text{otherwise.} \end{cases}$$

The m-dimensional up- Hodge Laplacian has nonzero elements only among upper incident m-simplices (simplices which are faces of a common m+1 simplex) The eigenvectors have support on the m-connected components

The m-dimensional down-Hodge Laplacian has nonzero elements only among lower incident m-simplices (simplifies sharing a m-1 face) The eigenvectors have support on the (m-1)—connected components

m-connected components



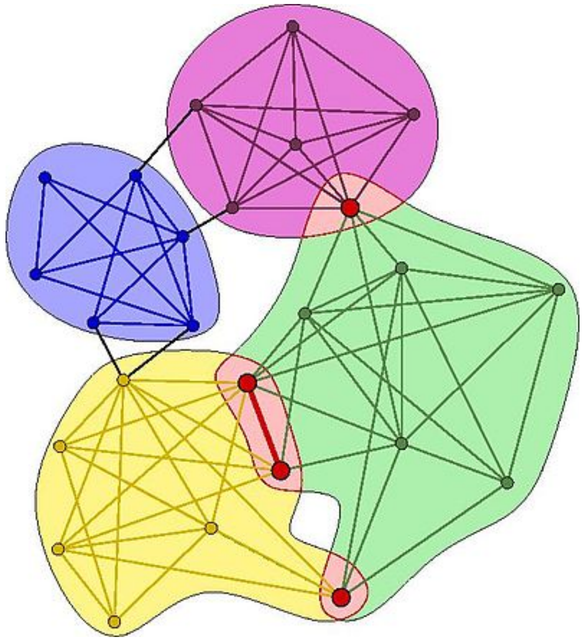
Expression of the matrix elements of the Hodge Laplacians

$$\mathbf{L}_m(i,j) = \begin{cases} k_{m+1,m}(\alpha_i^m) + m + 1, & i = j. \\ 1, & i \neq j, \alpha_i^m \not\frown \alpha_j^m, \alpha_i^m \smile \alpha_j^m, \alpha_i^m \sim \alpha_j^m. \\ -1, & i \neq j, \alpha_i^m \not\frown \alpha_j^m, \alpha_i^m \smile \alpha_j^m, \alpha_i^m \nsim \alpha_j^m. \\ 0 & \text{otherwise.} \end{cases}$$

for $0 < m < d$

The matrix elements of the Hodge Laplacian is only non zero among lower adjacent simplices that are not upper-adjacent

