

Measure Theory Second Week

Outer Measures:

Let X be a set,

$\mathcal{P}(X)$ the collection of all subsets of X .

An outer measure $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ (defined on all subsets) is a function such that

(a) $\mu(\emptyset) = 0$,

(b) if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,

(c) if A_1, A_2, \dots is a sequence of subsets then

$$\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i) \text{ (subadditive).}$$

For outer measures μ that are not measures there is some sequence A_1, A_2, \dots of disjoint sets such that

$$\sum_{i=1}^{\infty} \mu(A_i) > \mu(\cup_{i=1}^{\infty} A_i).$$

For finitely additive measures μ that are not measures there is some sequence A_1, A_2, \dots of disjoint sets such that

$$\sum_{i=1}^{\infty} \mu(A_i) < \mu(\cup_{i=1}^{\infty} A_i).$$

If an outer measure on a sigma-algebra is finitely additive, then it is a countably additive measure on that sigma algebra.

cn In general, outer measures are not measures,

as they are defined on all subsets;

usually measures require some restriction to a collection of measurable subsets.

Examples:

(a) $\mu(A) = 0$ if $A = \emptyset$ and

$\mu(A) = 1$ if $A \neq \emptyset$.

(b) $\mu(A) = 0$ if A is countable and

$\mu(A) = 1$ if A is uncountable.

(c) Let (X, \mathcal{A}, μ) be a measurable space.

Define $\mu^*(B) = \inf_{A \in \mathcal{A}, A \supset B} \mu(A)$.

– origin of the term outer measure.

Lebesgue outer measure:

λ^* is defined on all subsets of \mathbf{R} .

$$\lambda^*(A) =$$

$$\inf\left\{\sum_{i=1}^{\infty} b_i - a_i \mid \bigcup_{i=1}^{\infty} (a_i, b_i) \supset A\right\}.$$

Lemma (1.3.2): Lebesgue outer measure is an outer measure and assigns to every interval its length.

Proof: The empty set is covered by any collection of open intervals, hence also of lengths $\epsilon/2, \epsilon/4, \dots$,

therefore $\lambda^*(\emptyset) = 0$.

If $A \subseteq B$ then any collection of intervals covering B also covers A .

Hence the collection of coverings for A involves a larger collection than that for B ,

and therefore $\lambda^*(A) \leq \lambda^*(B)$.

Let $\epsilon > 0$ be given. Any covering collection used to define $\mu(A_i)$ to within $\frac{\epsilon}{2^i}$ also is a covering collection for $\cup_i A_i$.

Hence after taking the infimum on all coverings of $\cup_i A_i$ and ignoring the ϵ

it follows that $\lambda^*(\cup_i A_i) \leq \sum_i \lambda^*(A_i)$.

Finally, letting I be any interval from a to b with $b > a$, be in closed, open, or open on one end and closed on the other,

the sequence $(a - \epsilon, b + \epsilon)$ covers the interval, and so λ^* of the interval is no more than $b - a$.

On the other hand, it suffices to show that λ^* of the closed interval $[a, b]$ is at least $b - a$.

Because it is compact, any collection of covering open intervals can be reduced to a finite covering collection.

Now easy to show that if the lengths of this finite cover did not add up to at least $b - a$ they could not reach from a to b .

Definition: Let μ be an outer measure on X . A subset B is μ -measurable if for every subset A of X it holds that

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B).$$

Subadditivity of outer measures implies already that $\mu(A) \leq \mu(A \cap B) + \mu(A \setminus B)$,

so whenever $\mu(A) = \infty$ it is automatically true.

A *Lebesgue* measurable set is one that is measurable with respect to Lebesgue outer measure,

and the measure λ is the measure λ^* restricted to the Lebesgue measurable sets.

Lemma: (1.3.5) Let μ be an outer measure on X . Every subset B such that $\mu(B) = 0$ or $\mu(X \setminus B) = 0$ is μ -measurable.

Proof: We need only show for every subset A that $\mu(A) \geq \mu(A \cap B) + \mu(A \cap (X \setminus B))$.

