Topology, geometry and dynamics of higher-order networks

An introduction to simplicial complexes
Lesson I

LTCC module 2024

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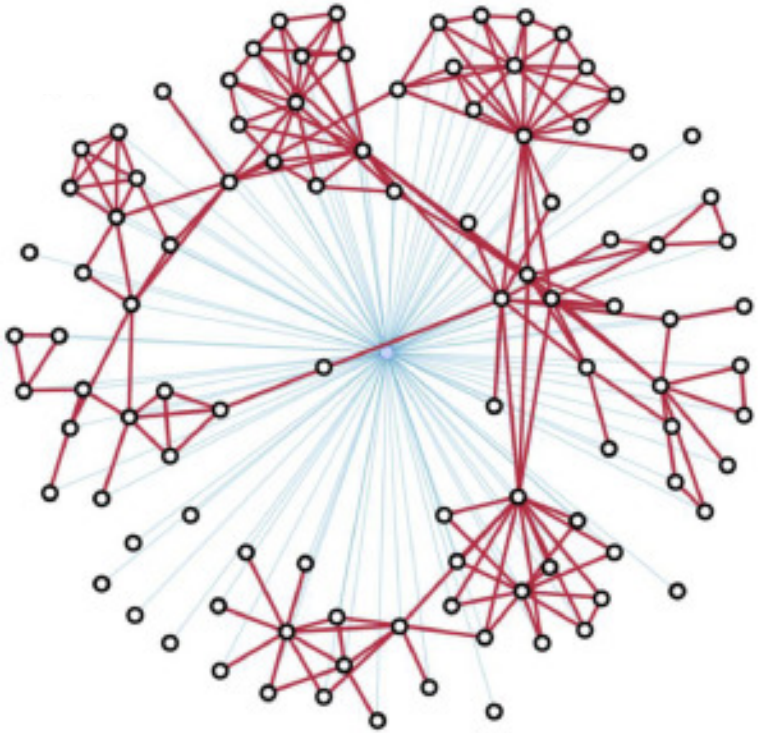
Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London

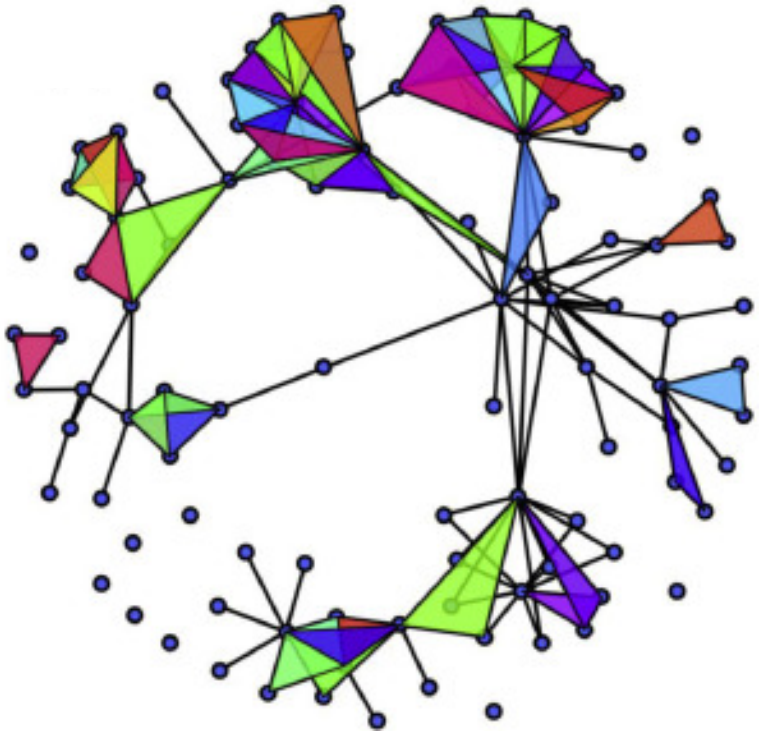
Queen Mary
University of London

Networks



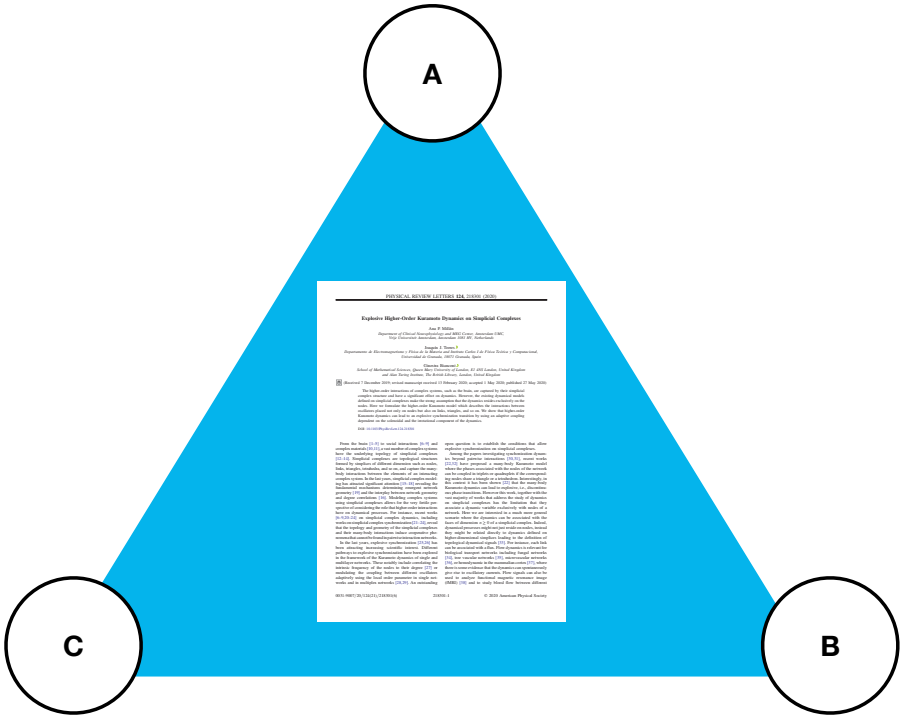
Simple network

Higher-order networks



Simplicial complex

Higher order network



PHYSICAL REVIEW LETTERS 124, 218301 (2020)

Explosive Higher-Order Kuramoto Dynamics on Simplicial Complexes

Ana P. Millán
Department of Clinical Neurophysiology and MIRA Center, Amsterdam UMC, Vrije Universiteit Amsterdam, Amsterdam, 1081 HV, Netherlands

Joaquín J. Torres
Departamento de Electromagnetismo y Física de la Materia and Institute Carlos I for Theoretical and Computational Physics, Universidad de Granada, 18071 Granada, Spain

Ginestra Bianconi
School of Mathematical Sciences, Queen Mary University of London, E1 4NS London, United Kingdom and Alan Turing Institute, The British Library, London, United Kingdom

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The higher-order interactions of complex systems, such as the brain, are captured by their simplicial complex structure and have a significant effect on dynamics. However, the existing dynamical models defined on simplicial complexes make the strong assumption that the dynamics evolves exclusively on the nodes of the network, while the dynamics of the links and of higher simplices is simply enslaved. Here we let oscillators placed not only on nodes but also on links, triangles, and so on. We show that higher-order Kuramoto dynamics can lead to an explosive synchronization transition by using an *adaptive coupling* dependent on the relaxational and the frustrational components of the dynamics.

DOI: 10.1103/PhysRevLett.124.218301

From the brain [1–3] to social interactions [4–7] and ecosystems [8–11], many complex systems are characterized by have the underlying topology of simplicial complexes [12–14]. Simplicial complexes are topological structures composed by nodes, links, triangles, tetrahedra, and so on, and capture the many-body interactions between the elements of an interacting complex system. The recent literature on simplicial complexes has also witnessed significant attention [15–18] revealing the fundamental mechanisms determining emergent network geometry [19–21], structural [22–24] and dynamical [25–27] properties and degree correlations [28]. Modeling complex systems using simplicial complexes allows for the very fertile perspective of considering the role that higher-order interactions have on dynamical processes. For instance, recent works [29,30] have shown that higher-order interactions can lead to explosive synchronization [31–33], reveal that the topology and geometry of the simplicial complexes can affect the stability of the synchronous state [34], and that nonmonotonic cascades in the dynamics can occur [35].

In the last years, explosive synchronization [29,30] has been observed in many complex systems [36–38] and different pathways to explosive synchronization have been explored in the framework of the Kuramoto dynamics of simple and multiplex networks [39–45]. Among them, the role of the intrinsic frequency of the nodes or their degree [46] or modulating the coupling between different oscillators adaptively using the local order parameter in simple networks and in multiplex networks [30,39]. An outstanding open question is to establish the conditions that allow explosive synchronization to occur in simplicial complexes.

Among the papers investigating synchronization dynamics beyond pairwise interactions [29,31], recent works [47,48] have studied the Kuramoto model on hypergraphs where the phases associated with the nodes of the network can be coupled in triplets or quadruplets if the corresponding hyperlinks are present in the hypergraph. In this framework, Ref. [47] has found that for many Euler-Kuramoto dynamics can lead to explosive, i.e. discontinuous, synchronization transitions. However, there is still a vast majority of works that address the study of dynamics on simplicial complexes but the limitation that they associate a dynamic variable exclusively with nodes of a network. Here we are interested in a much more general framework where the dynamical variables are defined on the faces of dimension $d \leq D$ of a simplicial complex. Indeed, dynamical processes might not just reside on nodes, instead they can be defined on links, triangles, and generally on higher-dimensional simplices leading to the definition of topological dynamical signals [49]. For instance, such link signals are relevant in the study of flow dynamics in biological transport networks including fungal networks [50], ant trail networks [51], microvascular networks [52], and cerebral blood flow networks [53]. Also, recent studies [54] have evidence that the dynamics can spontaneously give rise to oscillatory currents. Flow signals can also be used to analyze functional magnetic resonance image (fMRI) [55] and to study blood flow between different

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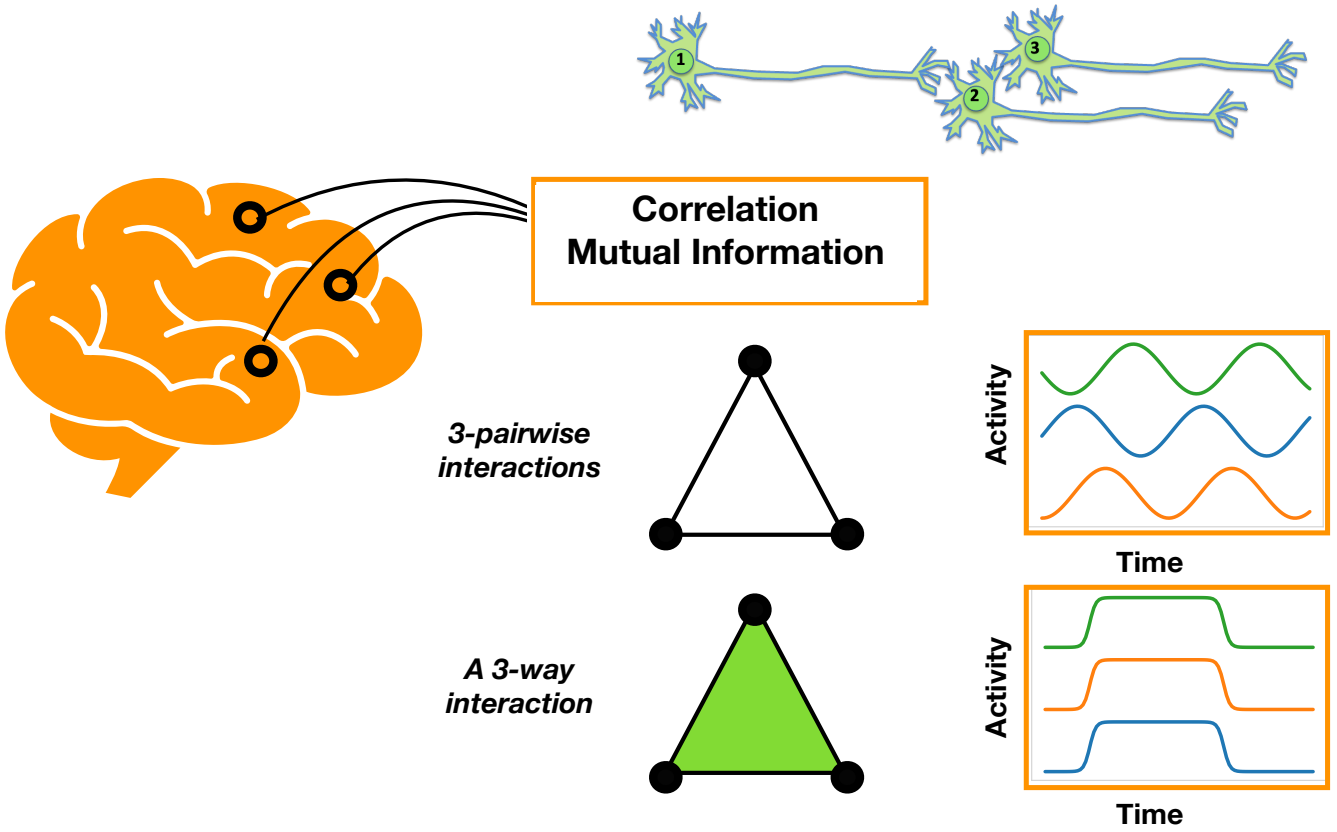
Collaboration network

Higher-order brain networks



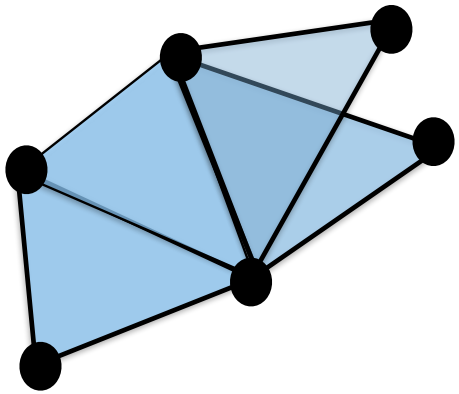
Simplicial complexes

Simplicial complexes are characterising the interactions between two ore more nodes and

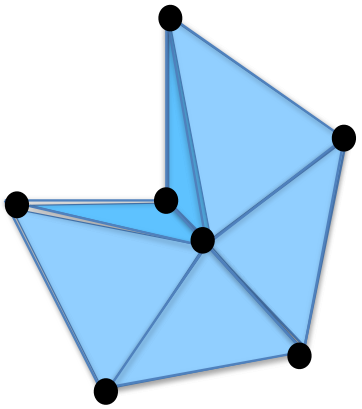
are formed by nodes, links, triangles, tetrahedra etc.