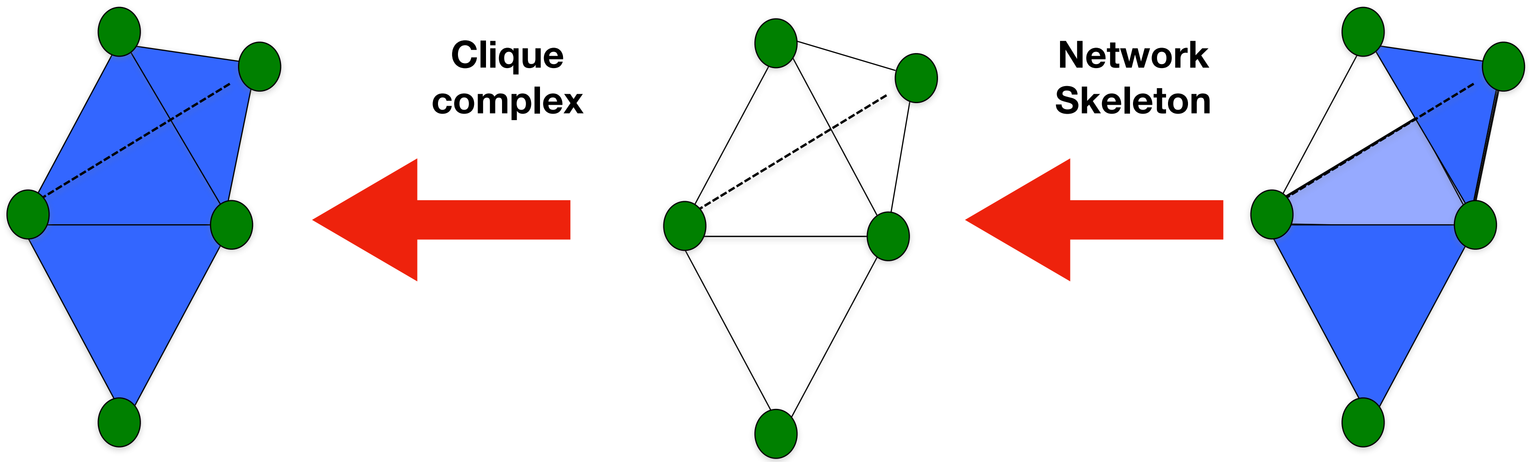
Clique communities



Palla et al. Nature 2005

The m-clique communities are the m-connected components of the clique complex of the network

