# Topology, geometry and dynamics of higher-order networks

*An introduction to simplicial complexes*
*Lesson II*

**LTCC module 2024**

**14 October 2024**

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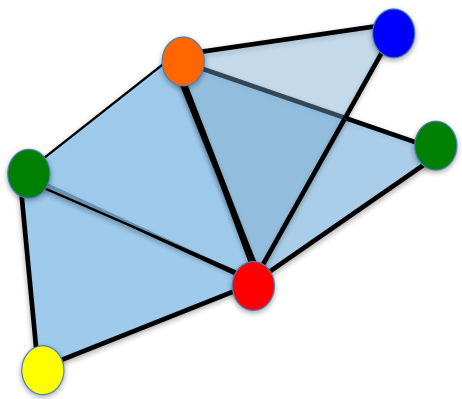
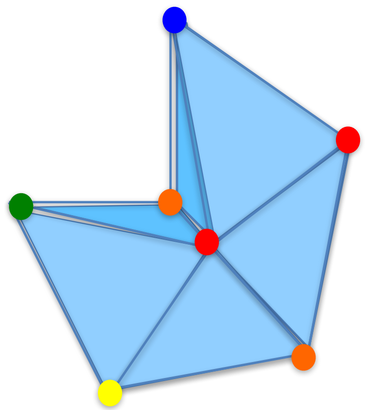
# Higher-order networks

**Higher-order networks are characterising the interactions between two ore more nodes and are  formed by nodes, links, triangles, tetrahedra etc.**



**d=2 simplicial complex**

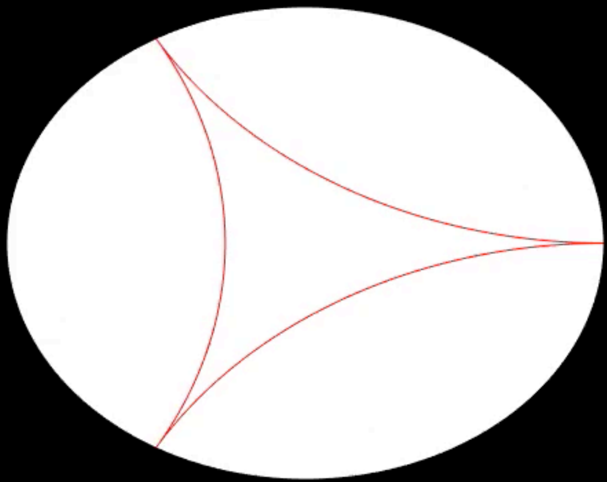


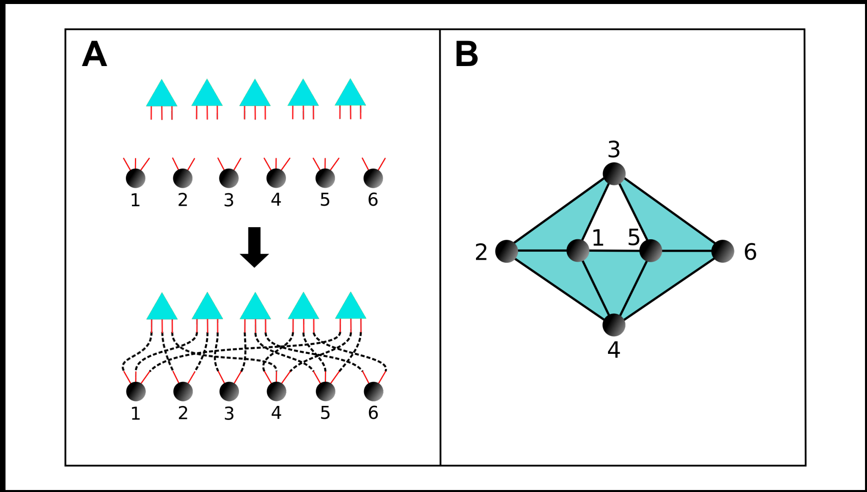
**d=3 simplicial complex**

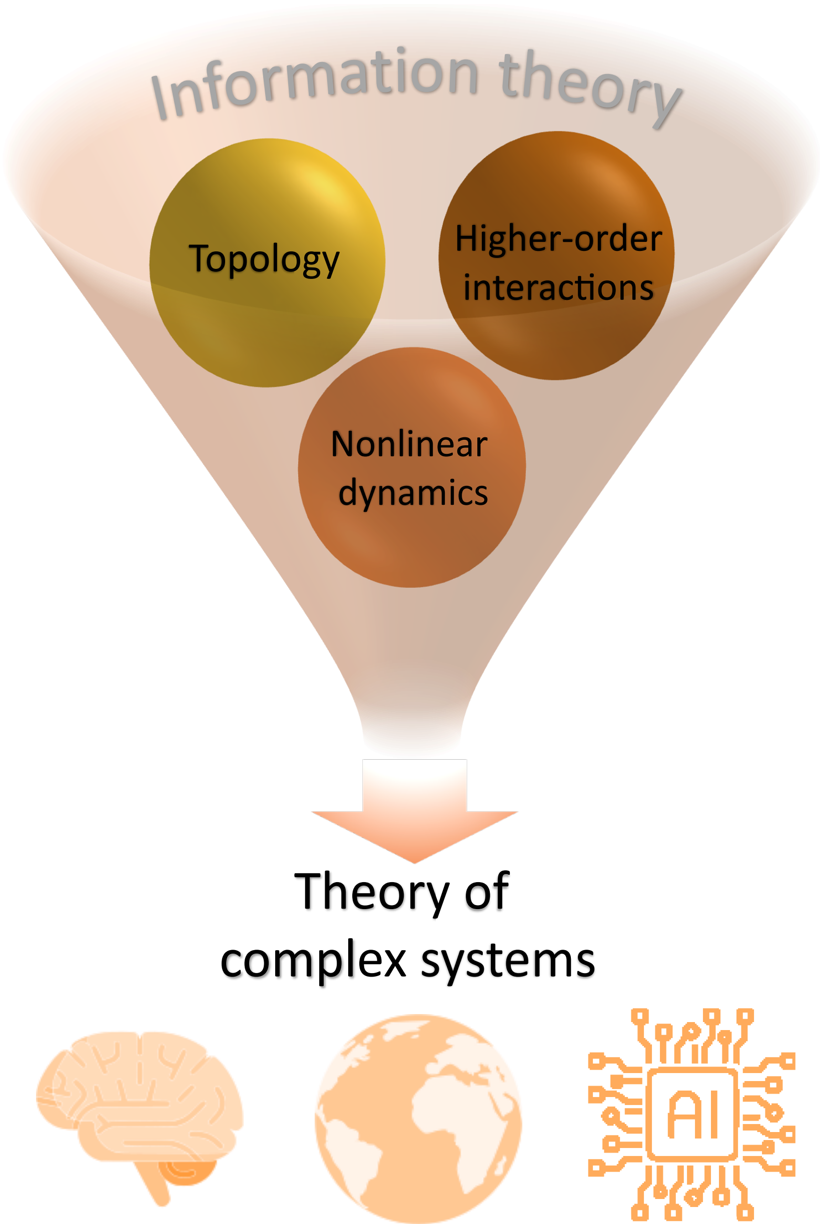
# Simplicial complex models

Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]