They have been widely studied in discrete quantum gravity approaches

They allow for topological and geometrical interpretation of higher-order interactions



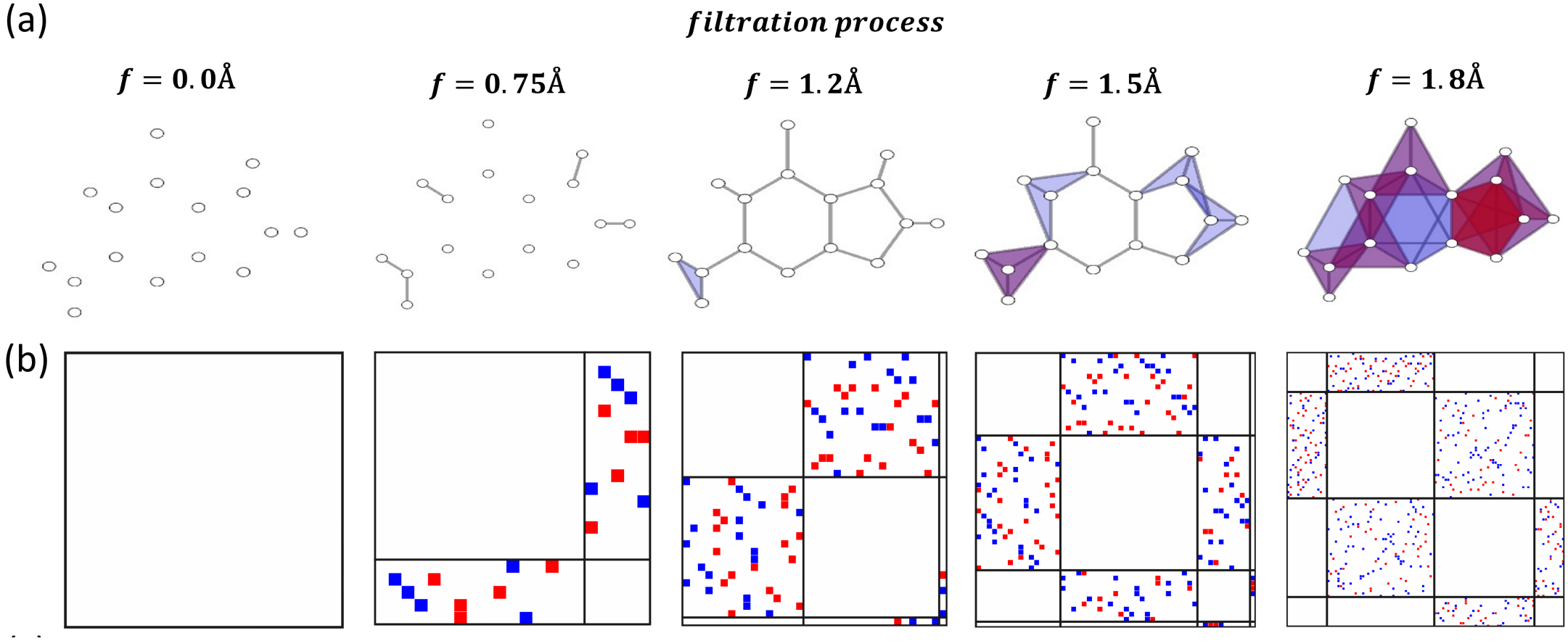
d=2 simplicial complex



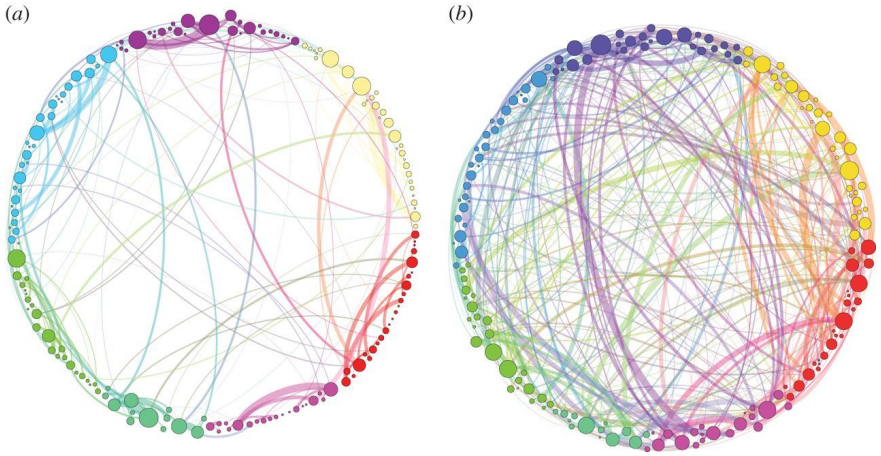
d=3 simplicial complex

Topology of Biomolecules

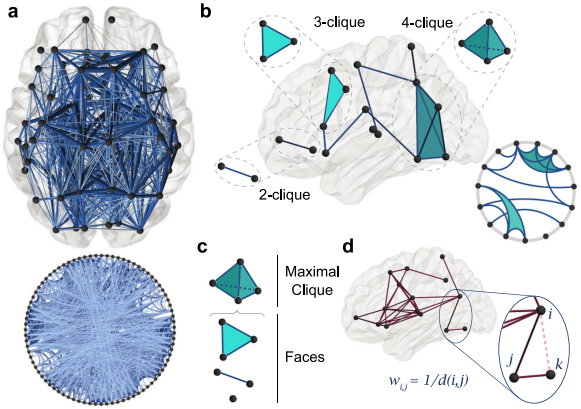
Wee et al. (2023)

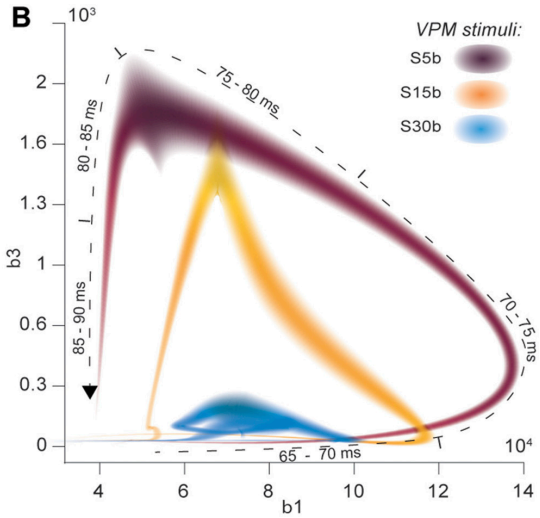


Topology
of higher-order brain networks



Petri et al.
"Homological scaffolds
of brain functional networks."
Journal of the Royal Society Interface
2014

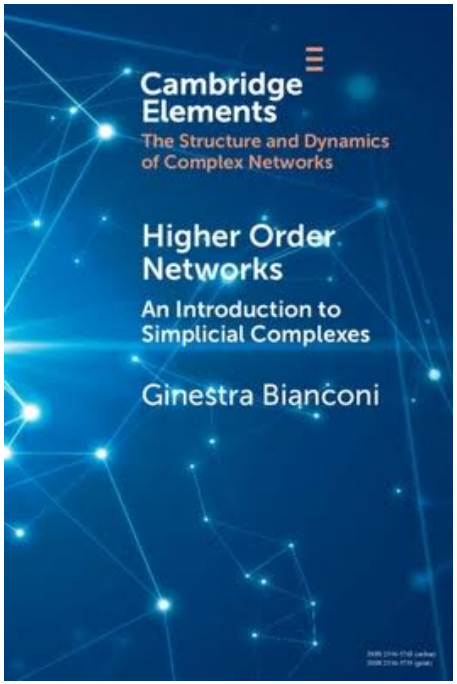




Reimann et al.
Cliques of neurons bound into cavities
provide a missing link between structure and function.
Frontiers in Computational Neuroscience 2017

Sizemore et al.
Cliques and cavities in the human connectome
Journal of Computational Neuroscience 2018