The skeleton of a simplicial complex and its clique complex

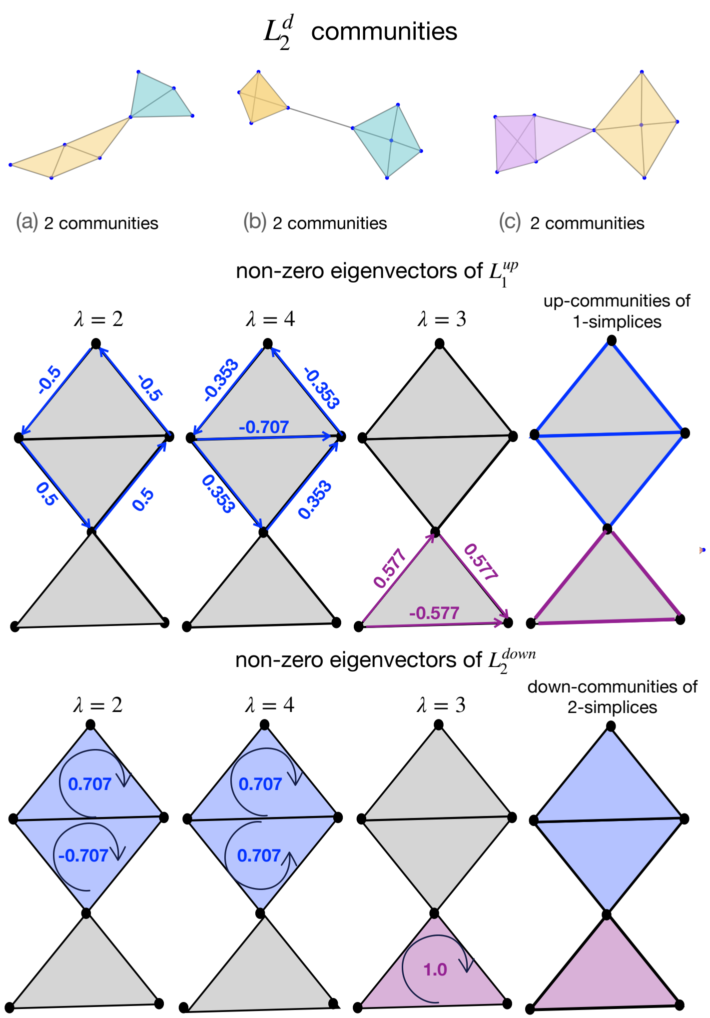


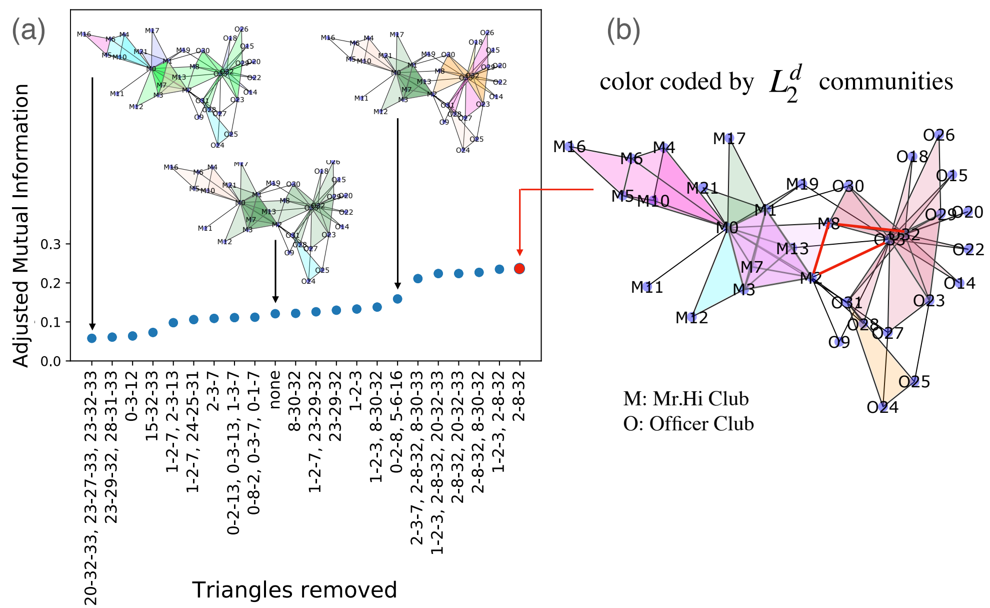
Attention!

By concatenating the operations you are not guaranteed to return to the initial simplicial complex

Inference of higher-order interactions

Higher-order communities





We can infer which higher-order interactions using higher-order communities and ground-truth community assignments

S. Khrisnagopal and GB (2021)

Weighted simplicial complexes

Metric matrices

We introduce the $N_m \times N_m$ metric matrices $\mathbf{G}_m^{-1}$ typically taken to be diagonal with elements

$$\mathbf{G}_m^{-1}(\alpha_i, \alpha_i) = w(\alpha_i)$$

where $w(\alpha_i)$ indicates the affinity weight (inverse of a "distance") associated to the simplex α_i

For a graph, typical choices of these matrices are

$$\mathbf{G}_1^{-1}([r,s],[r,s]) = w([r,s])$$ weight of the link

$$\mathbf{G}_0^{-1}([r],[r]) = \sum_{s \in Q_0(\mathscr{K})} w([r,s])$$ strength (weighted degree) of the node

Scalar product between co-chains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{G}_m^{-1} \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{i \in Q_m(\mathscr{K})} f_i [G_m^{-1}]_{ij} f_j$$

For $\mathbf{G}_m = \mathbf{I}$ we recover the standard L^2 norm.

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{m+1}^{\top}\mathbf{f} \equiv \bar{\mathbf{B}}_{m+1}\mathbf{f}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \longrightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

Ajoint operator δ_m^*

We define the matrix $\mathbf{B}_{m+1}^*$ as the matrix representing δ_m^*,

i.e. if $f' = \delta_m^* g$, then $\mathbf{f}' = \mathbf{B}_{m+1}^* \mathbf{g}$

From the definition it follows that

$$\boxed{\mathbf{B}_{m+1}^* = \mathbf{G}_m \bar{\mathbf{B}}_{m+1}^\top \mathbf{G}_{m+1}^{-1} = \mathbf{G}_m \mathbf{B}_{m+1} \mathbf{G}_{m+1}^{-1}}$$

Hence if $\mathbf{G}_m = \mathbf{I}, \mathbf{G}_{m+1} = \mathbf{I}$ then $\mathbf{B}_{m+1}^* = \mathbf{B}_{m+1}$

Proof

We define the matrix $\mathbf{B}^*_{m+1}$ as the matrix representing δ^*_m,

i.e. if $f' = \delta^*_m g$, then $\mathbf{f}' = \mathbf{B}^*_{m+1}\mathbf{g}$

We have the scalar product

$$\langle g, \delta_m f \rangle = \mathbf{g}\mathbf{G}^{-1}_{m+1}\bar{\mathbf{B}}_{m+1}\mathbf{f}$$
$$\langle \delta^*_m g, f \rangle = \mathbf{g}(\mathbf{B}^*)^{\top}_{m+1}\mathbf{G}^{-1}_m\mathbf{f}$$