With $\mu(B) = 0$ or $\mu(X \setminus B) = 0$ it follows by monotonicity.

If μ is an outer measure,

let \mathcal{M}_μ be the collection of μ measurable sets.

Theorem (1.3.6):

\mathcal{M}_μ is a sigma-algebra and

μ is a measure on \mathcal{M}_μ .

Proof: From the previous lemma and the definition of \mathcal{M}_μ , X is in \mathcal{M}_μ , $\mu(\emptyset) = 0$, and $A \in \mathcal{M}_\mu$ if and only if $X \setminus A \in \mathcal{M}_\mu$.

Next we show that \mathcal{M}_μ is an algebra and it is finitely additive.

Let $B_1, B_2 \in \mathcal{M}_\mu$; with closure by complementation already demonstrated, it suffices to show that $B_1 \cap B_2$ is also in \mathcal{M}_μ .

Let A be any subset: as B_2 is in \mathcal{M}_μ

$$\mu(A \cap B_1) =$$

$$\mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \setminus B_2) \text{ and}$$

$$\mu(A \setminus B_1) = \mu(A \cap (X \setminus B_1)) =$$

$$\mu((A \setminus B_1) \cap B_2) + \mu((A \setminus B_1) \setminus B_2).$$

As $B_1 \in \mathcal{M}_\mu$: $\mu(A) = \mu(A \cap B_1) + \mu(A \setminus B_1)$

$$\mu(A) = \mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \setminus B_2) + \mu((A \setminus B_1)) \geq$$

$$\mu(A \cap B_1 \cap B_2) + \mu(A \setminus (B_1 \cap B_2)) \geq \mu(A),$$

(by subadditivity)

hence $B_1 \cap B_2$ is also in \mathcal{M}_μ .

Furthermore, assuming $B_1, B_2 \in \mathcal{M}_\mu$ are disjoint,

and letting $A = B_1 \cup B_2$ be the set chosen,

we have $A \setminus B_1 = B_2$, $A \cap B_1 = B_1$

and $\mu(A) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$.

Therefore μ is finitely additive on \mathcal{M}_μ .

Let B_1, B_2, \dots be an infinite sequence of mutually disjoint members of \mathcal{M}_μ and let A be any subset:

It follows from finite additivity and induction that

$$\mu(A) = \sum_{i=1}^n \mu(A \cap B_i) + \mu(A \setminus (\cup_{i=1}^n B_i)).$$

Letting n go to infinity,

$$\mu(A) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A \cap B_i) + \lim_{n \rightarrow \infty} \mu(A \setminus (\cup_{i=1}^n B_i)).$$

By monotonicity $\lim_{n \rightarrow \infty} \mu(A \setminus (\cup_{i=1}^n B_i)) \geq \mu(A \setminus (\cup_{i=1}^\infty B_i))$

and by the definition of infinite sums

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A \cap B_i) = \sum_{i=1}^\infty \mu(A \cap B_i),$$

so $\mu(A) \geq \sum_{i=1}^\infty \mu(A \cap B_i) + \mu(A \setminus (\cup_{i=1}^\infty B_i))$

Therefore by the above and μ being an outer measure, $\mu(A) \geq$

$$\sum_{i=1}^{\infty} \mu(A \cap B_i) + \mu(A \setminus (\cup_{i=1}^{\infty} B_i)) \geq \\ \mu(A \cap (\cup_{i=1}^{\infty} B_i)) + \mu(A \setminus (\cup_{i=1}^{\infty} B_i)) \geq \mu(A).$$

It follows that $\cup_{i=1}^{\infty} B_i$ is in \mathcal{M}_{μ} .

Sigma-additivity on the disjoint sequence of the B_i follows from setting $A = \cup_{i=1}^{\infty} B_i$.

Starting from any sequence A_1, \dots of sets in \mathcal{M}_{μ} , by choosing the disjoint $B_i = A_i \setminus (\cup_{j=0}^{i-1} A_j)$ (with $A_0 = \emptyset$) we get $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$ in \mathcal{M}_{μ} . \square

Lemma: Every Borel subset of \mathbf{R} is Lebesgue measurable.