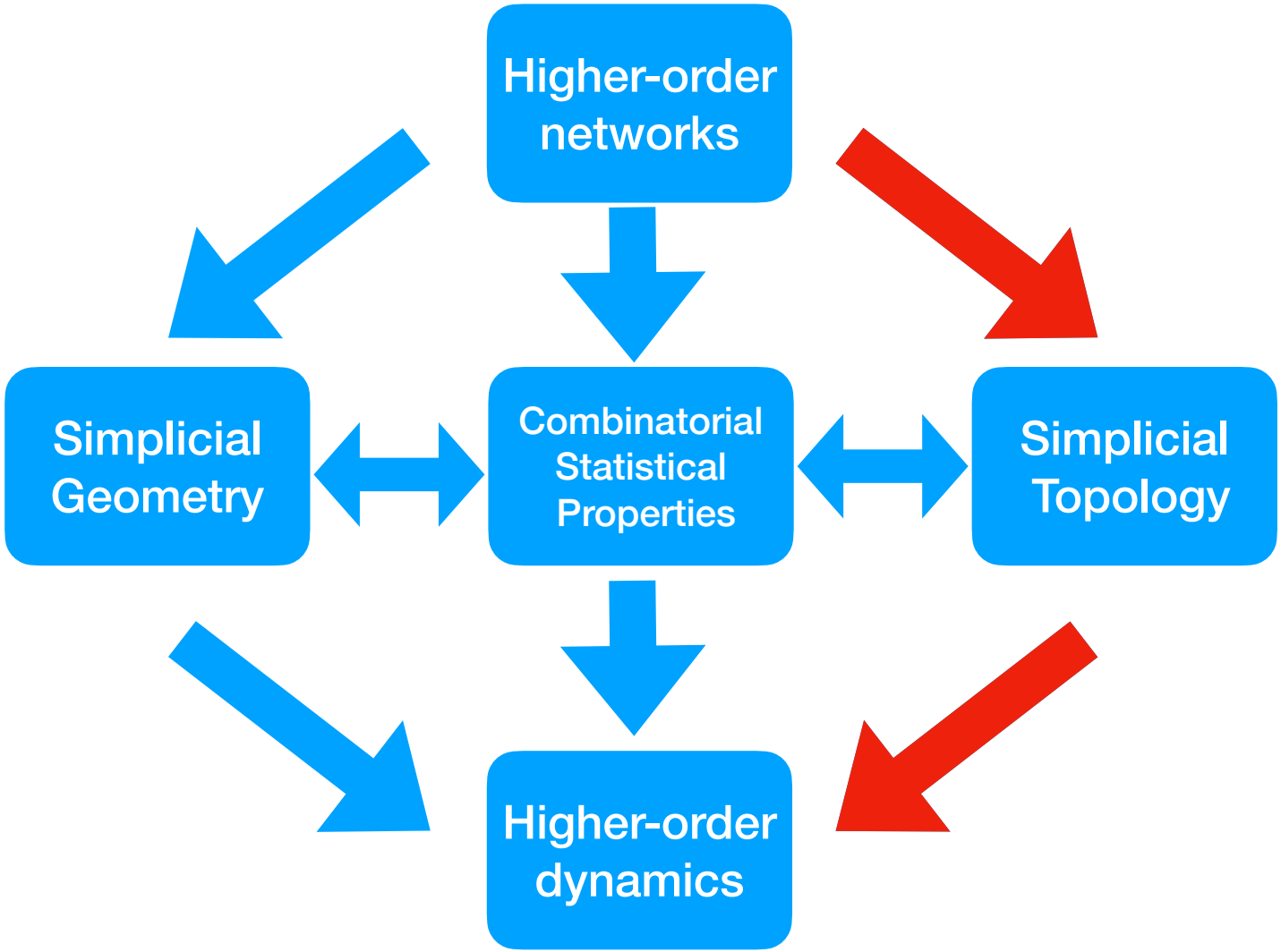
Higher-order networks



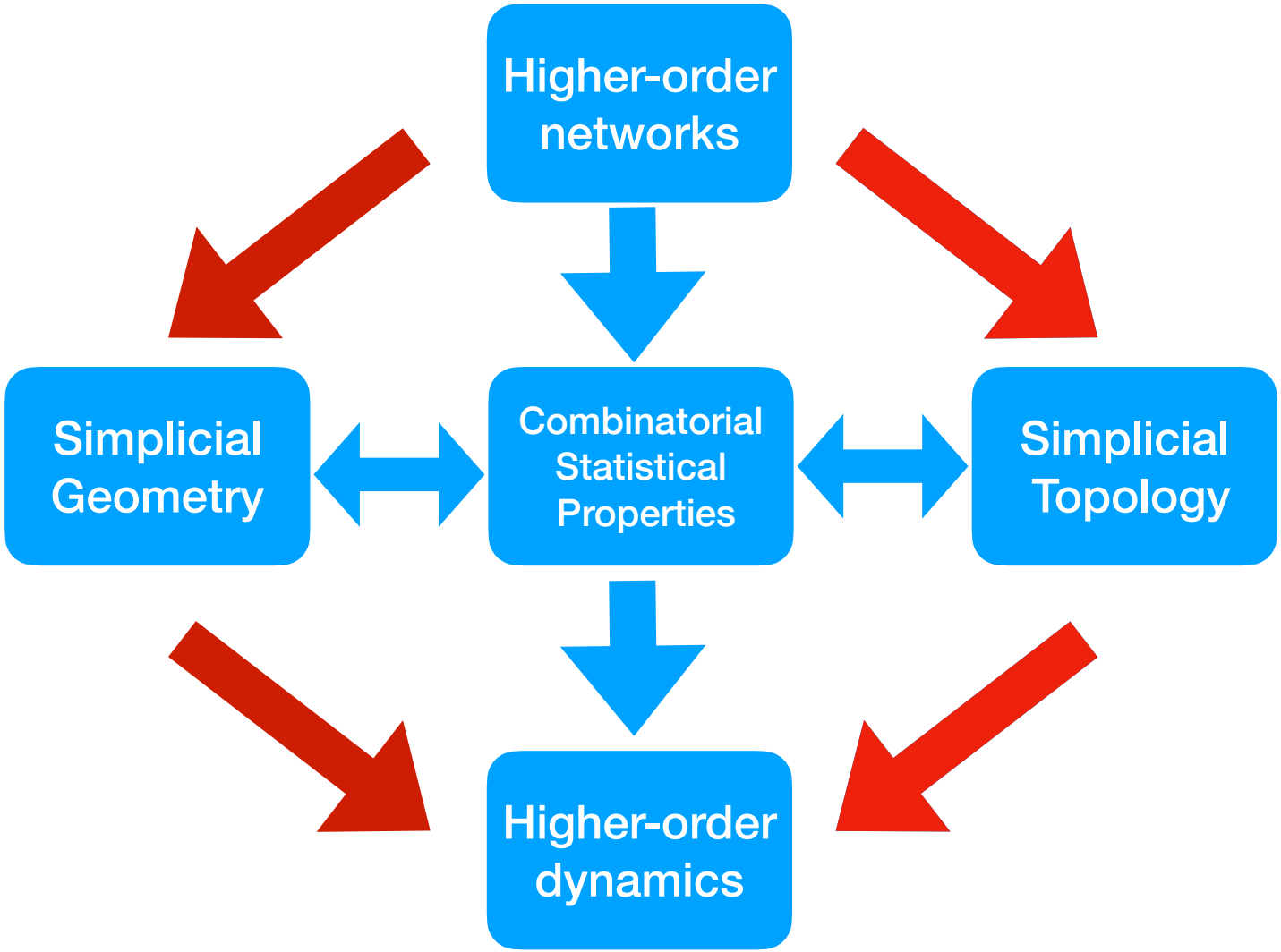
Network theory of higher-order networks

Providing a general view of the interplay between topology and dynamics

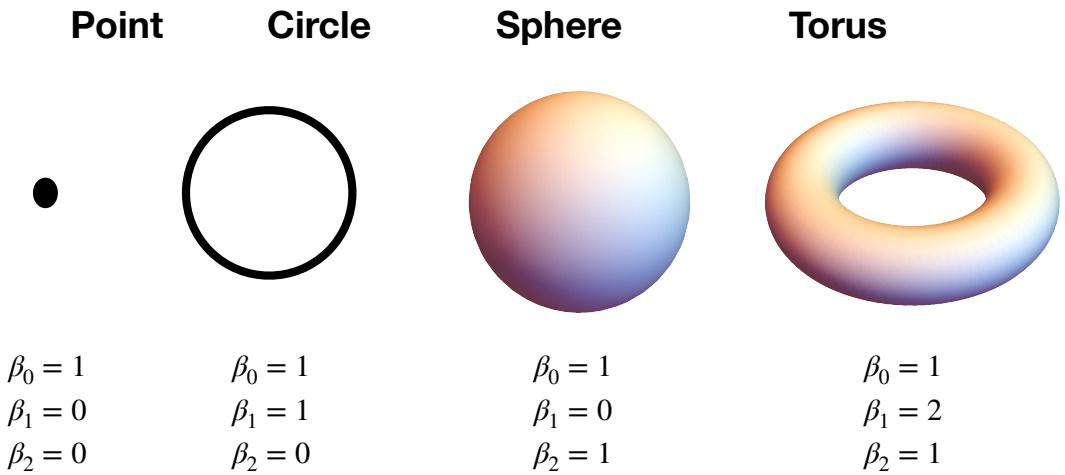
Higher-order structure and dynamics

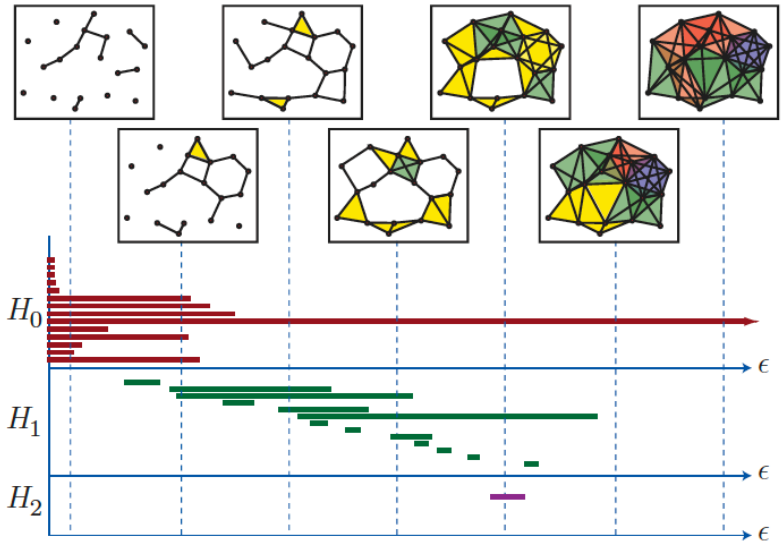


Higher-order structure and dynamics



What Is Topology?



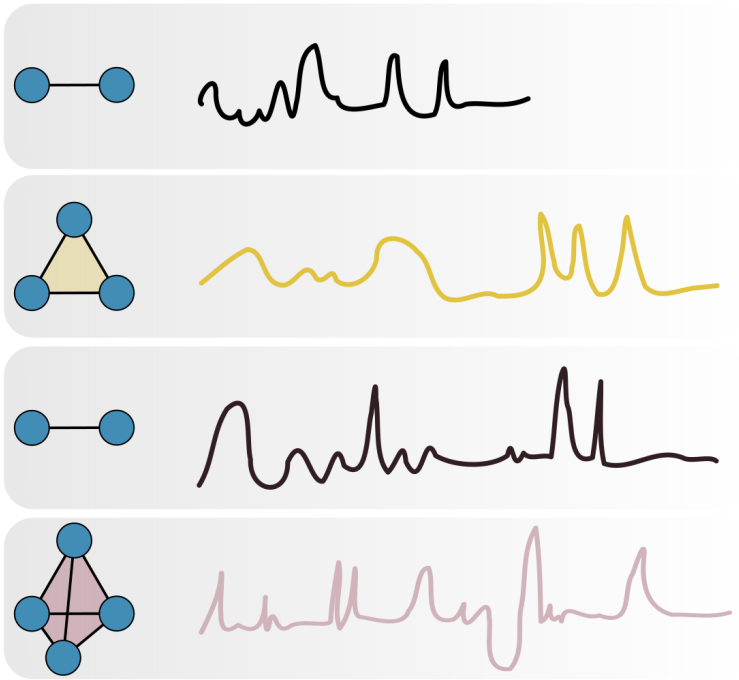


Ghrist 2008

Topological signals

Topological signals are not only defined on nodes but also on links, triangles and higher-order simplices

- Synaptic signal
- Edge signals in the brain
- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks

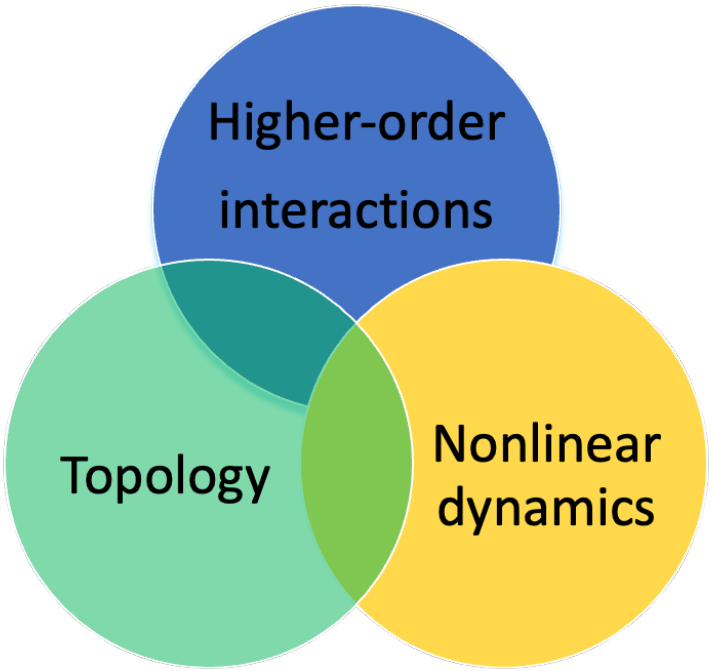


Battiston et al. Nature Physics 2021

Can we learn the
dynamics from the
complex system topology?

Can we learn the topology from the complex system dynamics?

Complexity challenge



Outline of the course:

1. Introduction to Homology and Cohomology
2. Hodge Laplacian and Introduction to Synchronization
3. Topological Kuramoto model
4. Weighted homology and Global topological synchronization
5. Dirac operator and dynamics

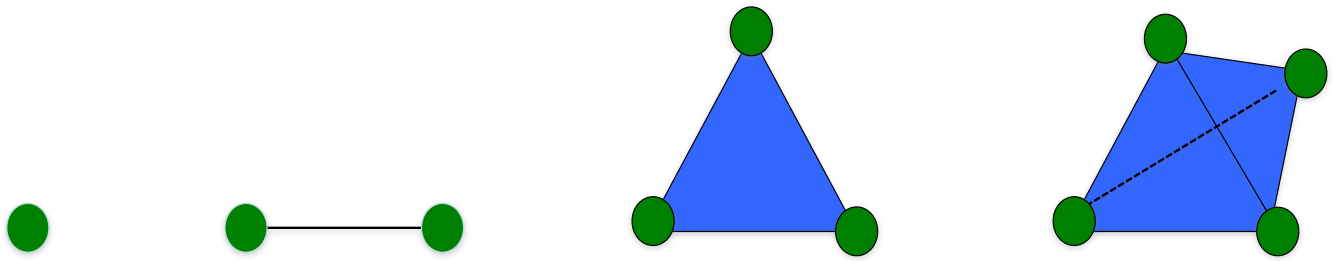
Lesson I:
Introduction to Algebraic Topology

- Introduction to simplicial complexes
- Introduction to algebraic topology
- Higher-order operators and their properties

1. Topological signals
2. Homology and boundary operators
3. Cohomology and coboundary operators

Introduction to Simplicial complexes

Simplices



0-simplex 1-simplex 2-simplex 3-simplex

Simplices

A d-dimensional simplex α (also indicated as a d-simplex α) is formed by a set of $(d+1)$ interacting nodes

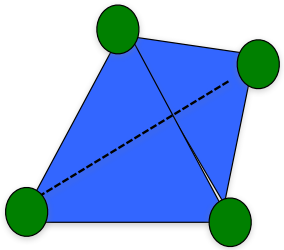
$$\alpha = [v_0, v_1, v_2 \ldots, v_d].$$

It describes a many body interaction between the nodes.
It allows for a topological and a geometrical interpretation of the simplex.

Faces of a simplex

FACES

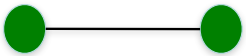
A face of a d-dimensional simplex α is a simplex α' formed by a proper subset of nodes of the simplex, i.e. $\alpha' \subset \alpha$.



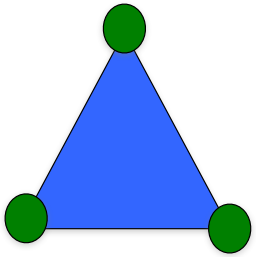
3-simplex

Faces

4 0-simplices



6 1-simplices



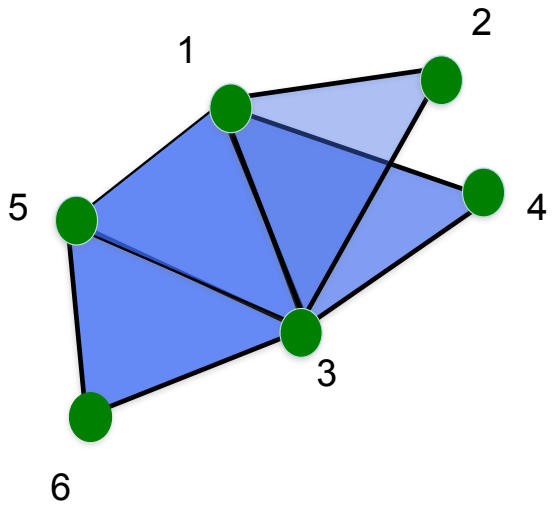
4 2-simplices

Simplicial complex

SIMPLICIAL COMPLEX

A simplicial complex $\mathcal{K}$ is formed by a set of simplices that is closed under the inclusion of the faces of each simplex.

The dimension d of a simplicial complex is the largest dimension of its simplices.

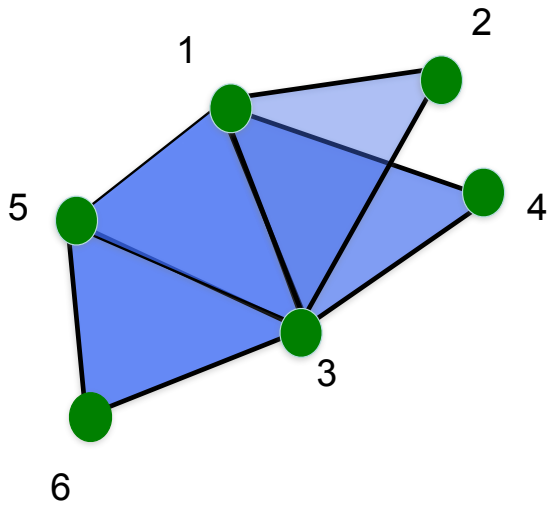


If a simplex α belongs to the simplicial complex $\mathcal{K}$ then every face of α must also belong to $\mathcal{K}$

$\mathcal{K} = \{[1], [2], [3], [4], [5], [6],$
$[1,2], [1,3], [1,4], [1,5], [2,3],$
$[3,4], [3,5], [3,6], [5,6],$
$[1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$

Dimension of a simplicial complex

The dimension of a simplicial complex $\mathscr{K}$
is the largest dimension of its simplices



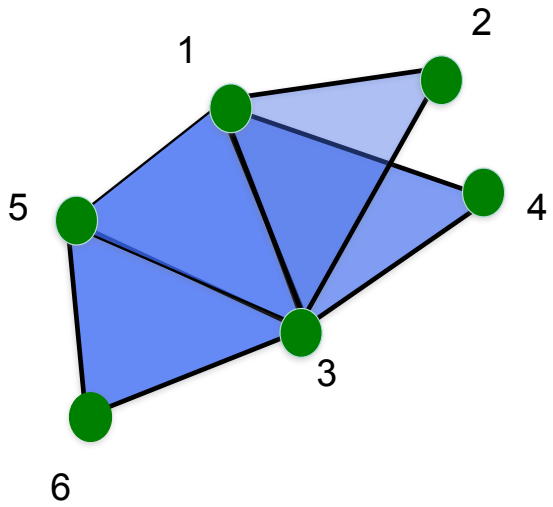
This simplicial complex has dimension 2

$$\mathscr{K} = \{[1],[2],[3],[4],[5],[6],$$
$$[1,2],[1,3],[1,4],[1,5],[2,3],$$
$$[3,4],[3,5],[3,6],[5,6],$$
$$[1,2,3],[1,3,4],[1,3,5],[3,5,6]\}$$

Facets of a simplicial complex

FACET

A facet is a simplex of a simplicial complex that is not a face of any other simplex. Therefore a simplicial complex is fully determined by the sequence of its facets.