If follows that for any **f** and **g**

$$\mathbf{g}\mathbf{G}^{-1}_{m+1}\bar{\mathbf{B}}_{m+1}\mathbf{f} = \mathbf{g}(\mathbf{B}^*)^{\top}_{m+1}\mathbf{G}^{-1}_m\mathbf{f}$$

Hence

$$\mathbf{B}^*_{m+1} = \mathbf{G}_m\bar{\mathbf{B}}^{\top}_{m+1}\mathbf{G}^{-1}_{m+1} = \mathbf{G}_m\mathbf{B}_{m+1}\mathbf{G}^{-1}_{m+1}$$

Weighed Hodge Laplacian

The Hodge-Laplacians

The m-dimensional Hodge-Laplacian L_m is defined as

$$L_m = L_m^{up} + L_m^{down}$$

where up and down m-dimensional Hodge Laplacians are given by

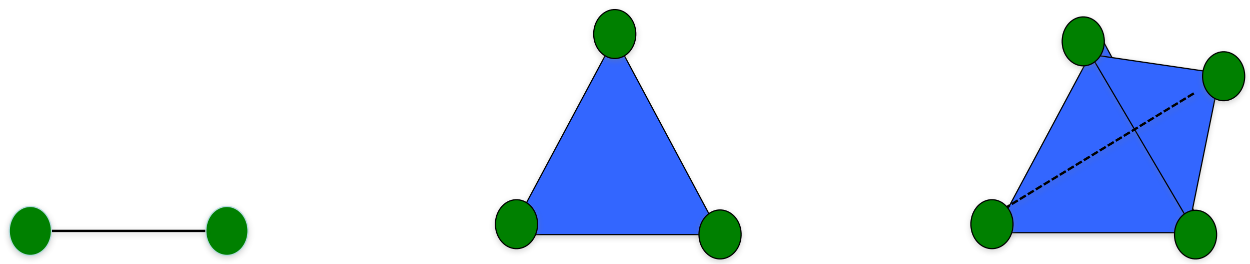
$$\begin{aligned} L_m^{up} &= \delta_m^* \delta_m, \\ L_m^{down} &= \delta_{m-1} \delta_{m-1}^*. \end{aligned}$$

The weighted Hodge Laplacian obeys Hodge decomposition

Hypergraphs or simplicial complexes?

The dilemma about the respresentation of higher-order network data

Hyperedges



2-hyperedge　3-hyperedge　4-hyperedge

An m-hyperedge is set nodes

$$\alpha = [i_1, i_2, i_3, \ldots i_m]$$

-it indicates the interactions between the m-nodes

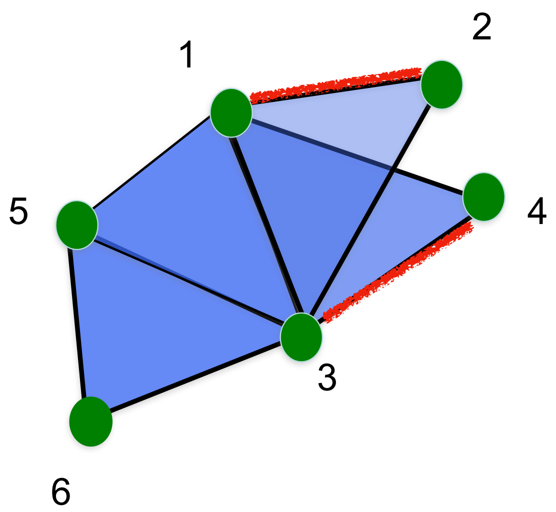
Hypergraphs

HYPERGRAPH

A hypergraph $\mathcal{G} = (V, E_H)$ is defined by a set V of N nodes and a set E_H of hyperedges, where a $(m+1)$-hyperedge indicates a set of $m+1$ nodes