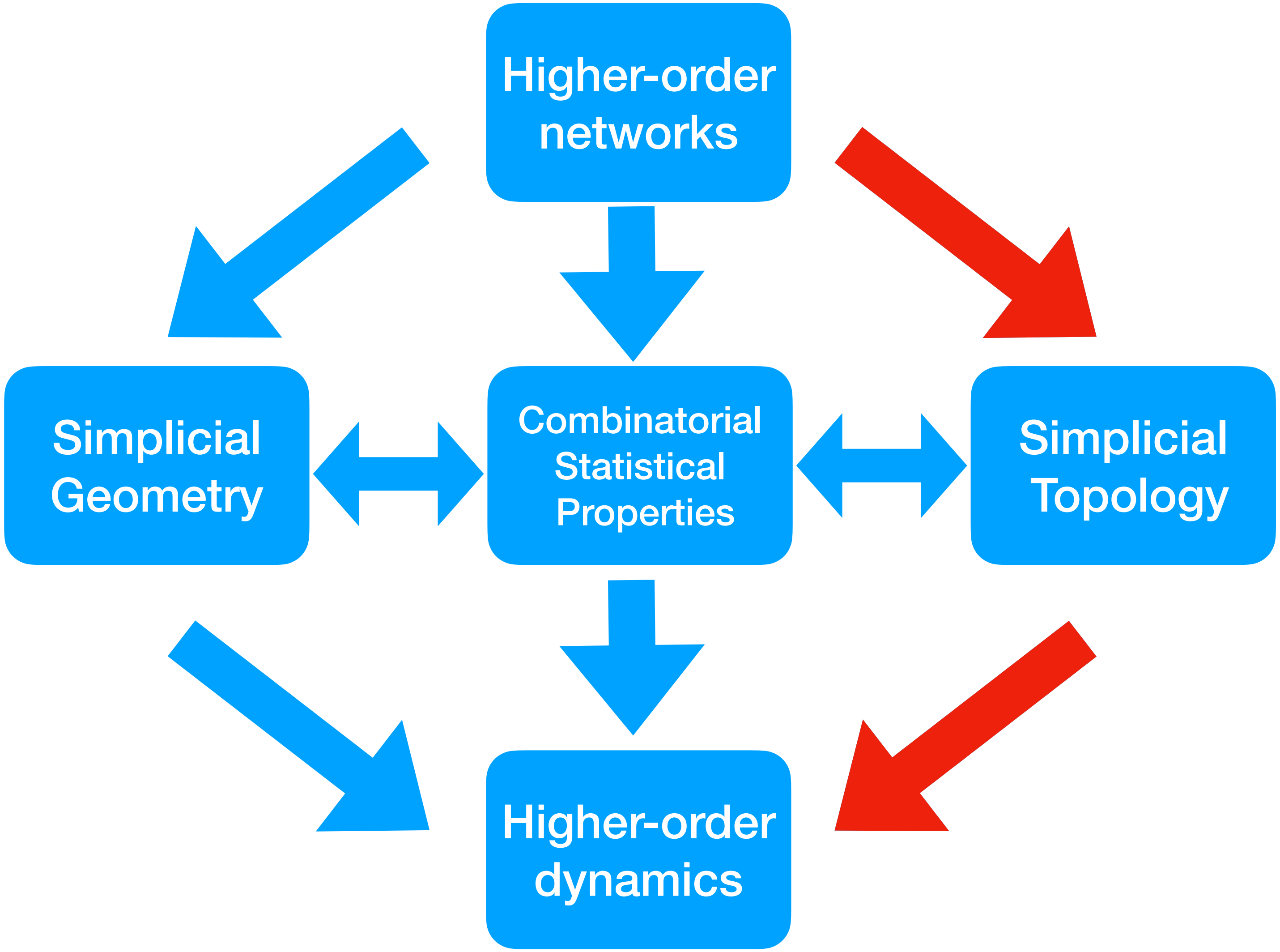







# Complexity challenge

# Higher-order structure and dynamics



# Lesson IIa:
# Hodge Laplacians

- Introduction to algebraic topology:
- Hodge Laplacians

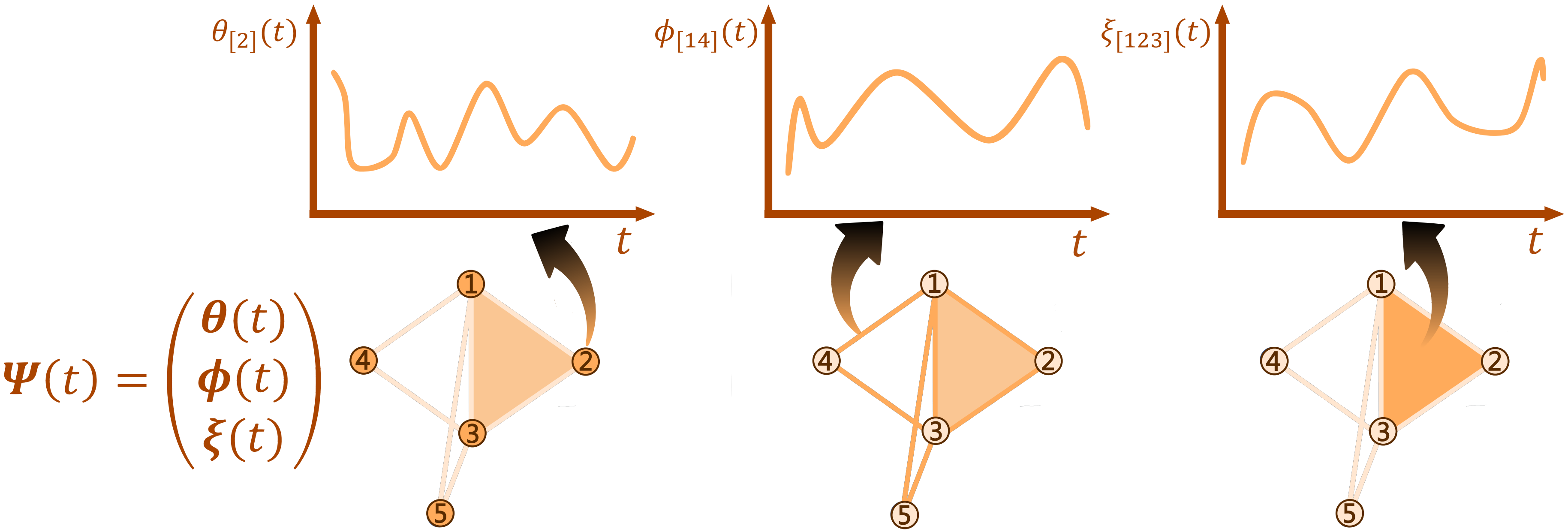
  -Graph Laplacian

  -Properties of the Hodge Laplacian

  -Connection with topology

# Topological signals

**Beyond the node centered description of network dynamics**
**The dynamical state of a simplicial complex includes**
**node, edge, and higher-order topological signals**



# Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

***Topological signals***
***are cochains or vector fields***

# Cochains

### $m$-cochains

A $m$-dimensional cochain $f \in C^m$ is a linear function $f : C_m \to \mathbb{R}$, that associates to every $m$-chain of the simplicial complex a value in $\mathbb{R}$.

**m-cochain** $f \in C^m$

**Given the m-chain**
$c_m \in C_m$

$$c_m = \sum_{r \in Q_m(\mathcal{K})} c_m^r \alpha_r^m, \text{ **with** } c_m^r \in \mathbb{Z}$$

$$\boxed{f(c_m) = \sum_{r \in Q_m(\mathcal{K})} c_m^r f(\alpha_r^m), \text{ with } c_m^r \in \mathbb{Z}}$$

# Cochains:properties

$m$-cochains

A $m$-dimensional cochain $f \in C^m$ is a linear function $f : C_m \longrightarrow \mathbb{R}$, that associates to every $m$-chain of the simplicial complex a value in $\mathbb{R}$.

