Measure Theory   Fourth Week

A function f is continuous if

$f^{-1}(A)$ is open

for every open set A.

A function f is measurable if

$f^{-1}(A)$ is measurable

for every measurable set A.

$f : X \to Y$ requires a concept of measurable for both X and Y.

When $f : X \to [-\infty, \infty]$ the concept of measurable in $[-\infty, \infty]$ is Borel measurable.

Lemma (2.1.1)

Let $(X, \mathcal{A})$ be a measurable space,

and let $A \in \mathcal{A}$.

For a function $f : A \to [-\infty, +\infty]$ the following are equivalent:

(a) for every real number t the set $\{x \in A \mid f(x) \leq t\}$ belongs to $\mathcal{A}$,

(b) for every real number t the set $\{x \in A \mid f(x) < t\}$ belongs to $\mathcal{A}$,

(c) for every real number t the set $\{x \in A \mid f(x) \geq t\}$ belongs to $\mathcal{A}$,

(d) for every real number t the set $\{x \in A \mid f(x) > t\}$ belongs to $\mathcal{A}$.

$\{x \in A \mid f(x) < t\} =$

$\cup_{i=1}^{\infty}\{x \in A \mid f(x) \leq t - \frac{1}{i}\}$

shows that (a) implies (b).

The symmetric argument shows that (c) implies (d).

Closure by complementation shows that (a) is equivalent to (d) and (c) is equivalent to (b).

A circle of implication is completed.

Definition: A function is measurable with respect to $\mathcal{A}$ if any/all of the above conditions are satisfied.

Notice that this is equivalent to $f^{-1}(B) \in \mathcal{A}$ for every Borel subset $B \subseteq \mathbf{R}$.

How to show this: define $\mathcal{B}$ as the collection of all subsets in $\mathbf{R}$ such that

$B \in \mathcal{B}$ if and only if $f^{-1}(B) \in \mathcal{A}$.

It is easy to see that $\mathcal{B}$ is a sigma-algebra, and it contains the intervals, so it contains the Borel sets.

Examples:

(a) Continuous real valued functions are Borel measurable.

(b) Non-decreasing functions $f : I \to \mathbf{R}$ are Borel measurable.

(c) A function is *simple* if it takes on only finitely many values.

A simple function $f : X \to [-\infty, +\infty]$ is measurable if $f^{-1}(\alpha)$ is measurable for each of the finitely many values α.

Lemma (2.1.3): Let $(X, \mathcal{A})$ be a measurable space and A a subset of X in $\mathcal{A}$.

Let $f, g : A \to [-\infty, +\infty]$ be measurable functions.

The following sets belong to $\mathcal{A}$:

$\{x \in A \mid f(x) < g(x)\}$,

$\{x \in A \mid f(x) \leq g(x)\}$

and $\{x \in A \mid f(x) = g(x)\}$.

Proof:

$$\{x \mid f(x) < g(x)\} =$$

$$\cup_{r\in\mathbf{Q}}(\{x \mid f(x) < r\} \cap \{x \mid r < g(x)\}),$$

$$\Rightarrow \ \{x \mid f(x) < g(x)\} \in \mathcal{A}.$$

By complementation,

$$\{x \mid f(x) \geq g(x)\} \in \mathcal{A}$$

and by symmetry

$$\{x \mid g(x) \geq f(x)\}, \{x \mid g(x) < f(x)\} \in \mathcal{A}.$$

$$\Rightarrow \{x \mid f(x) = g(x)\} =$$

$$\{x \mid g(x) \geq f(x)\} \backslash \{x \mid g(x) > f(x)\} \in \mathcal{A}.$$

If f, g have a common domain,

$$(f \vee g)(x) := \max(f(x), g(x))$$

$$f \wedge g)(x) := \min(f(x), g(x)).$$

If $f_1, f_2, \ldots$ is a sequence of functions on the same domain,

define the functions $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ pointwise,

and where it exists likewise $\lim_n f_n$.

Lemma: If f and g are measurable then $f \wedge g$ and $f \vee g$ are measurable.

Proof:

for every choice of t,

$\{x \mid (f \wedge g)(x) < t\} =$

$\{x \mid f(x) < t\} \cup \{x \mid g(x) < t\}$

$\{x \mid (f \vee g)(x) < t\} =$

$\{x \mid f(x) < t\} \cap \{x \mid g(x) < t\}$

Lemma: If the f_n are measurable

then the functions

$\sup_n f_n$,

$\inf_n f_n$,

$\limsup_n f_n$,

$\liminf_n f_n$

and $\lim_n f_n$ are measurable.