Proof: Given that the Lebesgue measurable sets define a sigma algebra,

and the Borel subsets are the smallest sigma algebra containing intervals of the form $I = (-\infty, c]$, given any subset A we need that

$$\lambda^*(A) = \lambda^*(A \cap I) + \lambda^*(A \setminus I).$$

We can break the i th open interval (a_i, b_i) covering A into two intervals, $(a_i, c + \frac{\epsilon}{2^i})$ and (c, b_i) whenever $a_i < c < b_i$.

In this way we cover both $A \cap I$ and $A \setminus I$ and show that $\lambda^*(A \cap I) + \lambda^*(A \setminus I) \leq \lambda^*(A) + \epsilon$ for every $\epsilon > 0$;

together with subadditivity, the equality follows.

More on Lebesgue measure:

Lemma (regularity): Let B be a Lebesgue measurable subset of finite measure.

For every $\epsilon > 0$ there is an open set A and a compact set C such that $C \subseteq B \subseteq A$ and $\lambda(A \setminus C) < \epsilon$.

Proof:

As the measure $\lambda(B)$ is approximated by open covers,

there is an open cover of B whose union A has measure less than $\lambda(B) + \epsilon/3$

By sigma additivity,

there is an n large enough so that

$$\lambda(B \cap [-n, n]) > \lambda(B) - \epsilon/3.$$

Cover $[-n, n] \setminus B$ with an open set G so that $\lambda(G) > \lambda([-n, n] \setminus B) + \epsilon/3$.

$C = [-n, n] \setminus G$ is a closed set contained in B whose measure is more than $\lambda(B) - 2\epsilon/3$.

Lemma: Lebesgue measure is translation invariant,

meaning that for any given $r \in \mathbb{R}$,

a set A is Lebesgue measurable

if and only if $A + r := \{a + r \mid a \in A\}$ is Lebesgue measurable

and $\lambda^*(A) = \lambda^*(A + r)$.

Proof: Let $(I_i \mid i = 1, 2, \dots)$ be a collection of open intervals covering A .

The intervals $(I_i + r)$ cover $A + r$ and each interval has the same length.

This shows that $\lambda^*(A + r) \leq \lambda^*(A)$,

and the same argument shifting by $-r$ shows the opposite inequality.

Likewise the intersection property with any subset of \mathbf{R} that confirms that A and $X \setminus A$ are Lebesgue measurable

shows the same for $A + r$ and $(X \setminus A) + r$ after all sets are shifted by r .

Theorem: Given the axiom of choice,
there is a subset of $[0, 1)$ that is not Lebesgue
measurable.

Proof: Define an equivalence relation on
 $r, s \in [0, 1)$

by $r \sim s \Leftrightarrow r - s$ is rational.

Define addition modulo 1,

so that $b + c$ is $b + c - 1$ if $b + c \geq 1$.

List the rational numbers a_1, a_2, \dots in $[0, 1)$.

Let B be a set of representatives for the equivalence relation (Axiom of Choice)

meaning that B intersects every equivalence class one and only once,

or that for every $r \in [0, 1)$ there is one and only one i with $r + a_i \in B$.

This means that $\cup_{i=1}^{\infty} (B - a_i)$ partitions $[0, 1)$:

for every r there is some $b \in B$ and a_i such that $r = b - a_i$

and if $r \in (B - a_i) \cap (B - a_j) \neq \emptyset$ for distinct $a_i \neq a_j$

then $r = b_i - a_i = b_j - a_j$ for some $b_i, b_j \in B$ and the equivalence relation sharing both $r + a_i$ and $r + a_j$ have two representatives b_i and b_j in B , a contradiction.

Assume that B is Lebesgue measurable.

Notice that translation invariance holds also in the modulo arithmetic,

due to a secondary shift of the measurable subset that went over the value of 1.

So every $B + a_i$ must be Lebesgue measurable and have the same measure.

This measure can neither be 0 or anything positive,

as that would imply that the whole set $[0, 1)$ is either infinite in measure or zero in measure,

when it is really of measure one.