**Given a basis for the m-simplices of the simplicial complex,**
**A m-cochain can be expressed as a vector $\mathbf{f}$ of elements**

$$\boxed{f_r = f(\alpha_r^m) \ \forall \alpha_r^m \in Q_m(\mathscr{K})}$$

# $L^2$ norm between cochains

We define a scalar product between $m$-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathscr{K})} f_r^2$$

This scalar product can be generalised by introducing metric matrices (see lecture III)

# Coboundary operator

Coboundary operator $\delta_m$

The coboundary operator $\delta_m : C^m \longrightarrow C^{m+1}$ associates to every $m$-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

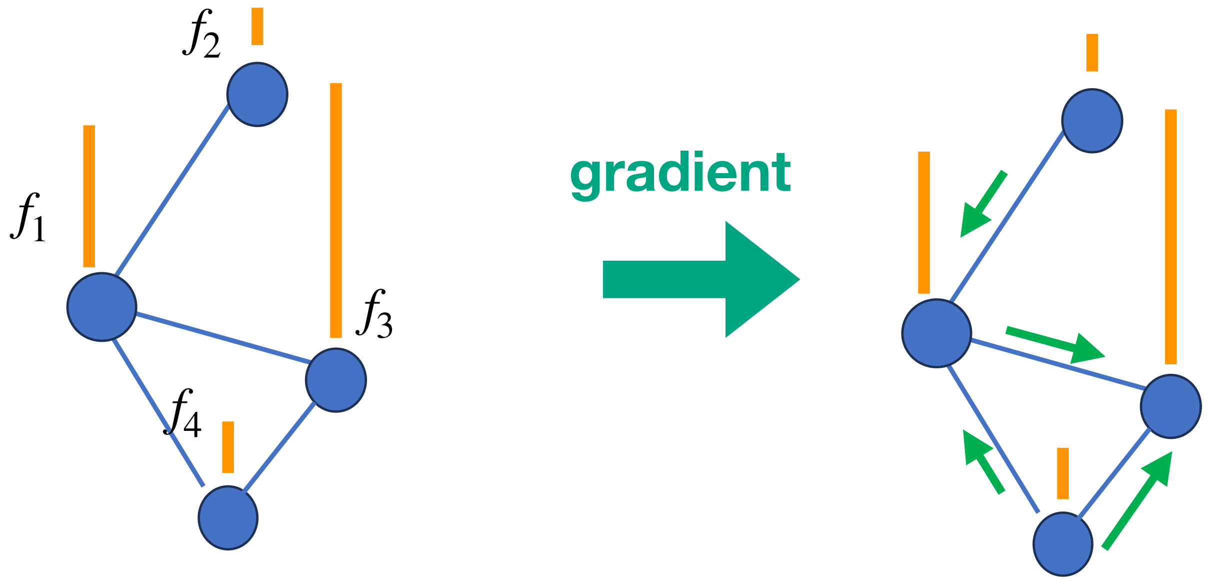
If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{m+1}^{\top}\mathbf{f} \equiv \bar{\mathbf{B}}_{m+1}\mathbf{f}$

# Discrete Gradient

If $f \in C^0$, then $g = \delta_0 f \in C^1$ indicates its discrete gradient

Indeed we have $\mathbf{g} = \mathbf{B}_{[1]}^{\top}\mathbf{f}$ which implies $g_{[r,s]} = f_s - f_r$

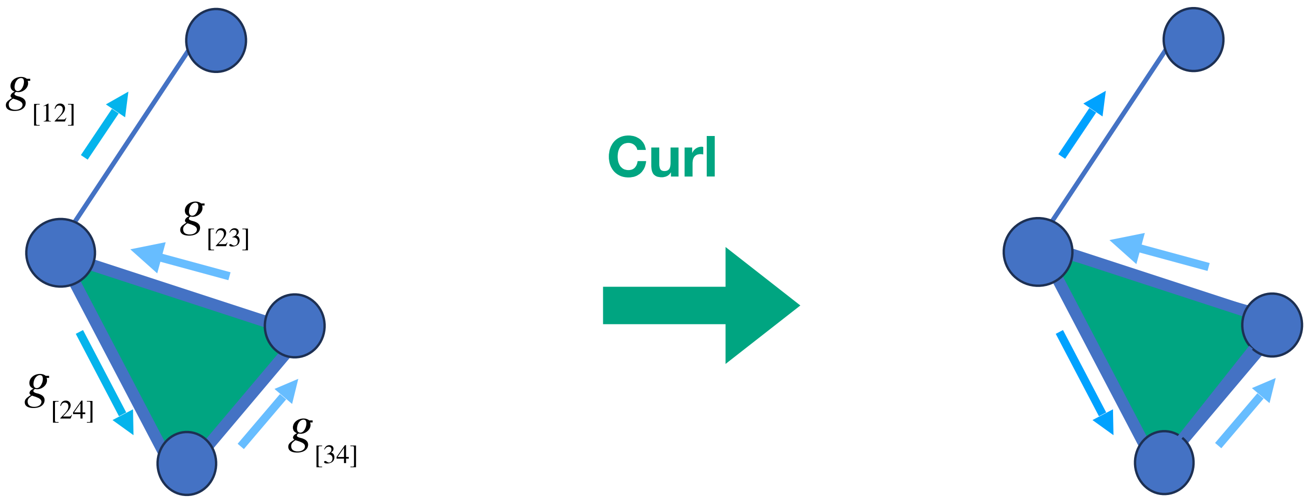


# Discrete Curl

If $f \in C^1$, then $h = \delta_1 g \in C^2$ indicates its discrete curl

Indeed we have $\mathbf{g} = \mathbf{B}_{[2]}^{\top}\mathbf{f}$ which implies

$$h_{[r,s,q]} = g_{[r,s]} + g_{[s,q]} - g_{[r,q]}$$



# Coboundary operator

We have that

$\delta_{m+1} \circ \delta_m = 0 \ \forall m \geq 1$ hence $\mathbf{B}^\top_{[m+1]} \mathbf{B}^\top_{[m]} = \mathbf{0}$

# Adjoint of the coboundary operator

Adjoint operator $\delta_m^*$

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \rightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

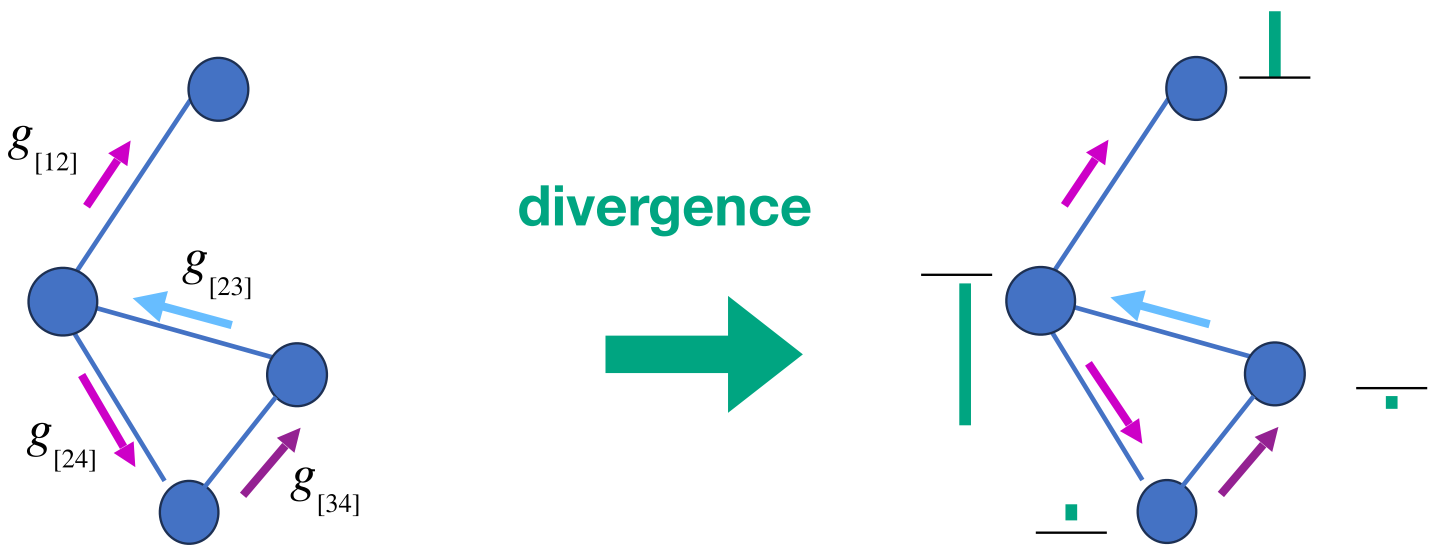
for any $f \in C^m$ and $g \in C^{m+1}$.