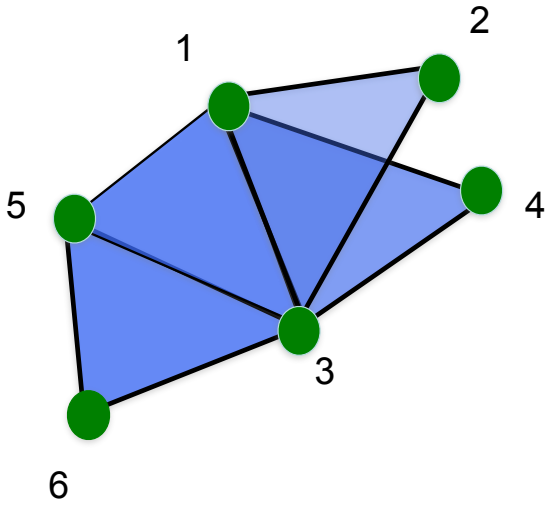
The facets of this simplicial complex are

$$\mathscr{K} = \{[1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Pure simplicial complex

PURE SIMPLICIAL COMPLEXES

A pure d-dimensional simplicial complex is formed by a set of *d*-dimensional simplices and their faces.
Therefore pure *d*-dimensional simplicial complexes admit as facets only *d*-dimensional simplices.



A pure d-dimensional simplicial complex is fully determined by an adjacency matrix tensor with (d+1) indices. For instance this simplicial complex is determined by the tensor

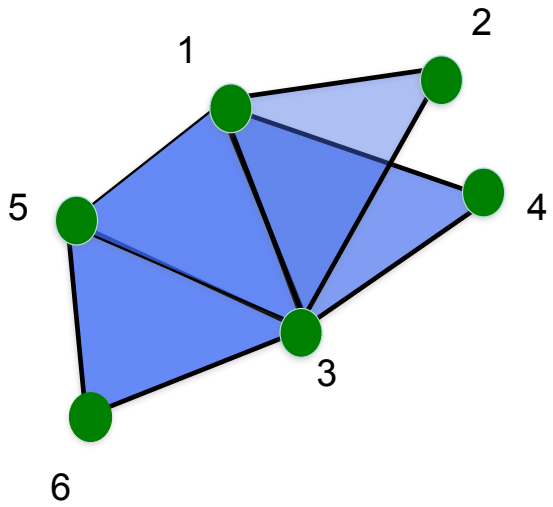
$$a_{rsp} = \begin{cases} 1 \text{ if } (r,s,p) \in \mathscr{K} \\ 0 \text{ otherwise} \end{cases}$$

Example

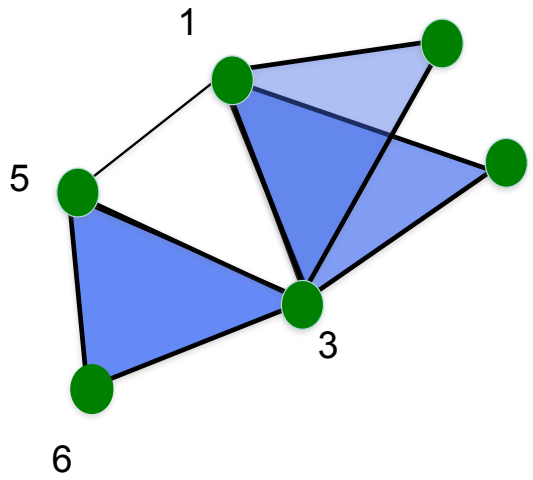
A simplicial complex $\mathscr{K}$ is pure

if it is formed by d-dimensional simplices

and their faces



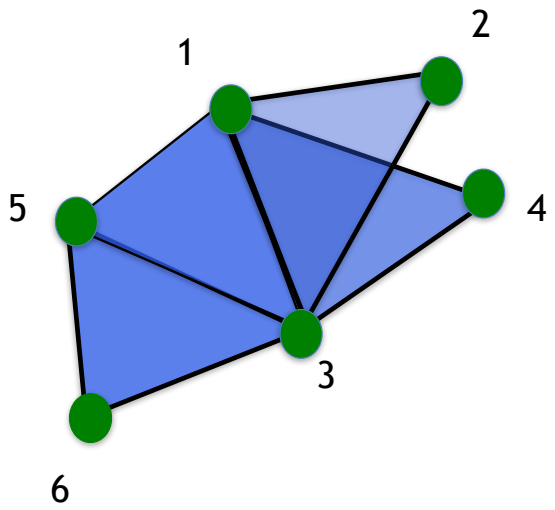
PURE SIMPLICIAL COMPLEX



SIMPLICIAL COMPLEX THAT IS NOT PURE

Generalized degree

The generalized degree $k_{m',m}(\alpha)$ of a m-face α
is given by the number
of m'-dimensional simplices incident to the m-face α.

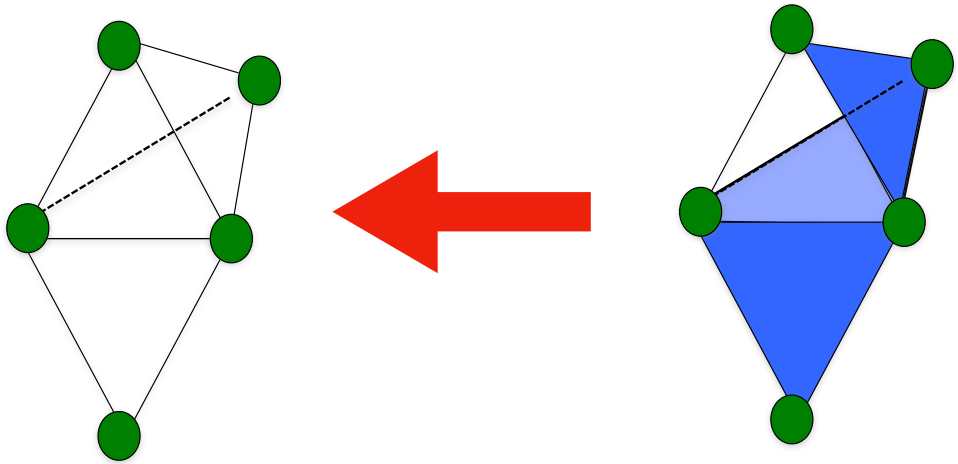


r	$k_{2,0}([r])$
[1]	3
[2]	1
[3]	4
[4]	1
[5]	2
[6]	1

$[r,s]$	$k_{2,1}([r,s])$
[1,2]	1
[1,3]	3
[1,4]	1
[1,5]	1
[2,3]	1
[3,4]	1
[3,5]	2
[3,6]	1
[5,6]	1

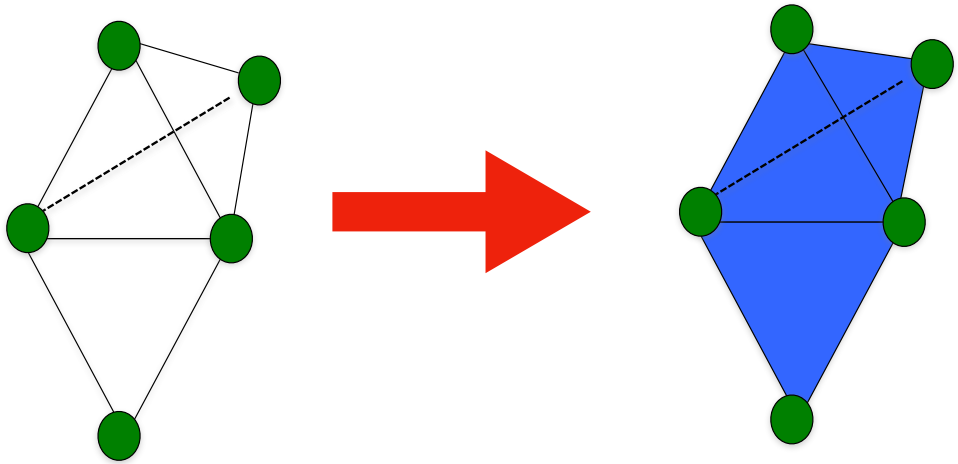
[Bianconi & Rahmede 2015, Courtney & Bianconi 2016]

Simplicial complex skeleton



From a simplicial complex is possible to generate a network salled the simplicial complex skeleton by considering only the nodes and the links of the simplicial complex

Clique complex



From a network is possible to generate a simplicial complex by Assuming that each clique is a simplex

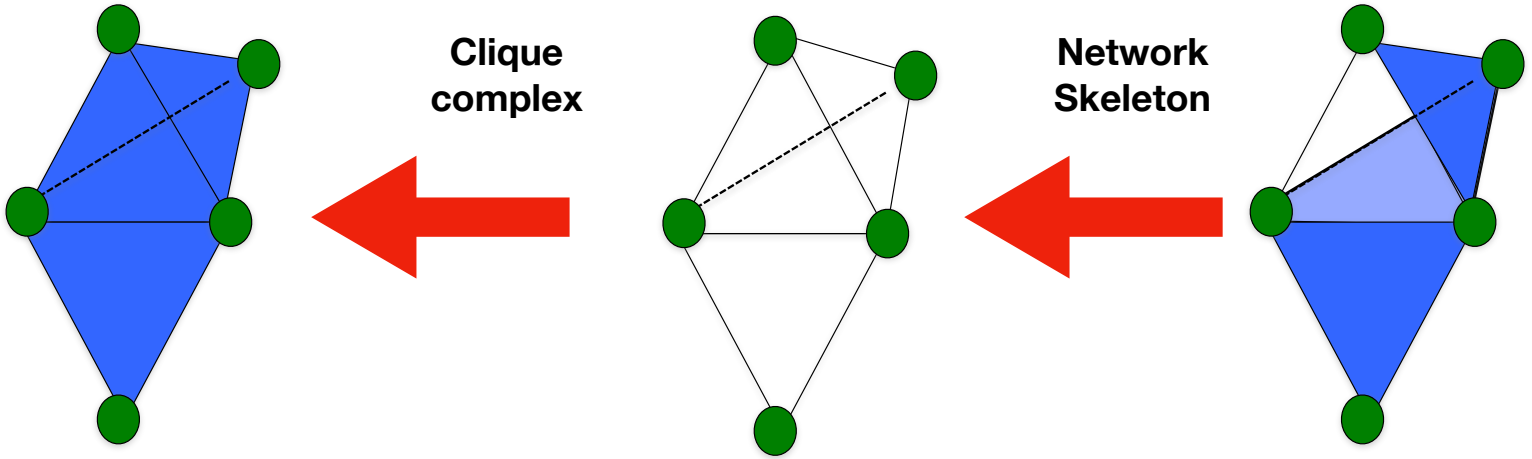
Note:

Poisson networks have a clique number that is 3 and actually on a finite expected number of triangles in the infinite network limit

However

Scale-free networks have a diverging clique number, therefore the clique complex of a scale-free network has diverging dimension. (Bianconi,Marsili 2006)

Concatenation of the operations



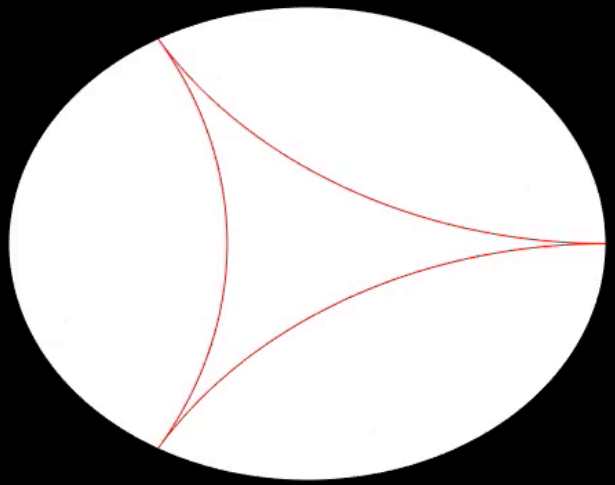
Attention!

