Recall: We defined the **d-dimensional torus T_{M^*}** to be $\mathrm{Spec}(\mathbb{C}[M])$ for $M \cong \mathbb{Z}^d$. Multiplication is defined in terms of comultiplication $\tilde{\mu}: \mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$, $z^n \mapsto z^n \otimes z^m$, which reduces to usual multiplication in $(\mathbb{C}^*)^d$ on closed points.

Two lattices associated to $T_{M^*} = \mathrm{Spec}(\mathbb{C}[M])$ play a central role in toric geometry.

We have already seen the character lattice M. Here is an alternative description of this lattice:

Def: Let T be a torus. The **character lattice of T** is $\mathrm{Hom}_{\text{Alg gp sch}}(T, \mathbb{C}^*)$.
Here $\mathbb{C}^*$ is shorthand for $\mathrm{Spec}(\mathbb{C}[L])$, where L is a rank 1 lattice. So we are looking at ring homomorphisms $\mathbb{C}[L] \xrightarrow{\varphi} \mathbb{C}[M]$ that respect the comultiplication.

$$\begin{tikzcd}
z^l \arrow[d, mapsto] \arrow[rrr, mapsto] & & & \varphi(z^l) \arrow[d, mapsto] \\
& \mathbb{C}[L] \arrow[r, "\varphi"] \arrow[d, "\tilde{\mu}"'] & \mathbb{C}[M] \arrow[d, "\tilde{\mu}"] & \\
& \mathbb{C}[L] \otimes_{\mathbb{C}} \mathbb{C}[L] \arrow[r, "\varphi \times \varphi"'] & \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] & \\
z^l \otimes z^l \arrow[rrr, mapsto] & & & \varphi(z^l) \otimes \varphi(z^l)
\end{tikzcd}$$

In order to have
$\tilde{\mu}(\varphi(z^l)) = \varphi(z^l) \otimes \varphi(z^l)$,
(Recall $\tilde{\mu}$ is a homomorphism)
φ must be induced by a lattice homomorphism $L \to M$.

So $\mathrm{Hom}_{\text{Alg gp sch}}(T, \mathbb{C}^*) = \mathrm{Hom}_{\mathbb{Z}}(L, M) \cong M$.

Exercise: It's common practice to think in more classical terms when doing toric geometry. So, working with, say, complex Lie groups rather than affine algebraic group schemes, can you show that $\mathrm{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*) \cong \mathbb{Z}^d$?

Hint: Try viewing $\mathbb{C}^*$ as $\exp(\mathbb{C})$. Then what is $(\mathbb{C}^*)^d$? Can you interpret the homomorphisms as certain linear maps between complex vector spaces?

The other lattice that plays a central role is the dual of the character lattice.

Def: The **cocharacter lattice of T** is $\mathrm{Hom}_{\text{Alg gp sch}}(\mathbb{C}^*, T)$.

Note that if N is the cocharacter lattice of T, then the closed points of T are identified with $N \otimes_{\mathbb{Z}} \mathbb{C}^*$ by evaluation: $z^n \otimes \lambda \mapsto z^n(\lambda)$.

For this reason, it is common to denote this torus by T_N.

Affine Toric Varieties

optional, but common, requirements

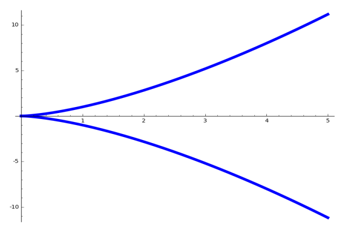
Def: A **toric variety** is a $\underbrace{\text{normal, finite-type}}$ variety X together with an open dense algebraic torus $T \subset X$ such that the action of T on itself (multiplication) extends to an action of T on X. If X is also an affine scheme, the toric variety is called an **affine toric variety**.

Def: A scheme is **normal** if all of its local rings are integrally closed domains.

A scheme (over $\mathrm{Spec}(\mathbb{C})$) is **finite type** if it has a finite cover of affine open subschemes $\mathrm{Spec}(R_i)$ where each R_i is a finitely generated $\mathbb{C}$-algebra.

Non-example of normal scheme: (Cuspidal cubic plane curve)

Let $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x^3 - y^2\rangle)$. You will show in the problem set that X fails to be normal. It is in fact an example of a non-normal toric variety.



Construction of affine toric varieties:

Let T_N be a d-dimensional torus. Write $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $M := N^*$, and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Def: A **convex rational polyhedral cone** (henceforth shortened to "**cone**") in $N_{\mathbb{R}}$ is a subset of the form

$$\sigma = \mathrm{Span}_{\mathbb{R}_{\geq 0}}\{ n_i : i \in I,\ n_i \in N\}$$

where I is a finite (possibly empty) indexing set.

(The adjective "rational" refers to the insistence that $n_i \in N$ rather than simply $N_{\mathbb{R}}$.)

If σ does **not** contain a line, it is said to be **strongly convex**.

Exercise: Can you give an equivalent definition involving an intersection of half spaces?

Def: Let σ be a cone in $N_{\mathbb{R}}$. The **dual cone** is $\sigma^\vee := \{ m \in M_{\mathbb{R}} : \langle n, m\rangle \geq 0\ \forall n \in \sigma\}$.

Def: A **semigroup** is a set S equipped with an associative binary operation and an identity element.