**It follows that if** $f' = \delta_m^* g$ **then** $\mathbf{f}' = \mathbf{B}_{[m+1]}\mathbf{g}$

# Discrete Divergence

If $g \in C^1$, then $f = \delta_0^* g \in C^0$ indicates its discrete divergence

Indeed we have $\mathbf{f} = \mathbf{B}_{[1]}\mathbf{g}$ which implies $f_r = \sum_s g_{[sr]} - \sum_s g_{[rs]}$



# Coboundary operator

We have that

$$\delta^*_{m-1} \circ \delta^*_m = 0 \ \forall m \geq 1 \text{ hence } \mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0}$$

# Hodge Laplacians

# Hodge Laplacian

The Hodge-Laplacians

The $m$-dimensional Hodge-Laplacian $L_m$ is defined as

$$L_m = L_m^{up} + L_m^{down}$$

where up and down $m$-dimensional Hodge Laplacians are given by

$$\begin{aligned} L_m^{up} &= \delta_m^* \delta_m, \\ L_m^{down} &= \delta_{m-1} \delta_{m-1}^*. \end{aligned}$$

**Note that $L_0^{down} = 0$ by definition**

# Hodge Laplacians

**The Hodge Laplacians describe diffusion**

**from n-simplices to m-simplices through (m-1) and (m+1)**

**simplices**

$$\mathbf{L}_{[m]} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]} + \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

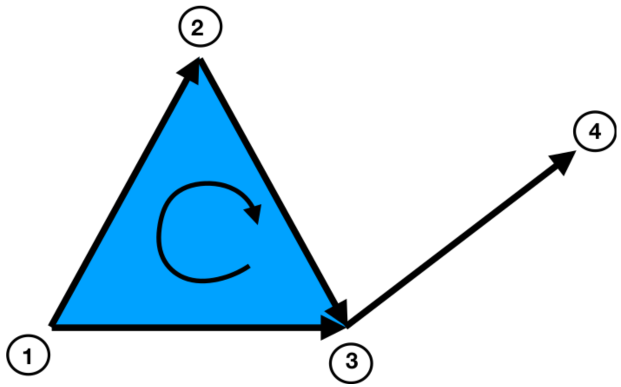
The higher order Hodge Laplacian can be decomposed as

$$\mathbf{L}_{[m]} = \mathbf{L}_{[m]}^{down} + \mathbf{L}_{[m]}^{up},$$

with

$$\mathbf{L}_{[m]}^{down} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]},$$
$$\mathbf{L}_{[m]}^{up} = \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}.$$

# Simplicial complexes and Hodge Laplacians




## Hodge Laplacians

The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices

**For a 2-dimensional simplicial complex we have**

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^{\top} \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^{\top}\mathbf{B}_{[2]}$$

# Properties of the Hodge Laplacians

- The Hodge Laplacians $L_m, L_m^{up}, L_m^{down}$ are semidefinite positive.

- They obey Hodge decomposition

- The dimension of the kernel of the Hodge Laplacian $L_m$ is the $m$-Betti number $\beta_m$

- The harmonic eigenvectors are related to the generators of the homology classes and can be chosen in such way that they localise on the $m-$holes

# The Hodge-Laplacians are semi-definitive positive

**The Hodge Laplacians $L_m, L_m^{up}, L_m^{down}$**

**are semidefinite positive.**

Indeed we have:

$$\langle f, L_m^{up} f \rangle = \langle f, \delta_m^* \delta_m f \rangle = \langle \delta_m f, \delta_m f \rangle \geq 0$$

$$\langle f, L_m^{down} f \rangle = \langle f, \delta_{m-1} \delta_{m-1}^* f \rangle = \langle \delta_{m-1}^* f, \delta_{m-1}^* f \rangle \geq 0$$

$$\langle f, L_m f \rangle = \langle f, L_m^{up} f \rangle + \langle f, L_m^{down} f \rangle \geq 0$$

# Hodge-decomposition

**We have**

$$L_m^{down} L_m^{up} = 0 \qquad \text{im} L_m^{up} \subseteq \ker L_m^{down}$$

$$L_m^{up} L_m^{down} = 0 \qquad \text{im} L_m^{down} \subseteq \ker L_m^{up}$$

$$\ker L_m = \ker L_m^{down} \cap \ker L_m^{up}$$

Indeed

$$L_m^{down} L_m^{up} = \delta_{m-1}\delta_{m-1}^{*}\delta_m^{*}\delta_m = 0$$

$$L_m^{up} L_m^{down} = \delta_m^{*}\delta_m\delta_{m-1}\delta_{m-1}^{*} = 0$$

# Hodge-decomposition

**Given that** $\mathbf{B}_{[m]}\mathbf{B}_{[m+1]} = \mathbf{0} \quad \mathbf{B}_{[m-1]}^{\top}\mathbf{B}_{[m]}^{\top} = \mathbf{0}$

**and that** $\mathbf{L}_{[m]}^{up} = \mathbf{B}_{[m+1]}\mathbf{B}_{[m+1]}^{\top}, \quad \mathbf{L}_{[m]}^{down} = \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m]}$

We have:

$$\mathbf{L}_{[m]}^{down}\mathbf{L}_{[m]}^{up} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}_{[m]}^{up} \subseteq \mathrm{ker}\mathbf{L}_{[m]}^{down}$$

$$\mathbf{L}_{[m]}^{up}\mathbf{L}_{[m]}^{down} = \mathbf{0} \qquad \mathrm{im}\mathbf{L}_{[m]}^{down} \subseteq \mathrm{ker}\mathbf{L}_{[m]}^{up}$$

# Hodge decomposition

The Hodge decomposition can be summarised as

$$C^m = \textbf{im}(\mathbf{B}_{[m]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})$$

This means that $\mathbf{L}_{[m]}, \mathbf{L}_{[m]}^{up}, \mathbf{L}_{[m]}^{down}$ are commuting and can be diagonalised simultaneously. In this basis these matrices have the block structure

$$\mathbf{U}^{-1}\mathbf{L}_{[m]}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[m]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[m]}^{up} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[m]}^{down}\mathbf{U} = \begin{pmatrix} \mathbf{D}_{[m]}^{down} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{U}^{-1}\mathbf{L}_{[m]}^{up}\mathbf{U} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{[m]}^{up} \end{pmatrix}$$

- **Therefore an eigenvector in the ker of $\mathbf{L}_{[m]}$ is also in the ker of both $\mathbf{L}_{[m]}^{up}, \mathbf{L}_{[m]}^{down}$**
- **An eigenvector corresponding to an non-zero eigenvalue of $\mathbf{L}_{[m]}$ is either a non-zero eigenvector of $\mathbf{L}_{[m]}^{up}$ or a non-zero eigenvector of $\mathbf{L}_{[m]}^{down}$**