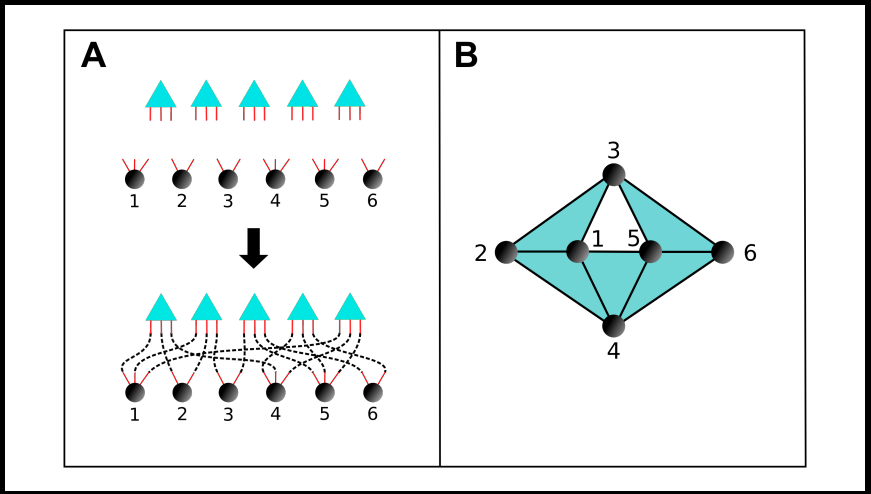
By concatenating the operations you are not guaranteed to return to the initial simplicial complex

Simplicial complex models

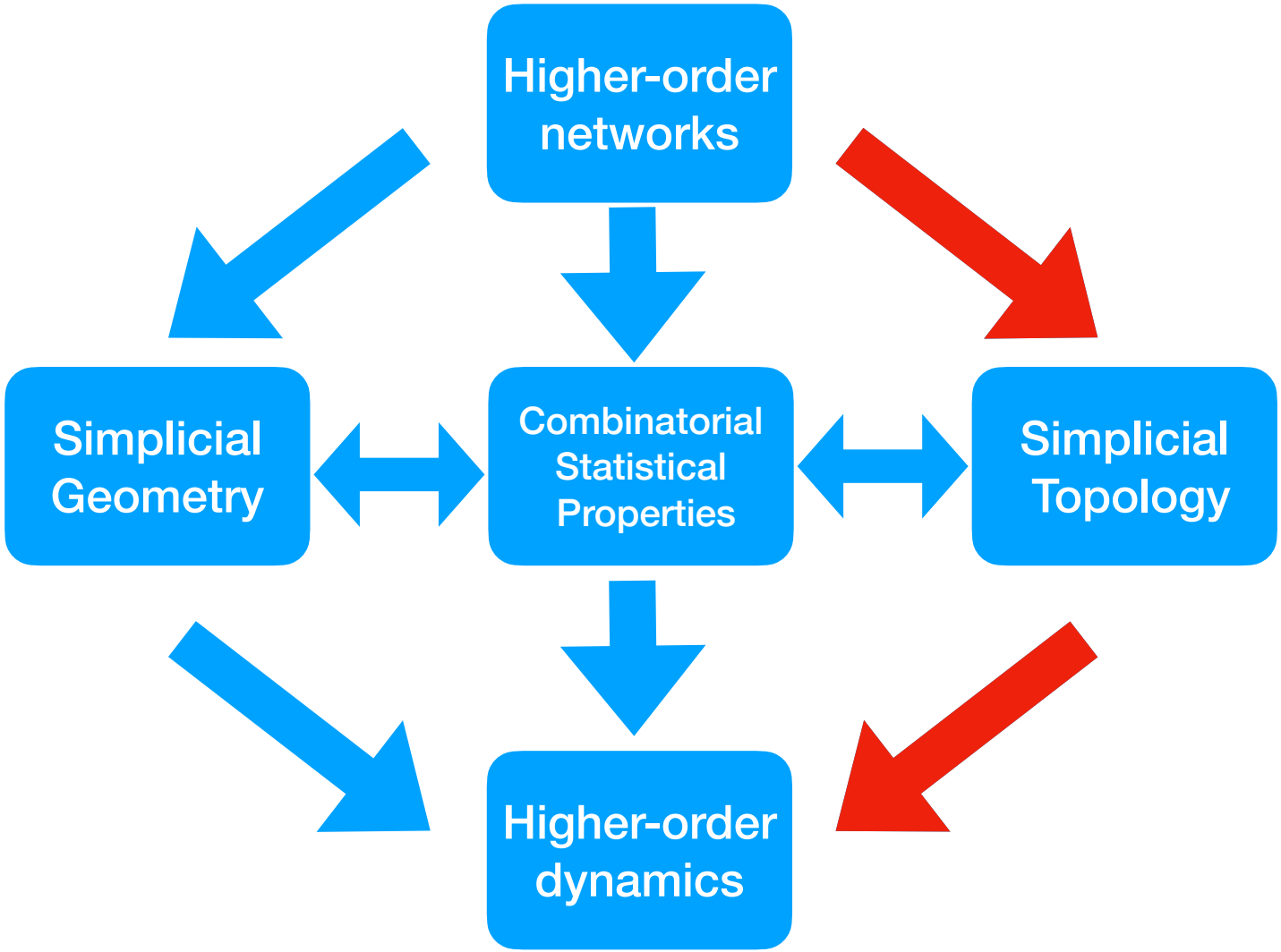
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]





Higher-order structure and dynamics



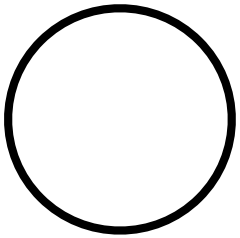
Introduction to Algebraic Topology

Betti numbers

Point

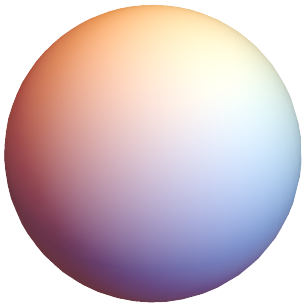
$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 0$

Circle



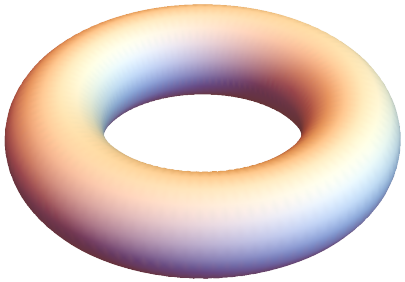
$\beta_0 = 1$
$\beta_1 = 1$
$\beta_2 = 0$

Sphere



$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 1$

Torus

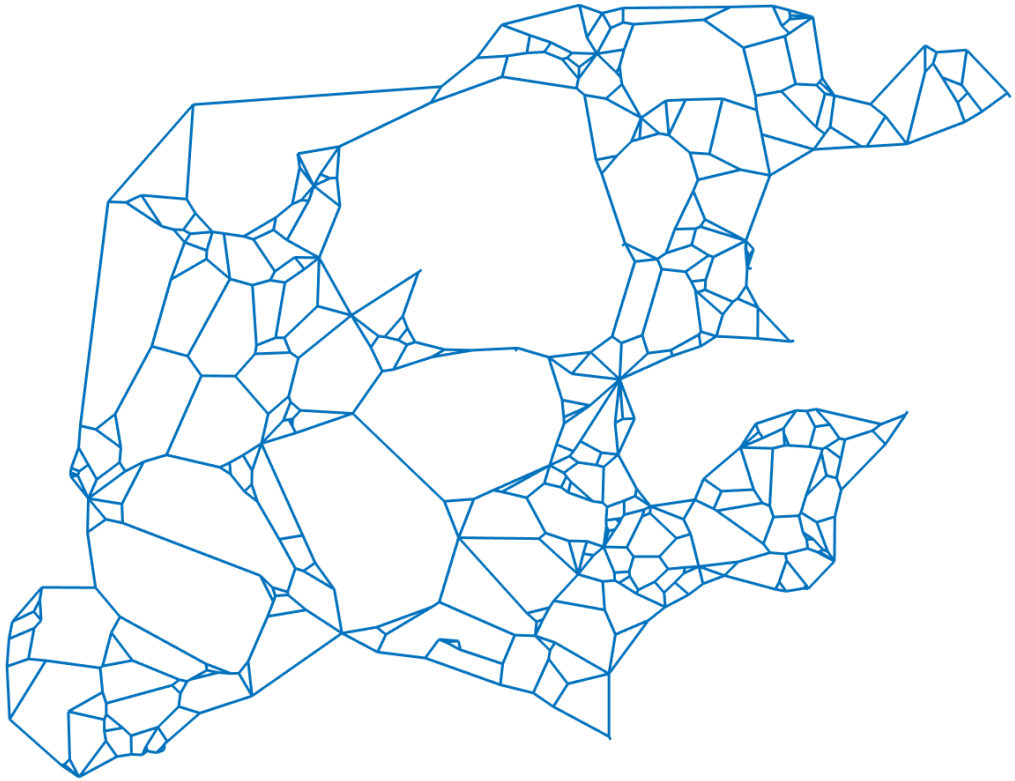


$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$

Euler characteristic

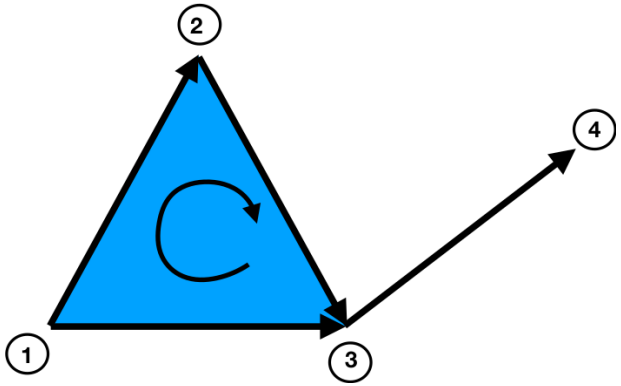
$$\chi = \sum_{m=0}^{d} (-1)^m \beta_m$$

Betti number 1



Fungi network from Sang Hoon Lee, et. al. Jour. Compl. Net. (2016)

Take Home Message: Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{c} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} \begin{array}{c} [1,2,3] \end{array} \\ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

Simplicial complex:notation

We consider a d-dimensional simplicial complex $\mathscr{K}$ having N_m

positively oriented simplices α_r^m

(or simply r) of dimension m.

We indicate the set of all the m positively oriented simplices of the simplicial complex $Q_m(\mathscr{K})$

Orientation of a simplex

A m-dimensional *oriented simplex* α is a set of $m+1$ nodes

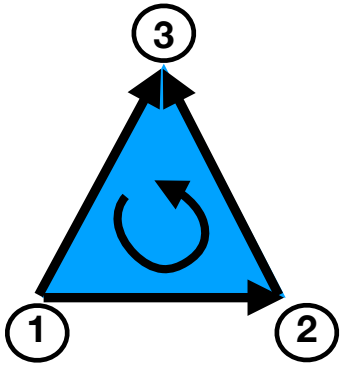
$$\alpha = [v_0, v_1, \ldots, v_m], \tag{3.1}$$

associated to an orientation wuch that

$$[v_0, v_1, \ldots, v_m] = (-1)^{\sigma(\pi)}[v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(m)}] \tag{3.2}$$

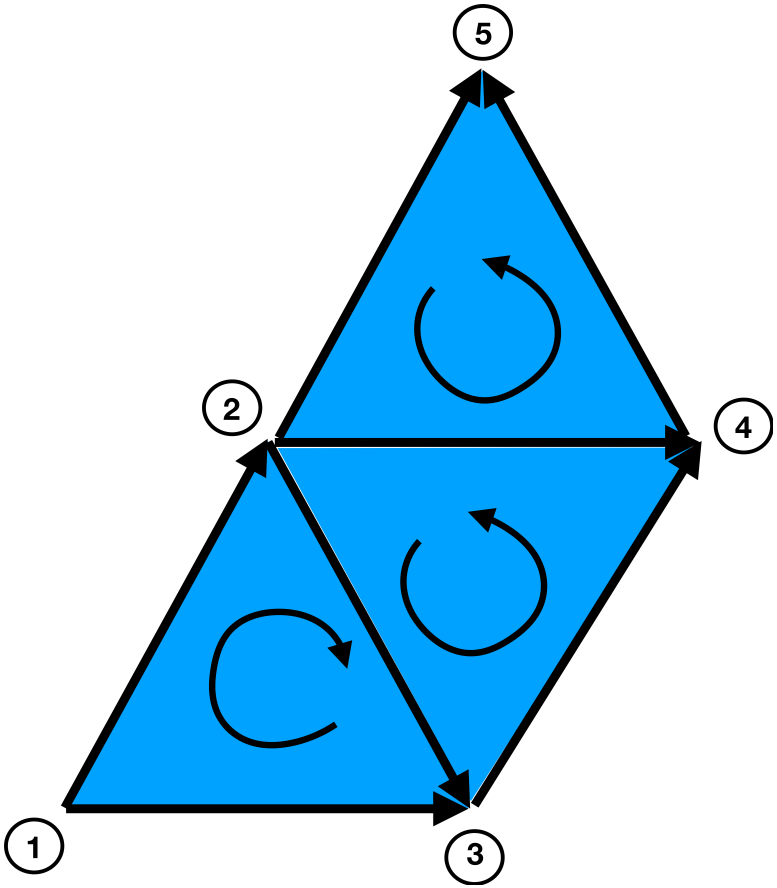
where $\sigma(\pi)$ indicates the parity of the permutation π.

$$[r, s] = -[s, r]$$



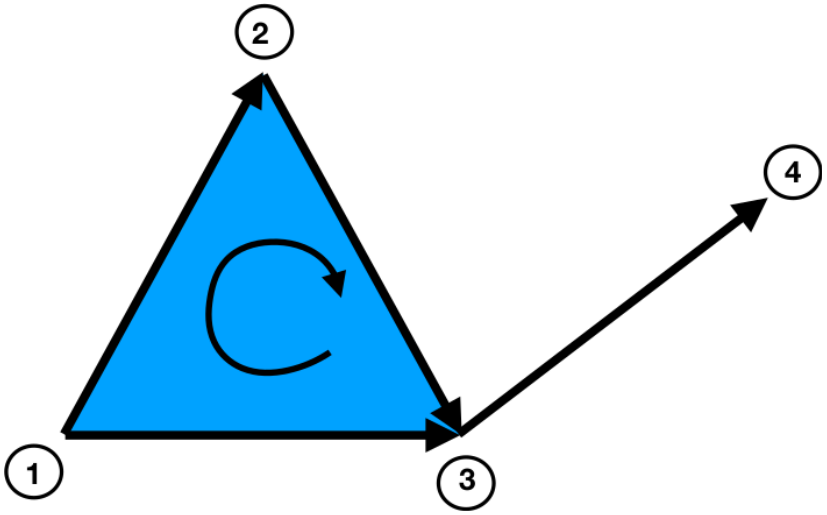
$$[r, s, q] = [s, q, r] = [q, r, s] = -[s, r, q] = -[q, s, r] = -[r, q, s]$$

Oriented simplicial complex



A typical choice of orientation of a simplicial complex, is to consider the orientation induced by the node labels, i.e. each simplex is oriented in an increasing (or decreasing) order of the node labels

Oriented simplicial complex



The set of positively oriented simplices on this simplicial complex are:

{[1,2,3], [1,2], [2,3], [1,3], [3,4], [1], [2], [3], [4]}

We adopt the convention that each 0-simplex is positively oriented

m-Chains

THE m-CHAINS

Given a simplicial complex, a m-chain C_m consists of the elements of a free abelian group with basis on the m-simplices of the simplicial complex.

