Measure Theory: Exercises 2

1. Show that for each bounded subset A of $\mathbf{R}$ that there is a Borel set B of $\mathbf{R}$ such that $A \subseteq B$ and $\lambda^*(B) = \lambda^*(A)$.

2. Show that a subset A of the real numbers is Lebesgue measurable if and only if for every finite length interval I it holds that $\lambda^*(A \cap I) + \lambda^*(I \backslash A) = \lambda^*(I)$.

3. Let A be a subset of $\mathbf{R}$. Show that the following are equivalent:

(a) A is Lebesgue measurable,

(b) A is the union of an F_σ and a set of Lebesgue measure zero,

(c) there is a set B that is an F_σ and satisfies $\lambda^*(A\Delta B) = 0$ (where Δ stands for symmetric difference).

4. Show that there is a closed subset C of $[0, 1]$ of positive Lebesgue measure that contains no open subset of $[0, 1]$.

5. Let μ be an outer measure defined on all subsets of X and let $\mathcal{A}$ be a sigma algebra such that μ is finitely additive on $\mathcal{A}$. Show that μ is also countably additive on $\mathcal{A}$.