Warning: You may have called this a **monoid** in your algebra classes, and possibly dropped the identity element requirement when using the term "semigroup".

Def: Let σ be a strongly convex cone in $N_{\mathbb{R}}$. The commutative semigroup S_σ is

$$S_\sigma := \sigma^\vee \cap M.$$

The affine toric variety U_σ is the spectrum of the semigroup algebra $\mathbb{C}[S_\sigma]$:

$$U_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma]).$$

Question: How do you describe T_N in this framework?

Answer: set $\sigma = \{0\}$. Then $\sigma^\vee = M$ and $U_\sigma = \operatorname{Spec}(\mathbb{C}[M]) =: T_N$.

Question: We'd like to recover T_N as an open subscheme of U_σ by localizing in some way. This works if the group completion of the semigroup S_σ is $S_\sigma^{gp} = M$. How do we know this holds?

Answer: This holds if and only if $\sigma^\vee$ is full dimensional. Full dimensionality of $\sigma^\vee$ follows from strong convexity of σ.

The torus action on U_σ:

We have claimed that T_N acts on U_σ. This means the multiplication action of T_N on itself should extend to an action $T_N \times_{\operatorname{Spec}(\mathbb{C})} U_\sigma \to U_\sigma$, or on the level of $\mathbb{C}$-algebras the comultiplication

$$\hat{\mu} : \mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M], \quad z^m \mapsto z^m \otimes z^m$$

should restrict to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S_\sigma] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]$, which it clearly does.

Normality of U_σ:

Def: A scheme $(X, \mathcal{O}_X)$ is **integral** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Lemma: If $V = \operatorname{Spec}(R)$ is an integral affine scheme, V is normal if and only if R is integrally closed.

Proof: First, note $R = \bigcap_{p \in V} \mathcal{O}_{V,p}$. To see this, take $s \in \bigcap_{p \in V} \mathcal{O}_{V,p}$ and let $I_s := \{r \in R : rs \in R\}$. For any $p \in V$, $s \in \mathcal{O}_{V,p} = R_p$, so there must be some $t \in R \setminus p$ such that $ts \in R$. Then $t \in I_s$ and $I_s \not\subseteq p$. In particular, I_s is an ideal not contained in any maximal ideal. $I_s = R$ and $s \in R$.

Now suppose each R_p is integrally closed. If x is integral over R, it is also integral over R_p, so in R_p for all $p \in V$. Then $x \in \bigcap_{p \in V} R_p = R$.

Next suppose R is integrally closed. If $x \in \mathrm{Frac}(R)$ is integral over R_p, we can write

$$x^n + \frac{r_{n-1}}{s_{n-1}} x^{n-1} + \cdots + \frac{r_1}{s_1} x + \frac{r_0}{s_0} = 0$$

where $s_i \in R \setminus p$. Then

$$(s_{n-1} \cdots s_0)^n x^n + (s_{n-1} \cdots s_0)^n \frac{r_{n-1}}{s_{n-1}} x^{n-1} + \cdots + (s_{n-1} \cdots s_0)^n \frac{r_0}{s_0} = 0,$$

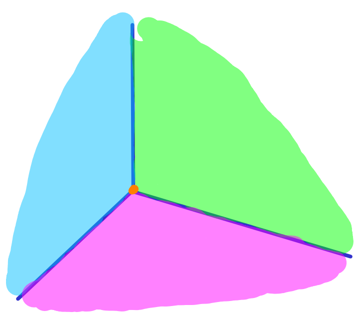
and $s_{n-1} \cdots s_0 x$ is integral over R, hence in R. So $x \in R_p$. ∎

You will complete the proof of normality of U_σ in the problem set.

Open toric subvarieties:

It may seem odd that we don't start directly with $\sigma^\vee$. But σ gives us a nice picture of the (toric) open subschemes of U_σ. Observe that the boundary of σ is a complex of lower dimensional cones.

Example: The boundary of a 3-dimensional simplicial cone consists of:



- Three 2-dimensional cones
- Three 1-dimensional cones
- One 0-dimensional cone

These boundary cones are called the **faces** of σ. For each face τ of σ, we have $\sigma^\vee \subset \tau^\vee$, $S_\sigma \subset S_\tau$, and $U_\tau \subset U_\sigma$. So inclusion of cones corresponds to inclusion of toric subvarieties.

You will describe U_σ and each U_τ in the problem set.

A distinguished point:

Proposition: Let $\sigma \subset N_{\mathbb{R}}$ be full dimensional. Then the $\mathbb{C}$-algebra homomorphism defined by

$$\mathbb{C}[S_\sigma] \xrightarrow{\varphi} \mathbb{C}$$
$$z^m \longmapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

determines a closed point $x_\sigma \in U_\sigma$ which is fixed by the T_N-action.

Proof: By definition, x_σ is the prime ideal $\varphi^{-1}(0)$ in $\mathbb{C}[S_\sigma]$. Since σ is full dimensional, $\sigma^\vee$ is strongly convex and $\sigma^\perp = \{0\} \subset M_{\mathbb{R}}$. That is, $\varphi(z^0) = 1$ while $\varphi(z^m) = 0$ for $m \neq 0$, and x_σ is the **maximal** ideal of non-invertible elements in $\mathbb{C}[S_\sigma]$ — it is a closed point.

Tune in next week for torus action!

(And find out what new cliff-hanger will leave you in suspense for the following week...)