Its elements can be represented as linear combinations of the of all oriented m-simplices

$$\alpha = [v_0, v_1, \ldots, v_m] \tag{3.6}$$

with coefficients in $\mathbb{Z}$.

m-chain $c_m \in C_m$

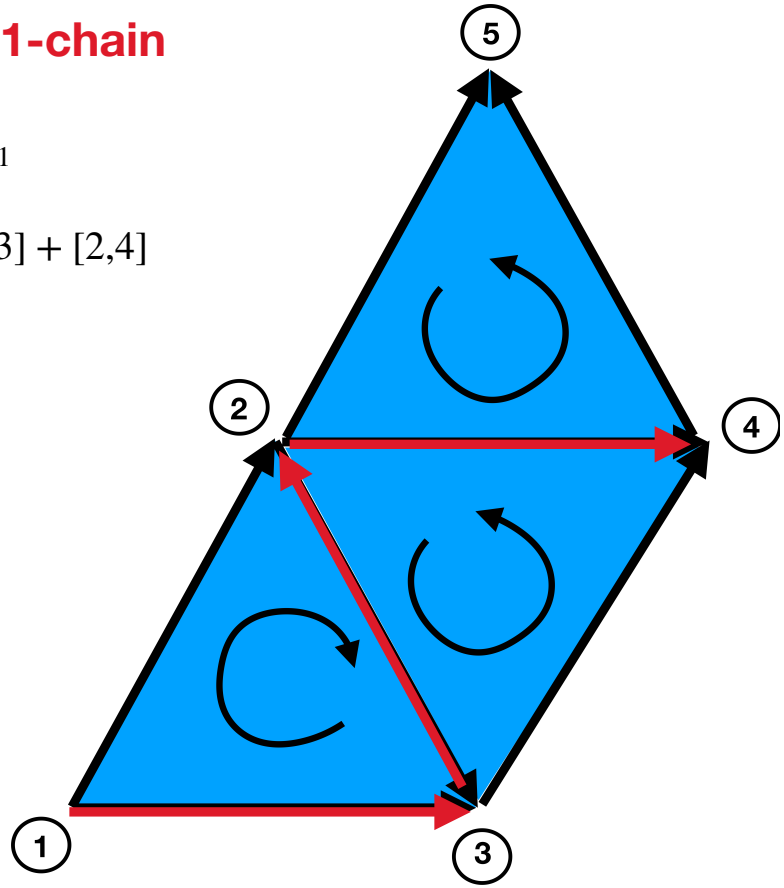
$$c_m = \sum_{\alpha_r \in Q_m(\mathcal{K})} c_m^r \alpha_r^m, \ \textbf{with } c_m^r \in \mathbb{Z}$$

Oriented simplicial complex and m-chains

Example of 1-chain

$$a \in \mathscr{C}_1$$

$$a = [1,3] - [2,3] + [2,4]$$



Boundary operator

THE BOUNDARY MAP

The boundary map ∂_m is a linear operator

$$\partial_m : C_m \to C_{m-1} \tag{3.8}$$

whose action is determined by the action on each m-simplex of the simplicial complex is given by

$$\partial_m [v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m]. \tag{3.9}$$

Boundary operator

The boundary map ∂_n is a linear operator

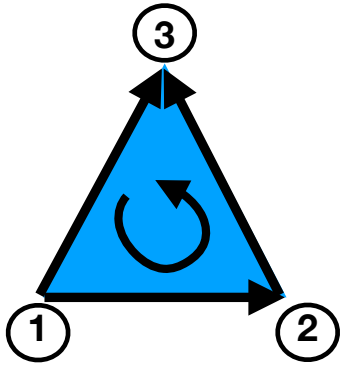
$$\partial_m : \mathscr{C}_m \to \mathscr{C}_{m-1}$$

whose action is determined by the action on each n-simplex of the simplicial complex

$$\partial_m[v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m] .$$

Therefore we have

$$\partial_1[1,2] = [2] - [1] .$$



$$\partial_2[1,2,3] = [2,3] - [1,3] + [1,2] .$$

Boundary operator

THE BOUNDARY MAP

The boundary map ∂_m is a linear operator

$$\partial_m : C_m \rightarrow C_{m-1} \tag{3.8}$$

whose action is determined by the action on each m-simplex of the simplicial complex is given by

$$\partial_m [v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m]. \tag{3.9}$$

From this definition it follows that the $\text{im}(\partial_m)$ corresponds to the space of $(m-1)$ boundaries and the $\ker(\partial_m)$ is formed by the cyclic m-chains.

Special groups

Boundary group $\hat{B}_m = \text{im}(\partial_{m+1})$
Cycle group $\hat{Z}_m = \ker(\partial_m)$

The boundary of a boundary is null

The boundary operator has the property

$$\partial_m \partial_{m+1} = 0 \quad \forall m \geq 1$$

Which is usually indicated by saying that the boundary of the boundary is null.

This property follows directly from the definition of the boundary, as an example we have

$$\partial_1 \partial_2 [r, s, q] = \partial_1([r, s] + [s, q] - [r, q]) = [s] - [r] + [q] - [s] - [q] + [r] = 0.$$

Proof

The boundary of the boundary is null.

Proof: Indicating with $\hat{v}_p$ the p$^{\text{th}}$ missing vertex we have

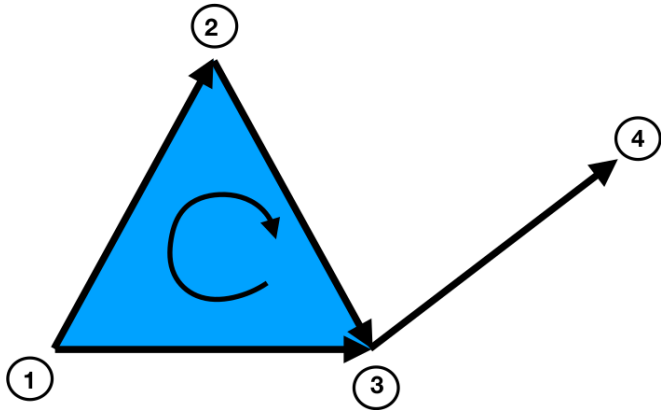
$$\begin{aligned}\partial_{m-1}\partial_m[v_0, v_1, \ldots, v_m] &= \sum_{p=0}^{m}(-1)^p\partial_{m-1}[v_0, v_1, \ldots\hat{v}_p\ldots v_m] \\ &= \sum_{p=0}^{m}(-1)^p\sum_{p'=0}^{p-1}(-1)^{p'}[v_0, v_1, \ldots\hat{v}_{p'}\ldots\hat{v}_p\ldots v_m] \\ &+\sum_{p=0}^{m}(-1)^p\sum_{p'=p+1}^{m}(-1)^{p'-1}[v_0, v_1, \ldots\hat{v}_p\ldots\hat{v}_{p'}\ldots v_m] = 0\end{aligned}$$

Incidence matrices

Given a basis for the m simplices and m-1 simplices the m-boundary operator

$$\partial_m[v_0, v_1 \ldots, v_m] = \sum_{p=0}^{m} (-1)^p [v_0, v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_m] \,.$$

is captured by the $N_{m-1} \times N_m$ incidence (or boundary) matrix $\mathbf{B}_{[m]}$



$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{c} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

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Boundary of the boundary is null

In terms of the incidence matrices the relation

$$\partial_m \partial_{m+1} = 0 \quad \forall m \geq 1$$

Can be expressed as

$$\mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0} \quad \forall m \geq 1 \qquad \mathbf{B}^{\top}_{[m+1]}\mathbf{B}^{\top}_{[m]} = \mathbf{0} \quad \forall m \geq 1$$

Homology groups

THE HOMOLOGY GROUPS

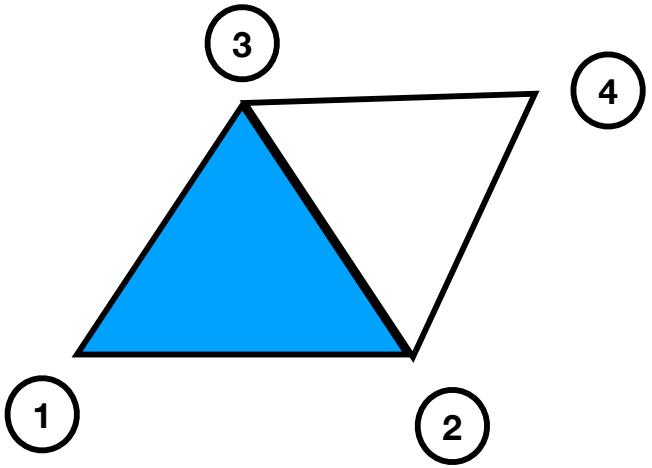
The homology group $\mathcal{H}_m$ is the quotient space

$$\mathcal{H}_m = \frac{\ker(\partial_m)}{\mathrm{im}(\partial_{m+1})}, \tag{3.14}$$

denoting homology classes of m-cyclic chains that are in the $\ker(\partial_m)$ and they do differ by cyclic chains that are not boundaries of $(m+1)$-chains, i.e. they are in $\mathrm{im}(\partial_{m+1})$.

It follows that $a \in \ker(\partial_m)$ is in the same homology class than $a + b \in \ker(\partial_m)$ with $b \in \mathrm{im}(\partial_{m+1})$

Homology



$$\mathscr{H}_1 = \mathbb{Z}$$

The two 1-chains

$$a = [2,4] - [3,4] - [2,3]$$

$$b = [1,2] + [2,4] - [3,4] - [1,3]$$

are in the same homology class

$$a \sim b$$

in fact

$$b = a + \partial_2[1,2,3] = [2,4] - [3,4] - [2,3] + [1,2] + [2,3] - [1,3]$$

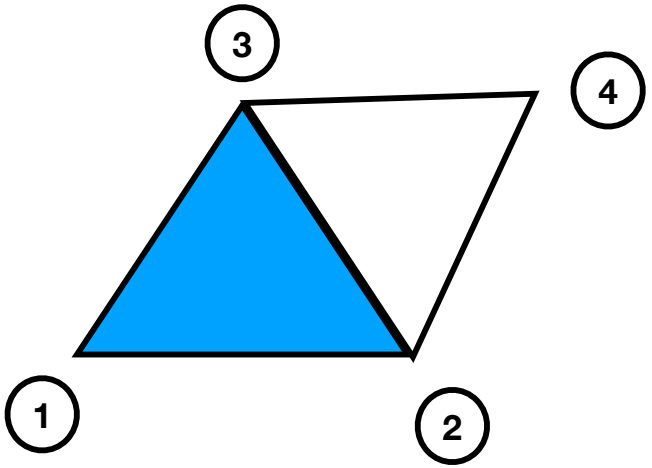
Betti numbers

BETTI NUMBERS

The Betti number β_m indicates the number of m-dimensional cavities of a simplicial complex and is given by the rank of the homology group $\mathcal{H}_m$, i.e.

$$\beta_m = \text{rank}(\mathcal{H}_m) = \text{rank}(\ker(\partial_m)) - \text{rank}(\text{im}(\partial_{m+1})). \tag{3.15}$$

Betti number



$$\mathscr{H}_1 = \mathbb{Z}$$

$$\beta_1 = \mathbf{dim}\mathscr{H}_1 = 1$$

The two 1-chains

$$a = [2,4] - [3,4] - [2,3]$$

$$b = [1,2] + [2,4] - [3,4] - [1,3]$$

are in the same homology class

$$a \sim b$$

in fact

$$b = a + \partial_2[1,2,3] = [2,4] - [3,4] - [2,3] + [1,2] + [2,3] - [1,3]$$

Euler characteristic

THE EULER CHARACTERISTIC AND THE EULER-POINCARÉ FORMULA

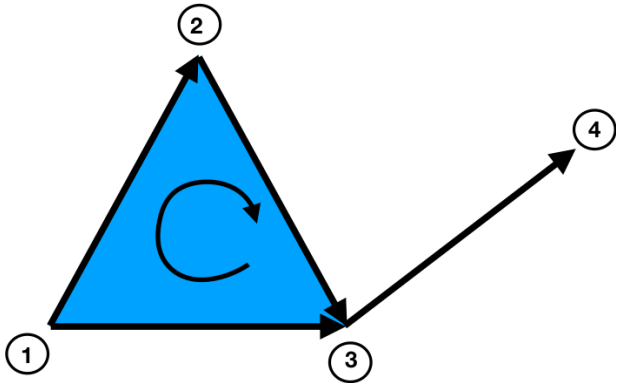
The Euler characterisic χ is defined as the alternating sum of the number of m-dimensional simplices, i.e.

$$\chi = \sum_{m \geq 0} s_m, \tag{3.16}$$

where s_m is the number of m-dimensional simplices in the simplicial complex. According to the Euler-Poincaré formula, the Euler characteristic χ of a simplicial complex can be expressed in terms of the Betti numbers as

$$\chi = \sum_{m \geq 0} (-1)^m \beta_m. \tag{3.17}$$

Boundary Operators



Boundary operators

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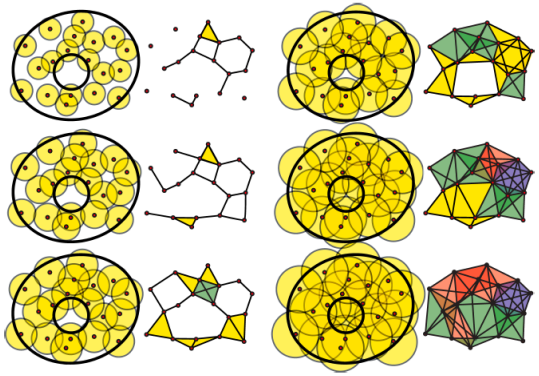
$$\mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