# Hodge decomposition

**We have**

$$C^m = \textbf{im}(\delta_{m-1}) \oplus \textbf{ker}(L_m) \oplus \textbf{im}(\delta_m^*)$$

**Any $m$-cochain can be decomposed in a unique way in**

$$c^m = \delta_{m-1}\omega^{(m-1)} + \omega^{m,(harm)} + \delta_m^*\omega^{(m+1)}$$

**Where $\omega^{(m-1)}, \omega^{(m+1)}$ are $m-1$ and $m+1$ cochains and $L_m\omega^{m,(harm)} = 0$**

For $m = 1$

$$c^1 = \delta_0\omega^{(0)} + \omega^{1(harm)} + \delta_1^*\omega^{(m+1)}$$

Representing the gradient flow, the harmonic component and the curl flow respectively

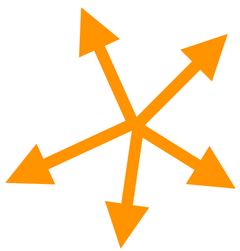
# Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

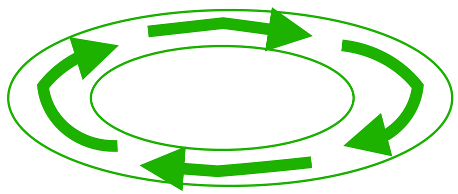
in a irrotational, harmonic and solenoidal components

$$\boxed{C^m = \textbf{im}(\mathbf{B}_{[m]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})}$$

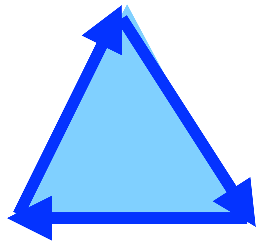
which in the case of topological signals of the links can be sketched as



**Irrotational component**
**Gradient Flow**



**Harmonic component**



**Solenoidal component**
**Curl Flow**

# Hodge decomposition

Every $m$-cochain (topological signal) can be decomposed in a unique way thanks to the Hodge decomposition as

$$\mathbb{R}^{D_m} = \text{im}(\mathbf{B}_{[m]}^{\top}) \oplus \text{ker}(\mathbf{L}_{[m]}) \oplus \text{im}(\mathbf{B}_{[m+1]})$$

therefore every $m$-cochain can be decomposed in a unique way as

$$\mathbf{x} = \mathbf{x}^{[1]} + \mathbf{x}^{[2]} + \mathbf{x}^{harm} \quad \text{With} \quad \begin{aligned} \mathbf{x}^{[1]} &= \mathbf{L}_{[m]}^{up}\mathbf{L}_{[m]}^{up,+}\mathbf{x} \\ \mathbf{x}^{[2]} &= \mathbf{L}_{[m]}^{down}\mathbf{L}_{[m]}^{down,+}\mathbf{x} \end{aligned}$$

# Betti numbers

**The dimension of the kernel of the Hodge Laplacian $L_m$ is the $m$-Betti number $\beta_m$, i.e.**

$$\textbf{dim}(\textbf{ker}(L_m)) = \beta_m$$

**Indeed, thanks to Hodge decomposition**

$$\begin{aligned}
\text{dim ker } \mathbf{L}_{[m]} &= \text{dim}(\text{ker } \mathbf{L}_{[m]}^{up} \cap \text{ker } \mathbf{L}_{[m]}^{down}) \\
&= \text{dim}(\text{ker } \mathbf{L}_{[m]}^{up}) - \text{dim}(\text{im } \mathbf{L}_{[m]}^{down}) \\
&= \text{dim}(\text{ker } \mathbf{L}_{[m]}^{down}) - \text{dim}(\text{im } \mathbf{L}_{[m]}^{up}) \\
&= \text{dim}(\text{ker } \mathbf{B}_{[m]}) - \text{dim}(\text{im } \mathbf{B}_{[m+1]}) \\
&= \text{rank} \mathscr{H}_m = \beta_m
\end{aligned}$$

# Harmonic eigenvectors of the Hodge Laplacian

**The dimension of the kernel of the Hodge Laplacian**

**is given by the corresponding Betti number**

$$\textbf{dim ker}(\mathbf{L}_{[m]}) = \beta_m$$

**The harmonic eigenvectors**

**are associated to the generators of the homology**

**They are in general non-uniform over the $m$-simplices of the simplicial complex**

# Digression
# on simple networks
# (Graphs)
# and on Graph Laplacians

# Graphs and networks

## Definition

A *graph* is an ordered pair $G = (V, E)$ comprising a set $V$ of *vertices* connected by the set $E$ of *edges*.

A graph is a 1-dimensional simplicial complex

## Definition

A *network* is the graph $G = (V, E)$ describing the set of interactions between the constituents of a complex system. The vertices of a network are called *nodes* and the edges *links*.

The *network size* $N=N_0$ is the total number of nodes in the network $N=|V|$. The total number of edges $N_1$ is given by $N_1=|E|$.

# Simple networks

**Adjacency matrix**

A simple network is fully determined by its adjacency matrix.

The adjacency matrix $a$ of a simple network is a $N \times N$ matrix of elements given by

$$a_{rs} = \begin{cases} 1 & \textbf{if } r \textbf{ is linked to } s \\ 0 & \textbf{otherwise.} \end{cases}$$

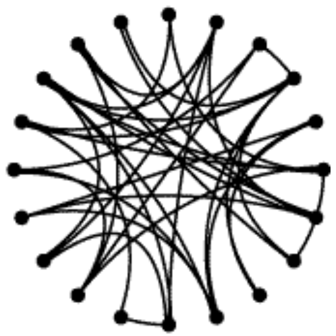
*The adjacency matrix of a simple network is symmetric.*

**Definition**

In a simple network the *degree* $k_i$ of node $i$ is given by the total number of links incident to node , i.e.

$$k_r = \sum_{s=1}^{N} a_{rs}$$

# Random graphs




Uncorrelated maximally random graphs with given degree sequence

Are generated by ensembles in which each edge $(r, s)$ is drawn independently with probability

$$\boxed{p_{rs} = \langle a_{rs} \rangle = \frac{k_r k_s}{\langle k \rangle N}}$$

# Graph Laplacian

The graph Laplacian matrix $\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}$ has elements

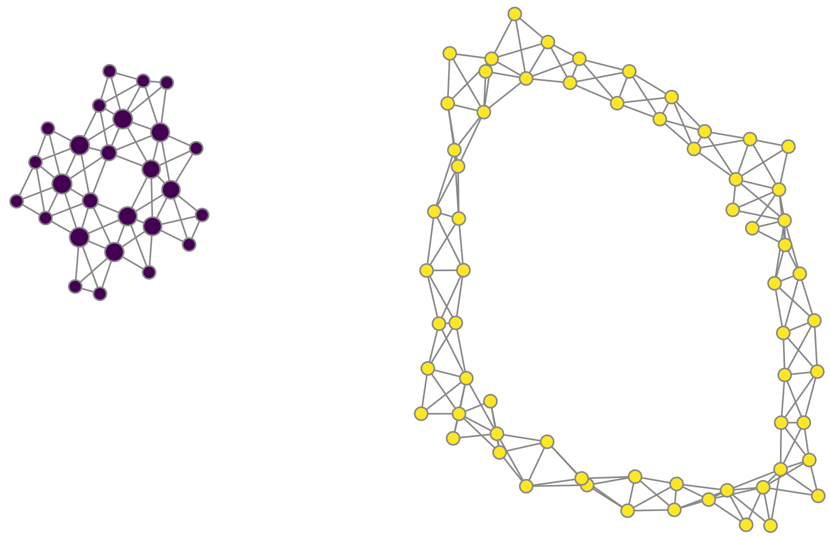
$$[L_{[0]}]_{ij} = \delta_{ij}k_i - a_{ij}$$