$$e = [v_0, v_1, v_2, \ldots, v_m],$$

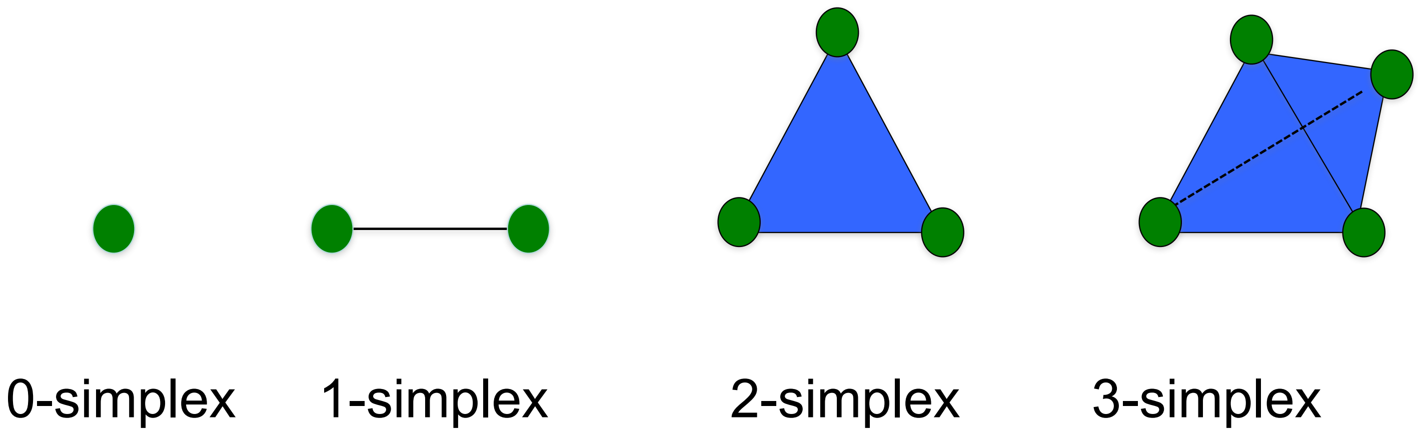
with generic value of $1 \leq m < N$.
An hyperdge describes the many-body interaction between the nodes.



Every hyperedge α formed by a subset of the nodes can belong or not to the hypergraph $\mathscr{H}$

$$\mathscr{H} = \{[1,2], [3,4], [1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Simplices



Simplices

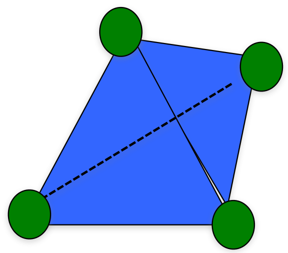
A d-dimensional simplex α (also indicated as a d-simplex α) is formed by a set of $(d+1)$ interacting nodes

$$\alpha = [v_0, v_1, v_2 \ldots, v_d].$$

It describes a many body interaction between the nodes.
It allows for a topological and a geometrical interpretation of the simplex.

Faces of a simplex

> **FACES**
>
> A face of a d-dimensional simplex α is a simplex α' formed by a proper subset of nodes of the simplex, i.e. $\alpha' \subset \alpha$.

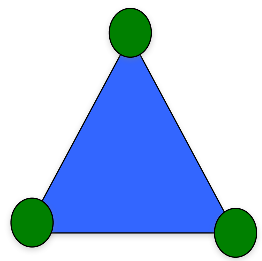


3-simplex

Faces

4 0-simplices

6 1-simplices

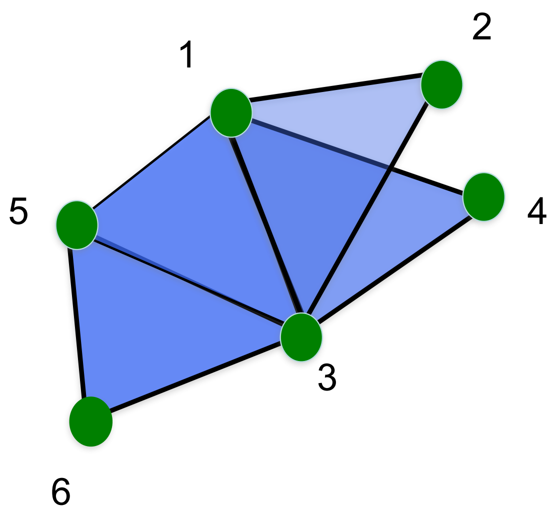


4 2-simplices

Simplicial complex

Simplicial complex

A simplicial complex $\mathcal{K}$ is formed by a set of simplices that is closed under the inclusion of the faces of each simplex.
The dimension d of a simplicial complex is the largest dimension of its simplices.