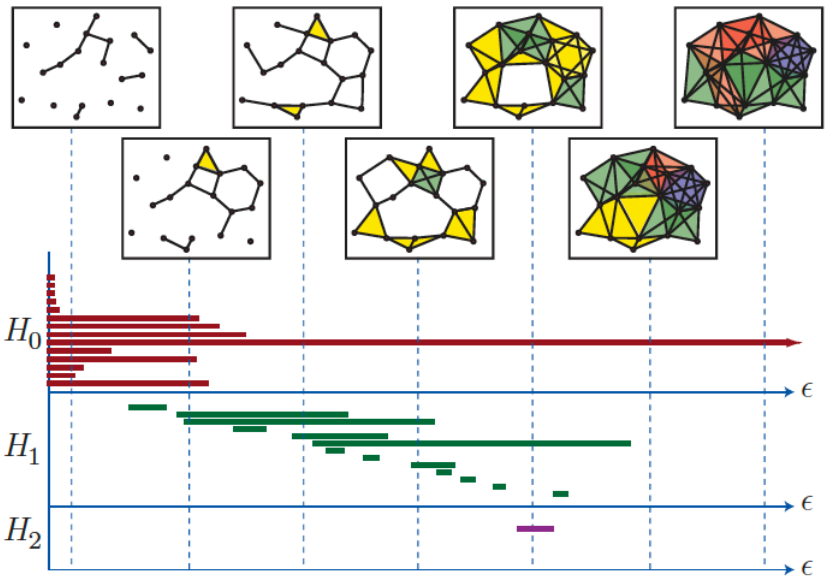
Persistent homology

Filtration: distance/weights



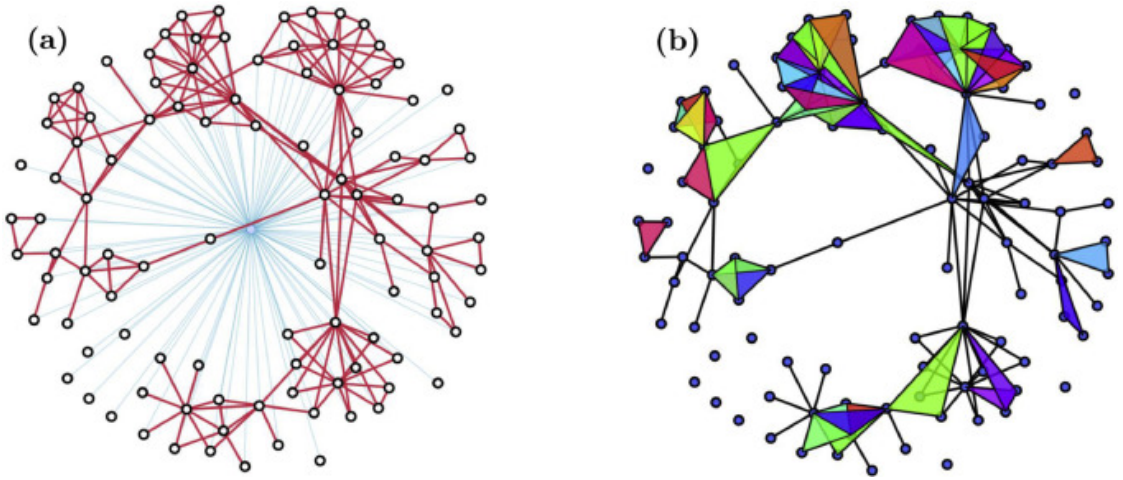
Ghrist 2008

Persistent homology Barcode



Topological clustering

The node neighbourhood is the clique simplicial complex formed by the set of all the neighbours of a node and their connections



Properties of the node

The degree k_r

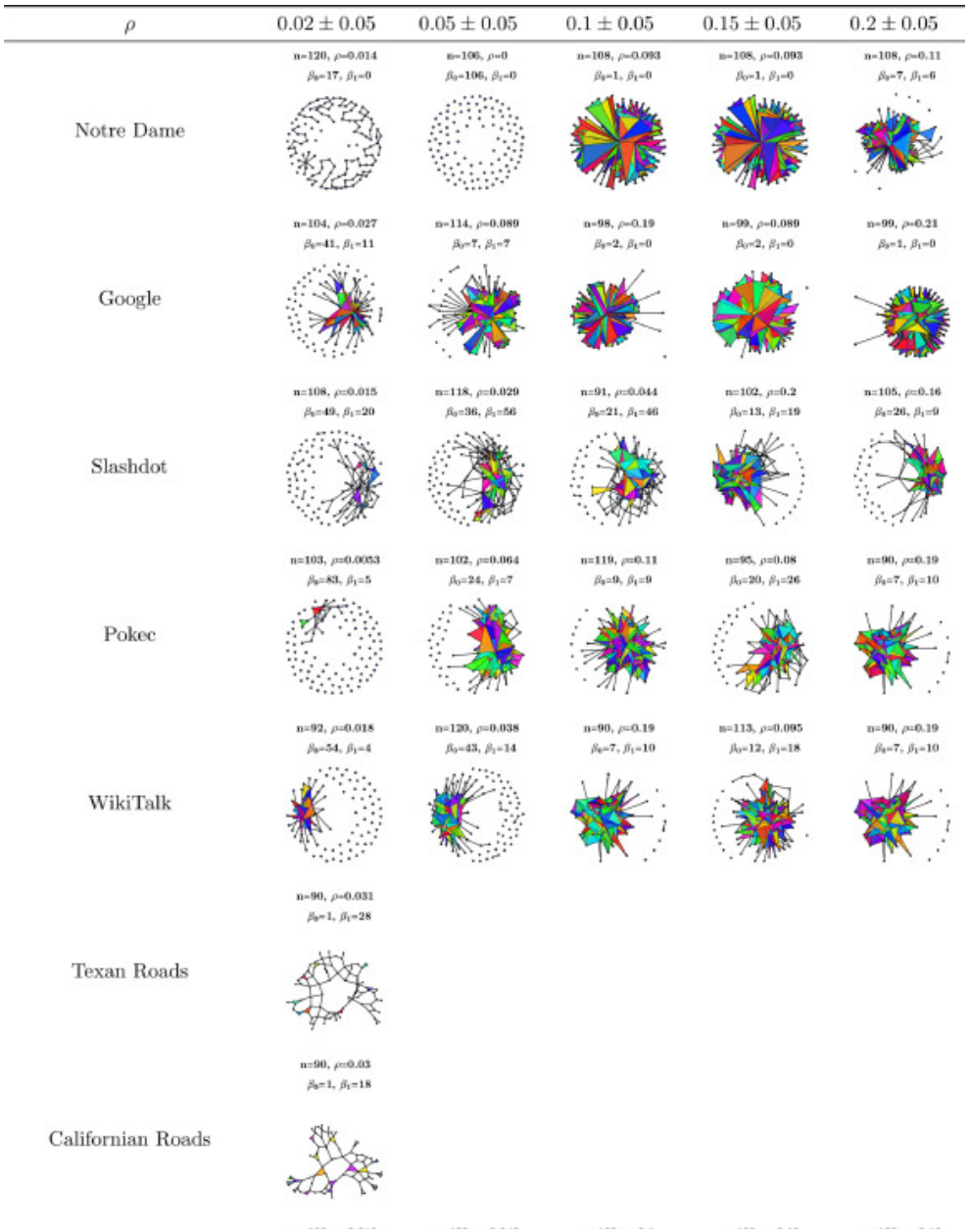
The local clustering coefficient C_r

AP Kartun-Giles et al. (2019)

Properties of the node neighbourhood

Number of nodes n

Density of the links ρ



Node neighbourhoods with the same number of nodes and the same density of links can have very different topology

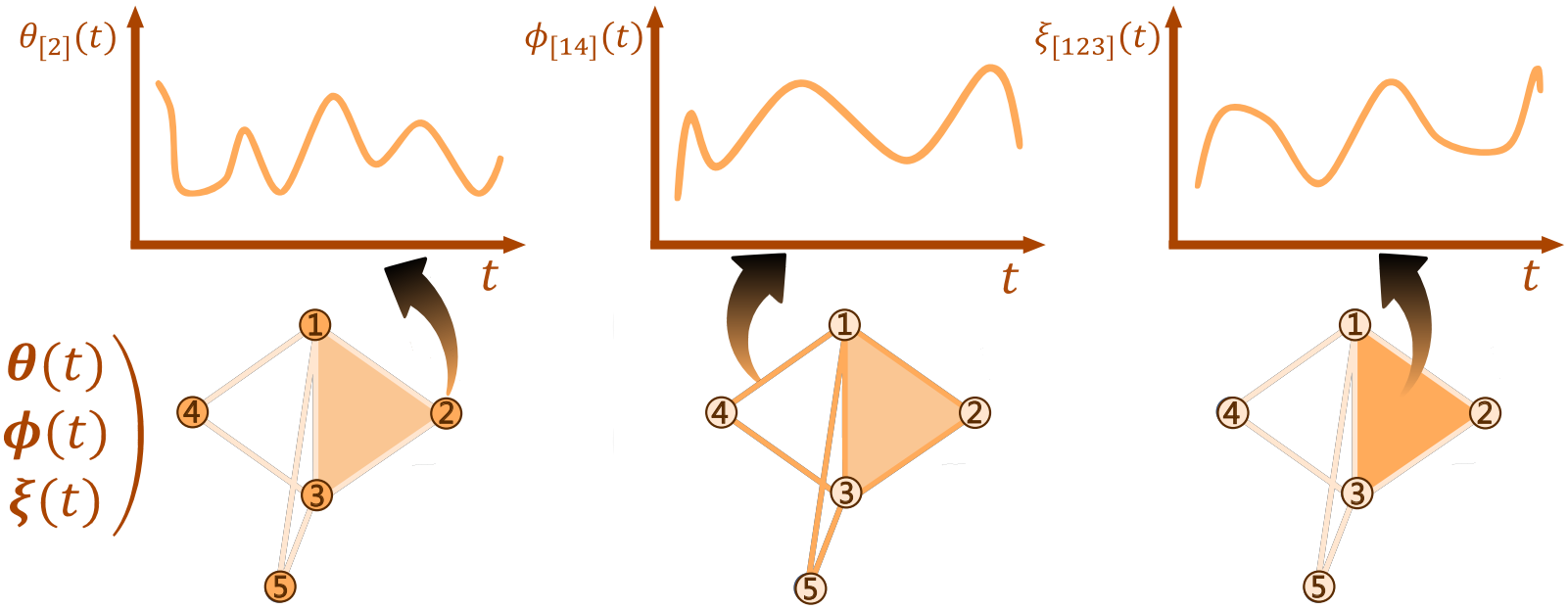
AP Kartun-Giles et al. (2019)

Topological signals, coboundary operators

Topological signals

Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals

$$\Psi(t) = \begin{pmatrix} \boldsymbol{\theta}(t) \\ \boldsymbol{\phi}(t) \\ \boldsymbol{\xi}(t) \end{pmatrix}$$

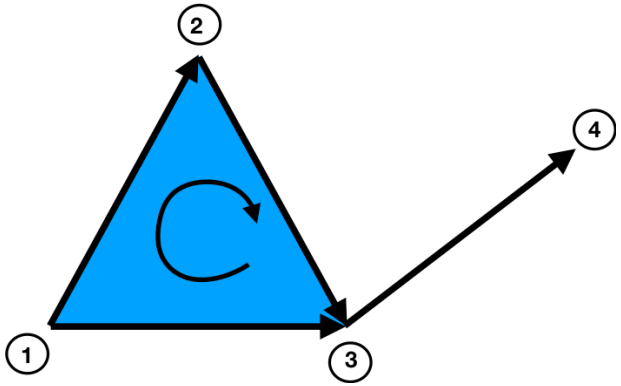


Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are cochains or vector fields

Boundary Operators



Boundary operators

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$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

Cochains

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \rightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

m-cochain $f \in C^m$

Given the m-chain $c_m \in C_m$

$$c_m = \sum_{r \in Q_m(\mathcal{K})} c_m^r \alpha_r^m, \text{ with } c_m^r \in \mathbb{Z}$$

$$f(c_m) = \sum_{r \in Q_m(\mathcal{K})} c_m^r f([\alpha_r^m]), \text{ with } c_m^r \in \mathbb{Z}$$

Oriented simplicial complex and m-chains

Example of 1-chain

$$a \in \mathscr{C}_1$$

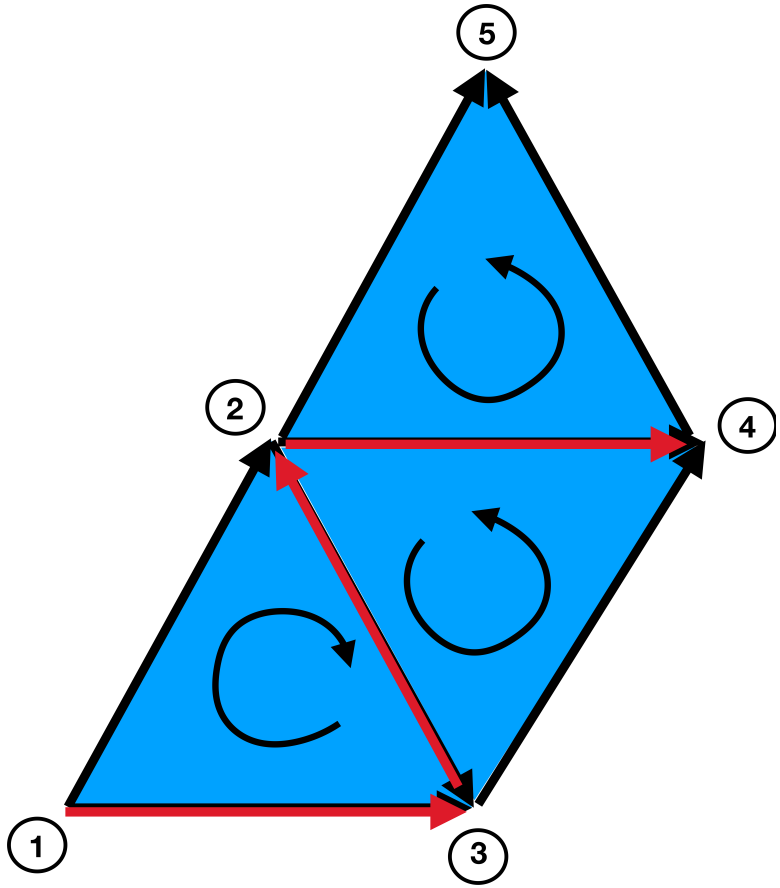
$$a = [1,3] - [2,3] + [2,4]$$

Example

Given $f \in C^1$

then

$$f(a) = f([1,3]) - f([2,3]) + f([2,4])$$



Cochains:properties

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \rightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