The graph Laplacian is a semi-definite positive matrix that in a connected network has eigenvalues

$$0 = \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_N$$

**The Laplacian is key for describing diffusion processes and the Kuramoto model on networks and constitutes a natural link between topology and dynamics**

# Harmonic eigenvectors of the graph Laplacian

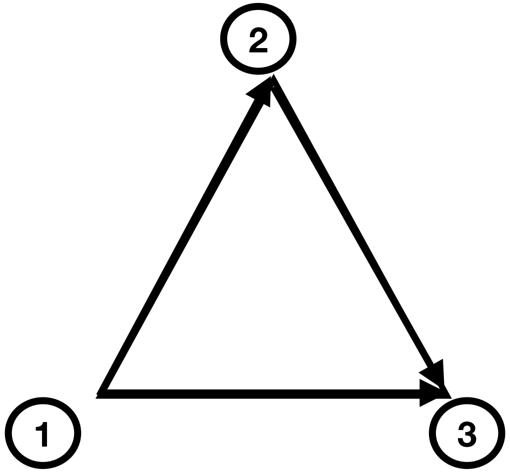


The quadratic form of the graph Laplacian reads

$$\mathbf{X}^\top \mathbf{L}_{[0]} \mathbf{X} = \frac{1}{2} \sum_{r,s} a_{rs} (X_r - X_s)^2$$

Therefore the harmonic eigenvectors of the graph Laplacian are constant on each connected component of the graph and zero everywhere else.

# Eigenvectors of the $\mathbf{L}_{[0]}$ Hodge Laplacian




$$\mathbf{B}_{[1]} = \begin{array}{cccc} & [1,2] & [2,3] & [1,3] \\ [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \mathbf{0} \qquad \mathbf{L}_{[0]} = \begin{array}{cccc} & [1,2] & [2,3] & [1,3] \\ [1,2] & 2 & -1 & -1 \\ [2,3] & -1 & 2 & -1 \\ [1,3] & -1 & -1 & 2 \end{array},$$

$$\mathbf{V}_{[0]} = \begin{array}{cccc} & \mu = 0 & \mu = 3 & \mu = 3 \\ [1] & 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{2} \\ [2] & 1/\sqrt{3} & 0 & 1/\sqrt{2} \\ [3] & 1/\sqrt{3} & 1/\sqrt{2} & 0 \end{array}, \qquad \mathbf{\Sigma}_{[0]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

# Connected network

**A connected network has a single eigenvector in the kernel of the graph Laplacian.**

**This eigenvector is constant on each node of the network, i.e.**

$$\mathbf{u} = \frac{1}{\sqrt{N}}\mathbf{1}$$

# Back to higher-order Hodge Laplacians

# Harmonic eigenvectors of the Hodge Laplacian

**The dimension of the kernel of the Hodge Laplacian**

**is given by the corresponding Betti number**

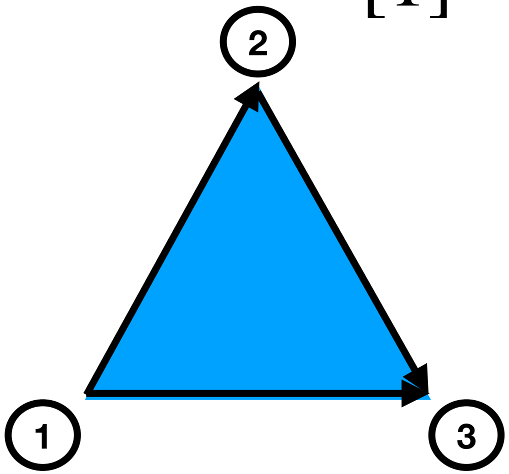
$$\textbf{dim ker}(\mathbf{L}_{[m]}) = \beta_m$$

**The harmonic eigenvectors**

**are associated to the generators of the homology**

**They are in general non-uniform over the $m$-simplices of the simplicial complex**

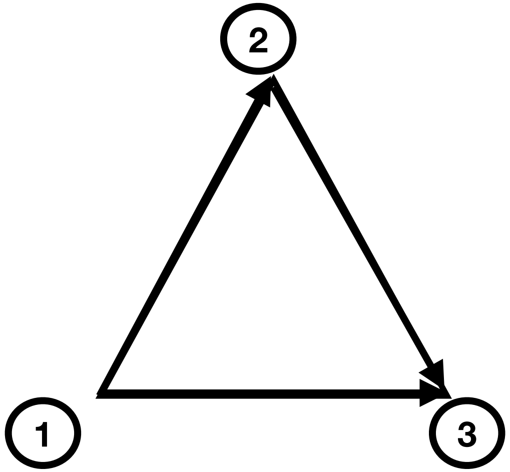
# Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian




$$\mathbf{B}_{[1]} = \begin{array}{cccc} & [1,2] & [2,3] & [1,3] \\ [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \quad \mathbf{B}_{[2]} = \begin{array}{cc} & [1,2,3] \\ [1,2] & 1 \\ [2,3] & 1 \\ [1,3] & -1 \end{array}. \quad \mathbf{L}_{[1]} = \begin{array}{cccc} & [1,2] & [1,3] & [1,3] \\ [1,2] & 3 & 0 & 0 \\ [2,3] & 0 & 3 & 0 \\ [1,3] & 0 & 0 & 3 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{cccc} & \mu = 3 & \mu = 3 & \mu = 3 \\ [1,2] & 1 & 0 & 0 \\ [2,3] & 0 & 1 & 0 \\ [1,3] & 0 & 0 & 1 \end{array}, \quad \mathbf{\Sigma}_{[1]} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

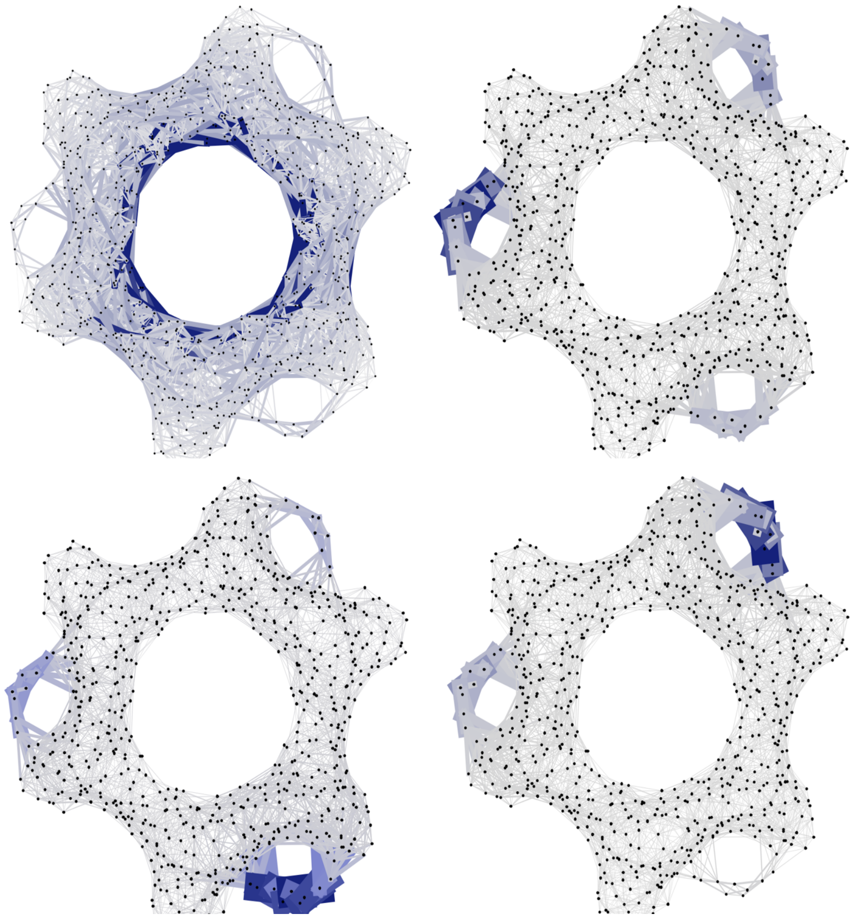
# Eigenvectors of the $\mathbf{L}_{[1]}$ Hodge Laplacian




