

LTCC — Special Functions and Applications

Mock exam 2025–26

1. Recall that the Jacobi polynomial of degree n is defined to be

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, & n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right),$$

where $\alpha > -1$ and $\beta > -1$.

(a) Show that $u_n(x) = P_n^{(\alpha, \beta)}(x)$ solves the ODE

$$-\frac{d}{dx} \left(p(x) \frac{du(x)}{dx} \right) = \lambda_n w(x) u_n(x),$$

where

$$p(x) = (1-x)^{\alpha+1}(1+x)^{1+\beta}, \quad w(x) = (1-x)^\alpha(1+x)^\beta, \quad \text{and} \quad \lambda_n = n(n+\alpha+\beta+1).$$

Solution: Now $y(z) := {}_2F_1 \left(\begin{matrix} -n, & n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; z \right)$ solves the hypergeometric equation

$$z(1-z) \frac{d^2 y}{dz^2} + [c - (a+b+1)z] \frac{dy}{dz} - aby = 0,$$

where $a = -n$, $b = n + \alpha + \beta + 1$ and $c = \alpha + 1$. That is,

$$z(1-z) \frac{d^2 y}{dz^2} + [\alpha + 1 - (\alpha + \beta + 2)z] \frac{dy}{dz} + n(n + \alpha + \beta + 1) y = 0,$$

Let $v(x) = y(z)$, where $z = (1-x)/2$. Then v solves

$$(1-x^2) \frac{d^2 v}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x] \frac{dv}{dx} + n(n + \alpha + \beta + 1)v = 0.$$

Since $u_n(x) = P_n^{(\alpha, \beta)}(x)$ is a constant multiple of $v(x)$, it also satisfies the same equation. The result follows on noting that

$$\begin{aligned} \frac{d}{dx} \left(p(x) \frac{du_n(x)}{dx} \right) &= p(x) \left[u_n'' + \frac{[\beta - \alpha - (\alpha + \beta + 2)x]}{(1-x)^{\alpha+1}(1+x)^{\beta+1}} u_n' \right] \\ &= -\frac{p(x)}{1-x^2} n(n + \alpha + \beta + 1) u_n(x) \\ &= -\lambda_n w(x) u_n(x). \end{aligned}$$

(b) Show that if $m \neq n$ then

$$\int_{-1}^1 w(x) u_m(x) u_n(x) dx = 0.$$

That is, if $m \neq n$ then

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = 0.$$

Hint: first show that

$$-\frac{d}{dx} (p(x) \{u_m(x) u'_n(x) - u_n(x) u'_m(x)\}) = (\lambda_m - \lambda_n) w(x) u_m(x) u_n(x).$$

Solution: From the previous part, we have

$$-u_m(pu'_n)' = \lambda_n w u_m u_n \quad \text{and} \quad -u_n(pu'_m)' = \lambda_m w u_m u_n.$$

Taking the difference we have

$$\{u_m(pu'_n)' - u_n(pu'_m)'\} = (\lambda_m - \lambda_n) w u_m u_n.$$

Observe that the left side is

$$\{p(u_m u'_n - u'_m u_n)\}'.$$

The result follows on integrating from -1 to 1 and noting that $p(-1) = 0 = p(1)$ giving

$$(\lambda_m - \lambda_n) \int_{-1}^1 w(x) u_m(x) u_n(x) dx = 0.$$

Finally we observe that $\lambda_m - \lambda_n = 0$ if and only if

$$0 = m(m + \alpha + \beta + 1) - n(n + \alpha + \beta + 1) = (m - n)(m + n + \alpha + \beta + 1).$$

Since $\alpha > -1$ and $\beta > -1$, it follows that $m + n + \alpha + \beta + 1 > 0$ provided $(m, n) \neq (0, 0)$. Therefore, if $m \neq n$, then $\lambda_m - \lambda_n \neq 0$.