Proof:

for every t,

$\{x \mid (\sup_n f_n)(x) \leq t\} =$

$\cap_n \{x \mid f_n(x) \leq t\}$,

$\{x \mid (\inf_n f_n)(x) < t\} =$

$\cup_n \{x \mid f_n(x) < t\}$,

$\{x \mid (\limsup_n f_n)(x) > t\} =$

$\cup_{i=1}^{\infty} \cap_{n=1}^{\infty} \{x \mid \ \sup_{k=n}^{\infty} f_k(x) > t + \frac{1}{i}\}$

$\{x \mid (\liminf_n f_n)(x) < t\} =$

$\cup_{i=1}^{\infty} \cap_{n=1}^{\infty} \{x \mid \ \inf_{k=n}^{\infty} f_k(x) < t - \frac{1}{i}\}$.

And finally $\lim_n f_n$ is defined on the set where $\limsup_n f_n = \liminf_n f_n$,

which is measurable, and on this set

$\{x \mid (\lim_n f_n)(x) \geq t\}$ is equal either to

$\{x \mid (\limsup_n f)(x) \geq t\}$ or

$\{x \mid (\liminf_n f)(x) \geq t\}$.

Notice that we could also show that

$\{x \mid (\liminf_n f_n)(x) > t\} =$

$\cup_{i=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \{x \mid f_k(x) > t + \frac{1}{i}\}$.

Lemma:

Real valued measurable functions form a subspace, or

when f, g are real valued and measurable and r is a real number

then $f+g$ and rf are measurable functions.

Proof:

$r = 0$ trivial

$r > 0$: $\{x \mid rf(x) \leq t\} = \{x \mid f(x) \leq \frac{t}{r}\}$.

$r < 0$: $\{x \mid rf(x) \leq t\} = \{x \mid f(x) \geq \frac{t}{r}\}$.

$\{x \mid f(x) + g(x) < t\} =$

$\cup_{r \in \mathbf{Q}}(\{x \mid f(x) < r\} \cap \{x \mid g(x) < t - r\})$.

Lemma: When f, g are measurable real valued functions

then fg is measurable

and $\frac{f}{g}$ is measurable where $g \neq 0$.

Proof: First, f^2 is measurable, as for every $t > 0$

$\{x \mid f^2(x) < t\} =$

$\{x \mid -\sqrt{t} < f(x) < \sqrt{t}\}$.

Then notice that $(f+g)^2 = f^2 + g^2 + 2fg$,

so that the measurability of f^2 and g^2 implies the measurability of $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$.

The set where $g \neq 0$ is measurable,

and $\{x \mid \frac{f(x)}{g(x)} < t\} =$

$\{x \mid g(x) > 0\} \cap \{x \mid f(x) < tg(x)\}$

unioned with

$\{x \mid g(x) < 0\} \cap \{x \mid f(x) > tg(x)\}$ $\square$

Implied is also that $|f|$ is a measurable function if f is measurable,

since $|f(x)| = \max(f(x), -f(x))$.

Also any funtion can be broken into its positive and negative parts, both measurable:

$f^+(x) = \max(f(x), 0)$,

$f^-(x) = -\min(f(x), 0)$ and

$f = f^+ - f^-$.

Lemma: Let A be a measurable subset of X.

For every measurable function $f : A \to [0, \infty]$

there is an infinite sequence $f_1 \leq f_2 \leq \dots$ of simple functions with values in $[0, \infty)$ such that

$f(x) = \lim_{i \to \infty} f_i(x)$ for all $x \in A$.

Proof: For every $n = 1, 2, 3, \ldots$ and $0 \leq k < n2^n$

define $A_{n,k} = \{x \mid \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}$

and $A_{n,n2^n} = \{x \mid f(x) \geq n\}$.

Define $f_n(x) := \frac{k}{2^n}$ if $x \in A_{n,k}$.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

A property holds μ-almost everywhere if

the set of points where it does not hold is contained in a set of measure zero with respect to μ.

For example, the real valued function $f(t) = \frac{1}{t}$ is defined λ^*-almost everywhere,

as it is not defined only for 0.

The real numbers are almost everywhere irrational,

because the set of rational numbers is a set of measure zero.

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space such that μ is complete.

Let $f, g : X \to [-\infty, +\infty]$ be functions such that

f is $\mathcal{A}$-measurable and

$f = g$ μ-almost everywhere.

Then g is $\mathcal{A}$ measurable.

Proof: Let N be a subset with $\mu(N) = 0$ where $\{x \mid f(x) \neq g(x)\}$ is contained in N.

For every t,

$\{x \mid g(x) \leq t\} =$

$\{x \mid f(x) \leq t\} \cap (X \backslash N)$

unioned with $\{x \mid g(x) \leq t\} \cap N$,

both sets $\mathcal{A}$ measurable.

Lemma: $(X, \mathcal{A}, \mu)$ is a measure space with μ complete.

If $f_n : X \to [-\infty, +\infty]$ is a sequence of $\mathcal{A}$ measurable functions and $f : X \to [-\infty, +\infty]$ is a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere,

then f is a $\mathcal{A}$ measurable function.

Proof: Where it is defined, by previous lemma necessarily on a measurable set,

$\lim_{n\to\infty} f_n(x)$ is measurable.

Then by the previous lemma, f is also measurable.