$$\mathbf{B}_{[1]} = \begin{array}{c|ccc} & [1,2] & [2,3] & [1,3] \\ \hline [1] & -1 & 0 & -1 \\ [2] & 1 & -1 & 0 \\ [3] & 0 & 1 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \mathbf{0} \qquad \mathbf{L}_{[1]} = \begin{array}{c|ccc} & [1,2] & [2,3] & [1,3] \\ \hline [1,2] & 2 & -1 & 1 \\ [2,3] & -1 & 2 & 1 \\ [1,3] & 1 & 1 & 2 \end{array},$$

$$\mathbf{V}_{[1]} = \begin{array}{c|ccc} & \mu = 0 & \mu = 3 & \mu = 3 \\ \hline [1,2] & 1/\sqrt{3} & 1/\sqrt{2} & 0 \\ [2,3] & 1/\sqrt{3} & 0 & 1/\sqrt{2} \\ [1,3] & -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{2} \end{array}, \qquad \boldsymbol{\Sigma}_{[1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

# Harmonic eigenvectors



Wee et al. (2023)

# Expression of the matrix elements of the Hodge Laplacians

$$\mathbf{L}_m^{\text{up}}(r,s) = \begin{cases} k_{m+1,m}(\alpha_r^m), & r = s. \\ -1, & r \neq s, \alpha_r^m \frown \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ 1, & r \neq s, \alpha_r^m \frown \alpha_s^m, \alpha_r^m \nsim \alpha_s^m. \\ 0, & \text{otherwise.} \end{cases}$$

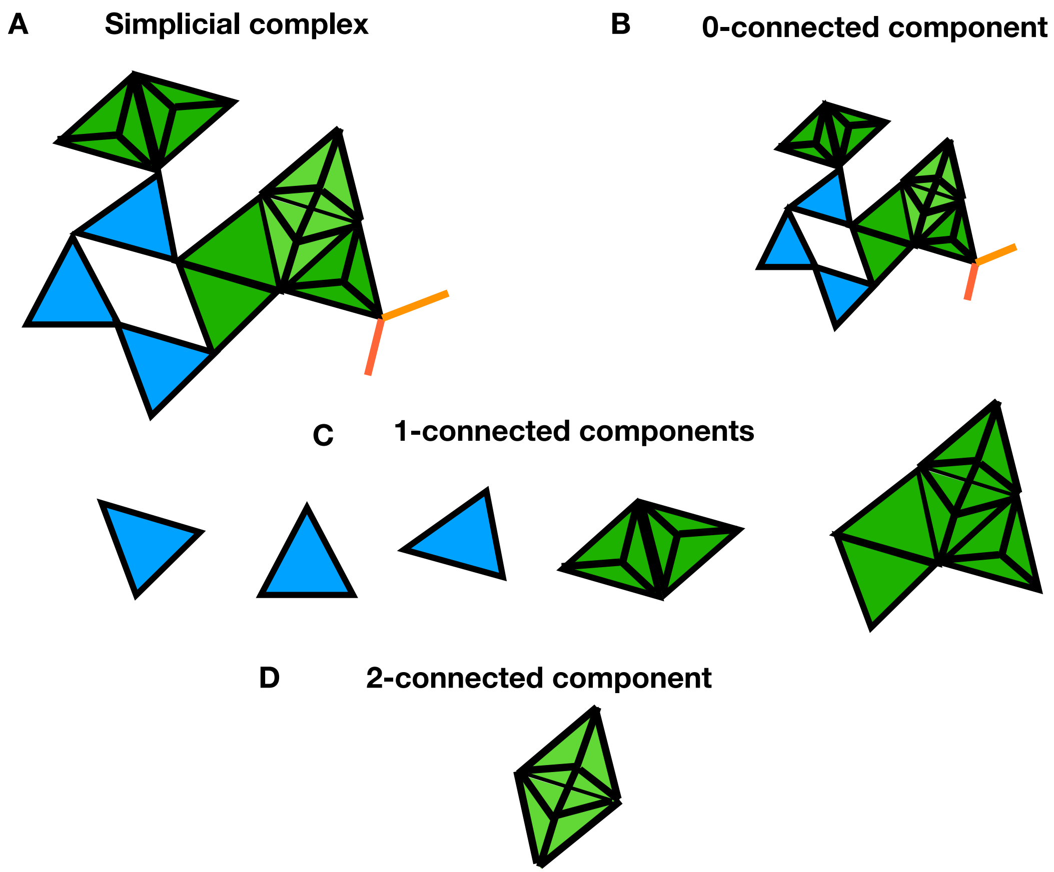
$$\mathbf{L}_m^{\text{down}}(r,s) = \begin{cases} m+1, & r = s. \\ 1, & r \neq s, \alpha_r^p \smile \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ -1, & r \neq s, \alpha_r^p \smile \alpha_s^m, \alpha_r^p \nsim \alpha_s^m. \\ 0, & \text{otherwise.} \end{cases}$$

**The m-dimensional up- Hodge Laplacian has nonzero elements**
**only among upper incident m-simplices**
**(simplices which are faces of a common m+1 simplex)**
**The eigenvectors have support on the m-connected components**

**The  m-dimensional down-Hodge Laplacian has nonzero elements**
**only among lower incident m-simplices**
**(simplifies sharing a m-1 face)**
**The eigenvectors have support on the (m-1)—connected components**

**Here ~ indicates similar orientation with respect to the lower-simplices**

# m-connected components



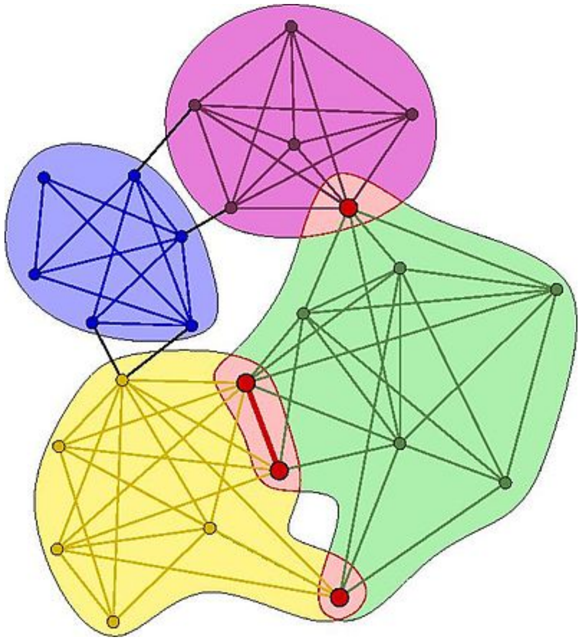
# Expression of the matrix elements of the Hodge Laplacians

$$\mathbf{L}_m(r,s) = \begin{cases} k_{m+1,m}(\alpha_r^m) + m + 1, & r = s. \\ 1, & r \neq s, \alpha_r^m \not\frown \alpha_s^m, \alpha_r^m \smile \alpha_s^m, \alpha_r^m \sim \alpha_s^m. \\ -1, & r \neq s, \alpha_r^m \not\frown \alpha_s^m, \alpha_r^m \smile \alpha_s^m, \alpha_r^m \nsim \alpha_s^m. \\ 0 & \text{otherwise.} \end{cases}$$

