

LTCC — Special Functions and Applications

Mock exam 2025–26

1. Recall that the Jacobi polynomial of degree n is defined to be

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, & n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right),$$

where $\alpha > -1$ and $\beta > -1$.

- (a) Show that $u_n(x) = P_n^{(\alpha, \beta)}(x)$ solves the ODE

$$-\frac{d}{dx} \left(p(x) \frac{du(x)}{dx} \right) = \lambda_n w(x) u_n(x),$$

where

$$p(x) = (1-x)^{\alpha+1}(1+x)^{1+\beta}, \quad w(x) = (1-x)^\alpha(1+x)^\beta, \quad \text{and} \quad \lambda_n = n(n + \alpha + \beta + 1).$$

- (b) Show that if $m \neq n$ then

$$\int_{-1}^1 w(x) u_m(x) u_n(x) dx = 0.$$

That is, if $m \neq n$ then

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) dx = 0.$$

Hint: first show that

$$-\frac{d}{dx} (p(x) \{u_m(x) u'_n(x) - u_n(x) u'_m(x)\}) = (\lambda_m - \lambda_n) w(x) u_m(x) u_n(x).$$

2. Show that if $u(x, t) = w(z)$, $z = x - ct$ is a travelling wave solution of the KdV

$$u_t + 6uu_x + u_{xxx} = 0,$$

then w satisfies an ODE of the form

$$(w')^2 + 2(w - \alpha)(w - \beta)(w - \gamma) = 0,$$

where α , β and γ are constants. Assuming that α , β and γ are real and that $\alpha < \beta < \gamma$, show that the travelling wave solution is given by

$$u(x, t) = \beta + (\gamma - \beta) \operatorname{cn}^2 \left\{ \left[\frac{1}{2}(\gamma - \alpha) \right]^{1/2} [x - 2(\alpha + \beta + \gamma)t + x_0]; k \right\},$$

where x_0 is an arbitrary constant and cn is the Jacobi elliptic function with modulus

$$k = \sqrt{\frac{\gamma - \beta}{\gamma - \alpha}}.$$

3. Suppose that the Riemann matrix B has a block diagonal structure

$$B = \begin{pmatrix} \tilde{B} & 0 \\ 0 & \hat{B} \end{pmatrix},$$

where $\tilde{B}$ and $\hat{B}$ are $m \times m$ and $n \times n$ matrices respectively. Show that

$$\theta(z|B) = \theta(\tilde{z}|\tilde{B})\theta(\hat{z}|\hat{B}),$$

where $z = (z_1, \dots, z_g)^T$, $\tilde{z} = (z_1, \dots, z_m)$ and $\hat{z} = (z_{m+1}, \dots, z_g)$.

4. Show that the Baker-Akhiezer function

$$\psi(P) \equiv \psi(x; P) = \frac{\sigma(u - u_1 - x)e^{x\zeta(u)}}{\sigma(u - u_1)\sigma(u_1 + x)}, \quad P = (\wp(u), \wp'(u)).$$

satisfies the Lamé equation $(-\partial_x^2 + 2\wp(x + u_1))\psi(x; P) = \lambda\psi(x; P)$, with $\lambda = \wp(u)$. Here $\wp$, ζ and σ are the Weierstrass functions defined in lectures. You may use the identity

$$x + y + z = 0 \quad \Rightarrow \quad \zeta'(x) + \zeta'(y) + \zeta'(z) + (\zeta(x) + \zeta(y) + \zeta(z))^2 = 0.$$