If a simplex α belongs to the simplicial complex $\mathscr{K}$ then every face of α must also belong to $\mathscr{K}$

$$\mathscr{K} = \{[1],[2],[3],[4],[5],[6],\\ [1,2],[1,3],[1,4],[1,5],[2,3],\\ [3,4],[3,5],[3,6],[5,6],\\ [1,2,3],[1,3,4],[1,3,5],[3,5,6]\}$$

Bare affinity weights

Bare affinity weights (blue) $\omega_\alpha \geq 0$

indicate which

higher-order interactions

are present in the data

The set of simplices with positive
bare affinity weights
does not need to be closed
under the inclusion of faces

Faccini et al. (2022)

Weighted simplicial complexes of d=1

Faccini et al. (2022)

Topological weights

For a d-dimensional simplex (link) α=[i,j]

$$w_{ij} = \omega_{ij} > 0$$

For a n-dimensional simplex α with $n < d$ (node) $\alpha = i$

$$w_i = \omega_i + \sum_{j \sim i} w_{ij} > 0$$

Weighted simplicial complexes

Faccini et al. (2022)

Topological weights

For a d-dimensional simplex α

$w_\alpha = \omega_\alpha$ >0

For a n-dimensional simplex α with $n < d$

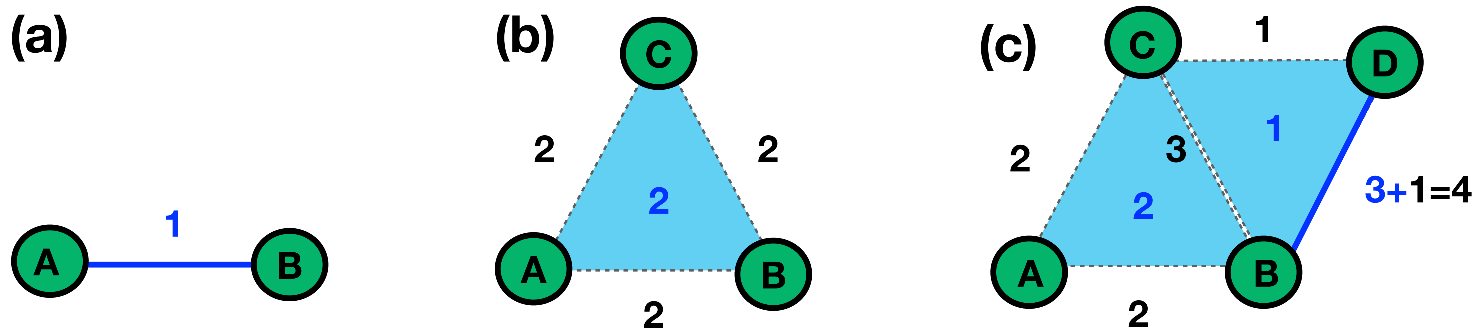
$$w_\alpha = \omega_\alpha + \sum_{\alpha' \supset \alpha} w_{\alpha'} > 0$$

Where α' is an $n + 1$ dimensional simplex

The relation between the bare affinity weights and the topological weights is invertible!!

With this convention weighted simplicial complexes do not involve any loss of information!!

Representation power of weighted simplicial complexes



Bare affinity weights (blue) $\omega_\alpha \geq 0$

1 Article: [A,B]

2 Articles: [A,B,C]

2 Articles: [A,B,C]
1 Article: [B,C,D]
3 Articles: [B,D]

Topological weights (black) $$w_\alpha = \omega_\alpha + \sum_{\alpha' \supset \alpha} w_{\alpha'} > 0$$

Faccini et al. (2022)

Lesson II:
Introduction to Algebraic Topology

- Introduction to algebraic topology
- Higher-order operators and their properties

1. Topological signals
2. The Hodge Laplacian and Hodge decomposition
3. Topology of weighed simplicial complexes