2. Show that if $u(x, t) = w(z)$, $z = x - ct$ is a travelling wave solution of the KdV

$$u_t + 6uu_x + u_{xxx} = 0,$$

then w satisfies an ODE of the form

$$(w')^2 + 2(w - \alpha)(w - \beta)(w - \gamma) = 0,$$

where α , β and γ are constants. Assuming that α , β and γ are real and that $\alpha < \beta < \gamma$, show that the travelling wave solution is given by

$$u(x, t) = \beta + (\gamma - \beta) \operatorname{cn}^2 \left\{ \left[\frac{1}{2}(\gamma - \alpha) \right]^{1/2} [x - 2(\alpha + \beta + \gamma)t + x_0]; k \right\},$$

where x_0 is an arbitrary constant and cn is the Jacobi elliptic function with modulus

$$k = \sqrt{\frac{\gamma - \beta}{\gamma - \alpha}}.$$

Solution: If we look for travelling wave solutions $u(x, t) = w(z)$, where $z = x - ct$, we get

$$-cw' + 6ww' + w''' = 0.$$

Integration gives

$$-cw + 3w^2 + w'' + c_1 = 0,$$

where c_1 is a constant. Multiplying by w' gives

$$w'w'' + (3w^2 - cw + c_1)w' = 0,$$

so a further integration yields

$$(w')^2 + 2w^3 - cw^2 + 2c_1w + c_2 = 0,$$

where c_2 is a constant. The desired result follows on noting that α , β and γ are the roots of the polynomial $2w^3 - cw^2 + 2c_1w + c_2$. Notice that $\alpha + \beta + \gamma = c/2$.

Now we follow the procedure given in lectures for transforming an elliptic integral to standard form. Let $w(z) = \gamma - v(z)^2$. Then we have

$$2(v')^2 = (v^2 - [\gamma - \alpha])(v^2 - [\gamma - \beta])$$

So

$$\frac{2(v')^2}{(\gamma - \alpha)(\gamma - \beta)} = \left(1 - \frac{v^2}{\gamma - \alpha}\right) \left(1 - \frac{v^2}{\gamma - \beta}\right).$$

Let $v = \sqrt{\gamma - \beta} V(z)$. Then

$$\frac{2(V')^2}{\gamma - \alpha} = (1 - V^2) \left(1 - \frac{\gamma - \beta}{\gamma - \alpha} V^2\right).$$

This gives

$$\int \frac{dV}{\sqrt{(1 - V^2)(1 - k^2 V^2)}} = \sqrt{\frac{\gamma - \alpha}{2}} \int dz, \quad k = \sqrt{\frac{\gamma - \beta}{\gamma - \alpha}}.$$

Therefore

$$V(z) = \operatorname{sn} \left(\sqrt{\frac{\gamma - \alpha}{2}} (z + x_0); k \right),$$

where x_0 is an arbitrary constant of integration. Therefore

$$u(x, t) = w(x - ct) = \gamma - (\gamma - \beta) \operatorname{sn}^2 \left(\sqrt{\frac{\gamma - \alpha}{2}} ([x - ct] + x_0); k \right).$$

The final result follows on noting that $\operatorname{sn}(\zeta; k)^2 + \operatorname{cn}(\zeta; k)^2 = 1$ and that $c = 2(\alpha + \beta + \gamma)$.

3. Suppose that the Riemann matrix B has a block diagonal structure

$$B = \begin{pmatrix} \tilde{B} & 0 \\ 0 & \hat{B} \end{pmatrix},$$

where $\tilde{B}$ and $\hat{B}$ are $m \times m$ and $n \times n$ matrices respectively. Show that

$$\theta(z|B) = \theta(\tilde{z}|\tilde{B})\theta(\hat{z}|\hat{B}),$$

where $z = (z_1, \dots, z_g)^T$, $\tilde{z} = (z_1, \dots, z_m)$ and $\hat{z} = (z_{m+1}, \dots, z_g)$.