Upon a change of orientation of a simplex the value of the cochain associated to a simplex changes sign

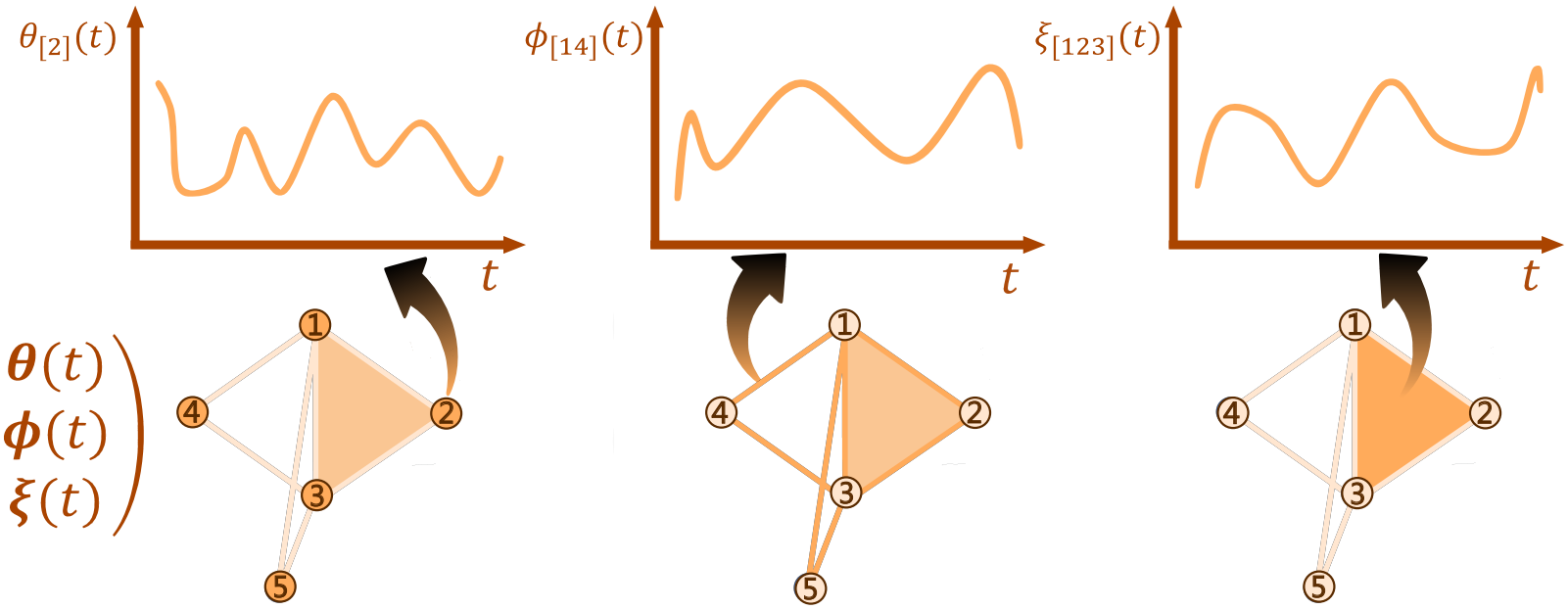
$$f([\alpha_r^m]) = -f(-[\alpha_r^m]) \; \forall \alpha_r^m \in Q_m(\mathcal{K})$$

For $m = 1$ a cochain can for instance express a flux along the links of the simplicial complex

Topological signals

Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals

$$\Psi(t) = \begin{pmatrix} \boldsymbol{\theta}(t) \\ \boldsymbol{\phi}(t) \\ \boldsymbol{\xi}(t) \end{pmatrix}$$



Cochains:properties

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \rightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

Given a basis for the m-simplices of the simplicial complex,
A m-cochain can be expressed as a vector $\mathbf{f}$ of elements

$$f_r = f([\alpha_r^m]) \ \forall \alpha_r^m \in Q_m(\mathcal{K})$$

L^2 norm between cochains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathcal{K})} f_r^2$$

This scalar product can be generalised by introducing metric matrices (see lecture III)

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{m+1}^{\top}\mathbf{f} \equiv \bar{\mathbf{B}}_{m+1}\mathbf{f}$

Coboundary operator

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The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

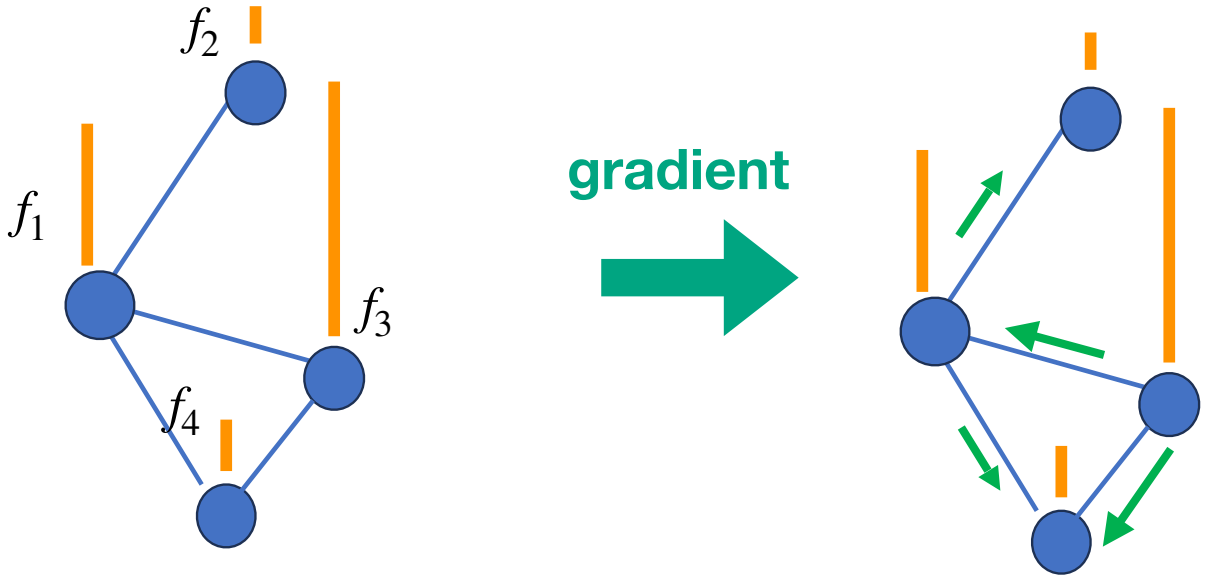
if follows that

$$\delta_{m+1} \circ \delta_m = 0 \ \forall m \geq 1 \text{ hence } \mathbf{B}_{[m+1]}^\top \mathbf{B}_{[m]}^\top = \mathbf{0}$$

Discrete Gradient

If $f \in C^0$, then $g = \delta_0 f \in C^1$ indicates its discrete gradient

Indeed we have $\mathbf{g} = \mathbf{B}_{[1]}^{\top}\mathbf{f}$ which implies $g_{[r,s]} = f_s - f_r$

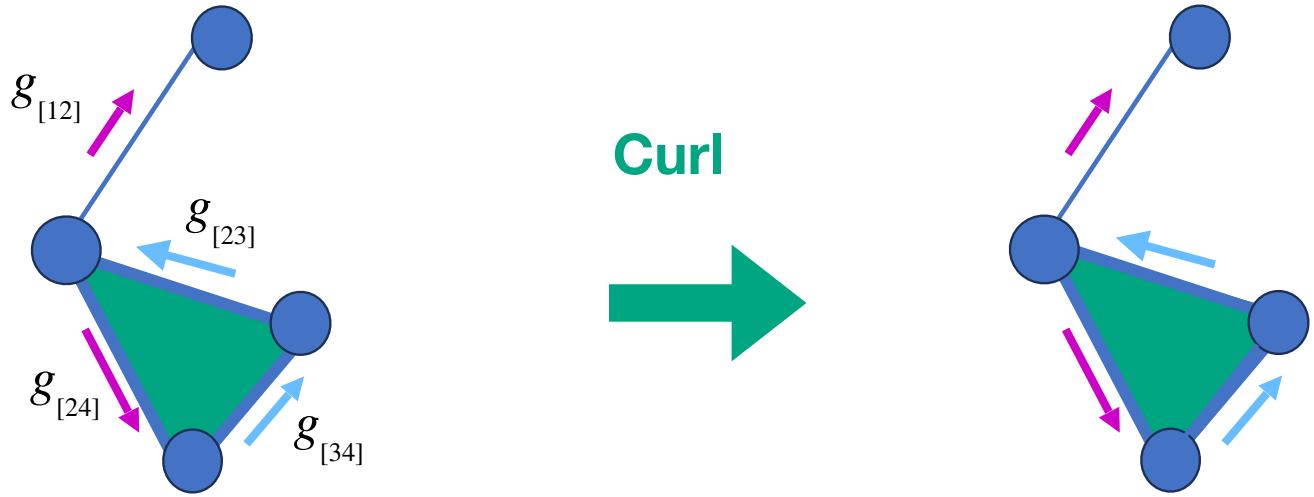


Discrete Curl

If $f \in C^1$, then $h = \delta_1 g \in C^2$ indicates its discrete curl

Indeed we have $\mathbf{g} = \mathbf{B}_{[2]}^{\top}\mathbf{f}$ which implies

$$h_{[r,s,q]} = g_{[r,s]} + g_{[s,q]} - g_{[r,q]}$$



Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \to C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

for any $f \in C^m$ and $g \in C^{m+1}$.

It follows that if $f' = \delta_m^* g$ then $\mathbf{f}' = \mathbf{B}_{[m+1]}\mathbf{g}$

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where $f \in C^m$ and $g \in C^{m+1}$.

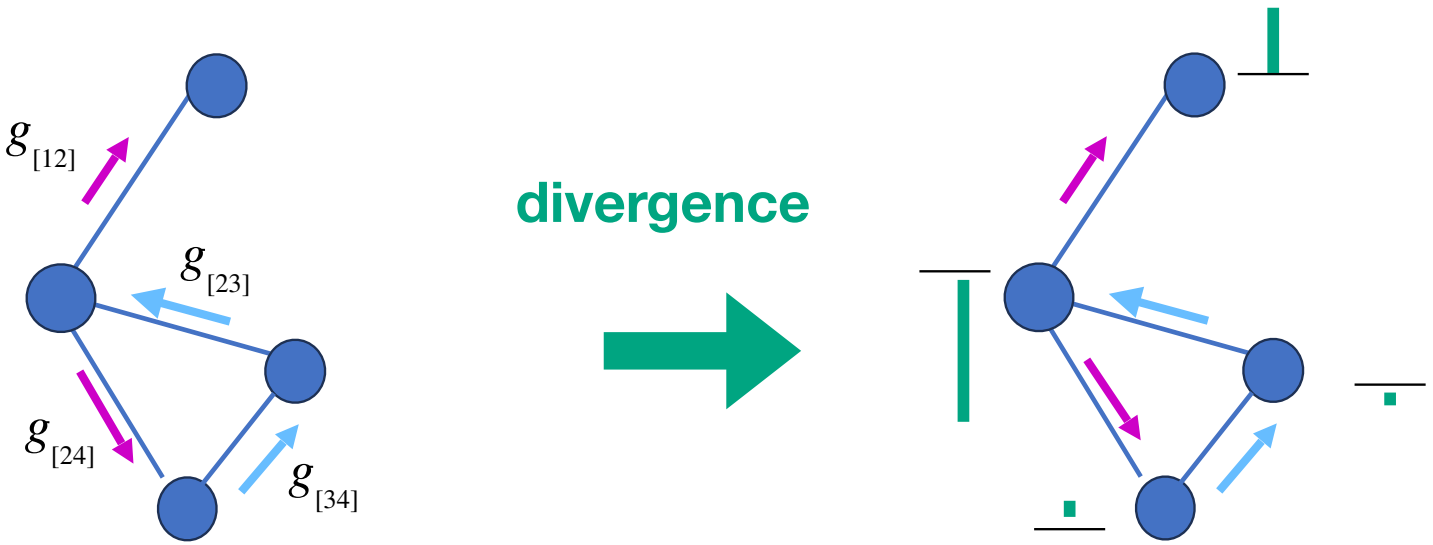
If follows that if $f' \in C^m$ is given by $f' = \delta_m^* g$.

Then $\mathbf{f}' = \bar{\mathbf{B}}_{[m+1]}^{\top} \mathbf{g} = \mathbf{B}_{[m+1]} \mathbf{g}$

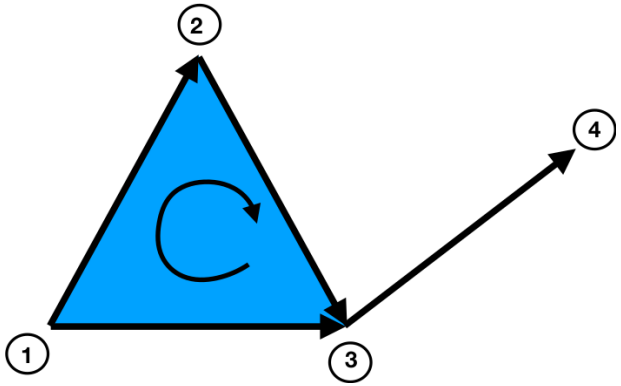
Discrete Divergence

If $g \in C^1$, then $f = \delta_0^* g \in C^0$ indicates its discrete divergence

Indeed we have $\mathbf{f} = \mathbf{B}_{[1]}\mathbf{g}$ which implies $f_r = \sum_s g_{[sr]} - \sum_s g_{[rs]}$



Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array},$$

$$\mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$