Solution: Let $N = (N_1, \dots, N_g)^T$, $\tilde{N} = (N_1, \dots, N_m)^T$ and $\hat{N} = (N_{m+1}, \dots, N_g)^T$. Then

$$\begin{aligned} \theta(z, B) &= \sum_{N \in \mathbb{Z}^g} \exp \left(\frac{1}{2} N^T B N + N^T z \right) \\ &= \sum_{N \in \mathbb{Z}^g} \exp \left(\frac{1}{2} \tilde{N}^T \tilde{B} \tilde{N} + \frac{1}{2} \hat{N}^T \hat{B} \hat{N} + \tilde{N}^T z + \hat{N}^T z \right) \\ &= \sum_{\tilde{N} \in \mathbb{Z}^m} \sum_{\hat{N} \in \mathbb{Z}^{g-m}} \exp \left(\frac{1}{2} \tilde{N}^T \tilde{B} \tilde{N} + \frac{1}{2} \hat{N}^T \hat{B} \hat{N} + \tilde{N}^T z + \hat{N}^T z \right) \\ &= \sum_{\tilde{N} \in \mathbb{Z}^m} \sum_{\hat{N} \in \mathbb{Z}^{g-m}} \exp \left(\frac{1}{2} \tilde{N}^T \tilde{B} \tilde{N} + \tilde{N}^T z \right) \exp \left(\frac{1}{2} \hat{N}^T \hat{B} \hat{N} + \hat{N}^T z \right) = \theta(\tilde{z}|\tilde{B})\theta(\hat{z}|\hat{B}). \end{aligned}$$

4. Show that the Baker-Akhiezer function

$$\psi(P) \equiv \psi(x; P) = \frac{\sigma(u - u_1 - x) e^{x\zeta(u)}}{\sigma(u - u_1) \sigma(u_1 + x)}, \quad P = (\wp(u), \wp'(u)).$$

satisfies the Lamé equation $(-\partial_x^2 + 2\wp(x + u_1)) \psi(x; P) = \lambda \psi(x; P)$, with $\lambda = \wp(u)$. Here $\wp$, ζ and σ are the Weierstrass functions defined in lectures. You may use the identity

$$x + y + z = 0 \quad \Rightarrow \quad \zeta'(x) + \zeta'(y) + \zeta'(z) + (\zeta(x) + \zeta(y) + \zeta(z))^2 = 0.$$

Solution: Recall that $\zeta'(z) = -\wp(z)$ and $\sigma'(z) = \zeta(z)\sigma(z)$. So taking the logarithmic derivative of ψ wrt x gives

$$\begin{aligned} \frac{\psi_x}{\psi} &= -\frac{\sigma'(u - u_1 - x)}{\sigma(u - u_1 - x)} + \zeta(u) - \frac{\sigma'(u_1 + x)}{\sigma(u_1 + x)} \\ &= -\zeta(u - u_1 - x) + \zeta(u) - \zeta(u_1 + x). \end{aligned}$$

Differentiating again wrt x gives

$$\begin{aligned} \frac{\psi_{xx}}{\psi} &= \left(\frac{\psi_x}{\psi} \right)^2 + \zeta'(u - u_1 - x) - \zeta'(u_1 + x) \\ &= [\zeta(u - u_1 - x) - \zeta(u) + \zeta(u_1 + x)]^2 + \zeta'(u - u_1 - x) - \zeta'(u_1 + x) \\ &= [\zeta(u - u_1 - x) + \zeta(-u) + \zeta(u_1 + x)]^2 + \zeta'(u - u_1 - x) - \zeta'(u_1 + x) \\ &= [-\zeta'(u - u_1 - x) - \zeta'(-u) - \zeta'(u_1 + x)] + [\zeta'(u - u_1 - x) - \zeta'(u_1 + x)] \\ &= \zeta'(u) - 2\zeta'(u_1 + x) = -\wp(u) + 2\wp(x + u_1), \end{aligned}$$

as required, where we have used the identity given in the question and the fact that ζ is an odd function.