**for** $0 < m < d$

**The matrix elements of the Hodge Laplacian is only non zero among lower adjacent simplices that are not upper-adjacent**

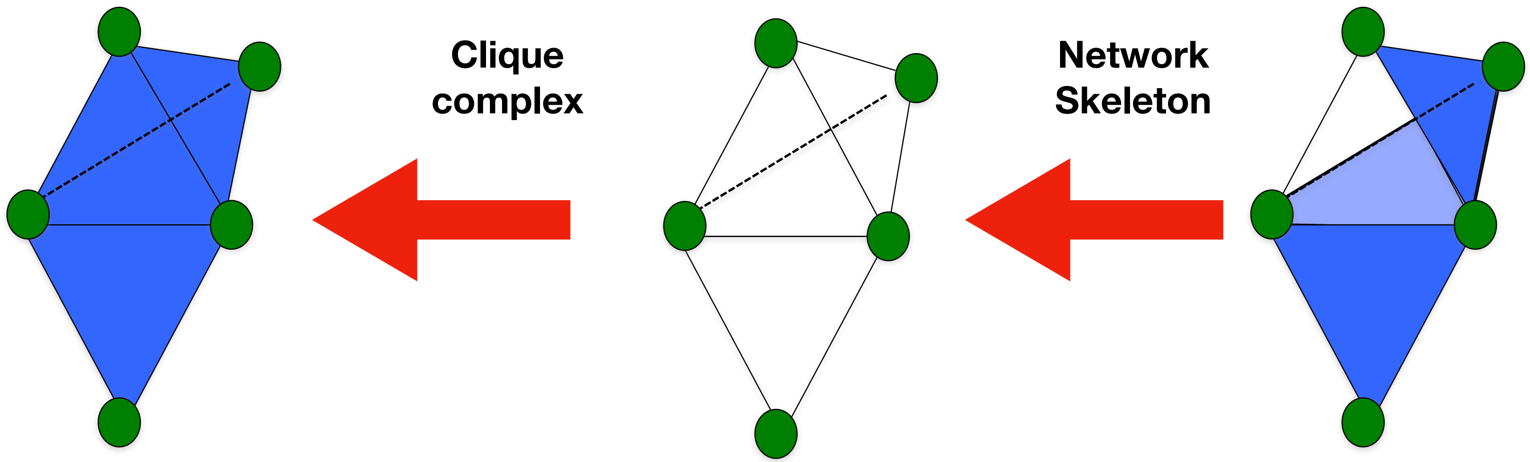
# Clique communities



Palla et al. Nature 2005

The m-clique communities are the m-connected components of the clique complex of the network

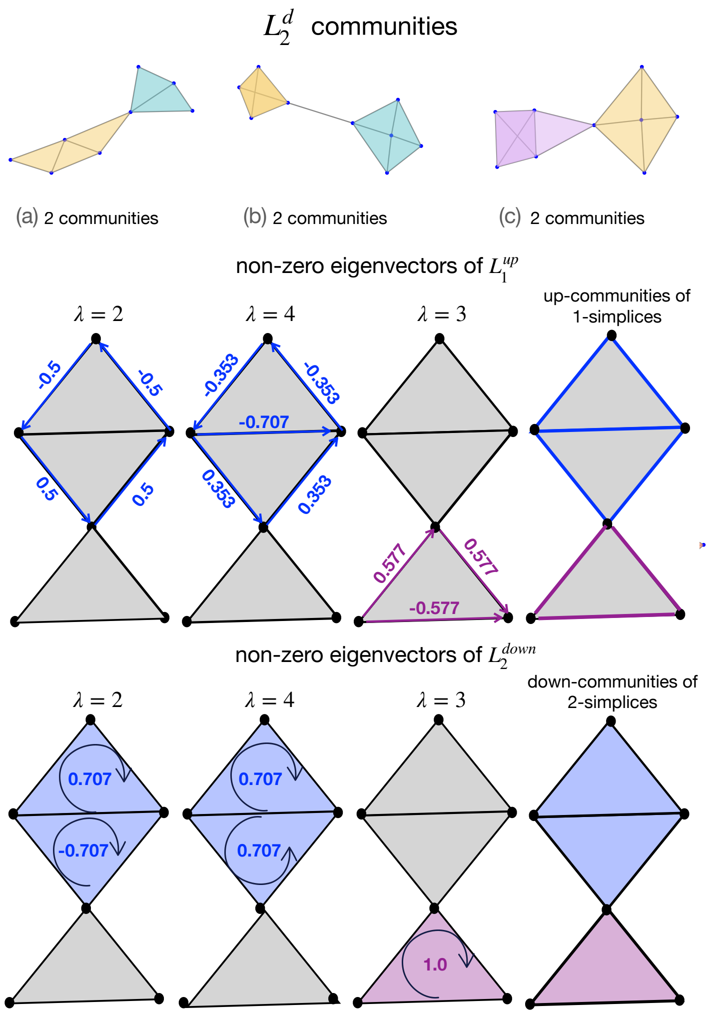
# The skeleton of a simplicial complex and its clique complex



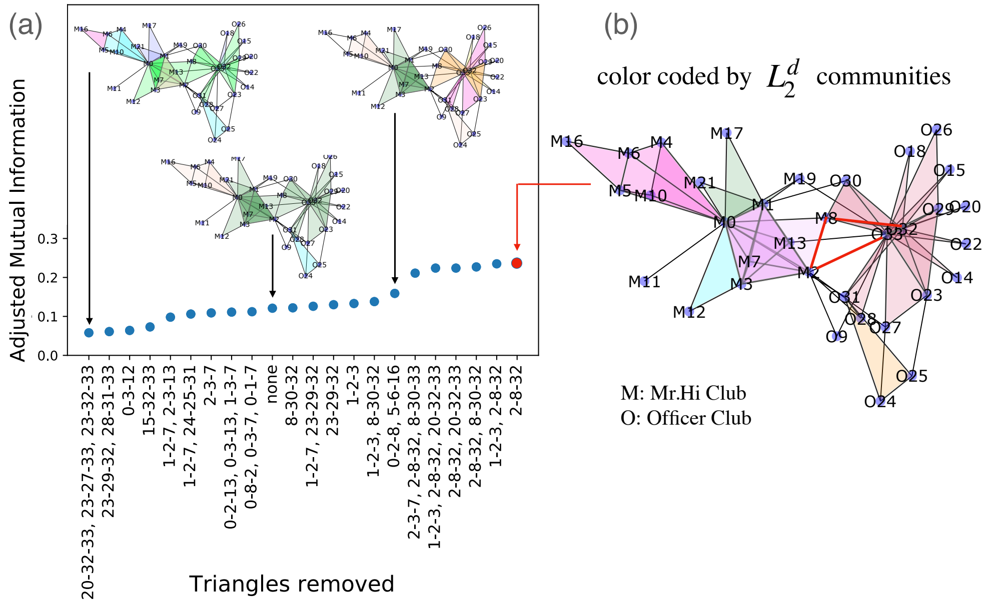

**Attention!**

**By concatenating the operations you are not guaranteed to return to the initial simplicial complex**

## Higher-order communities



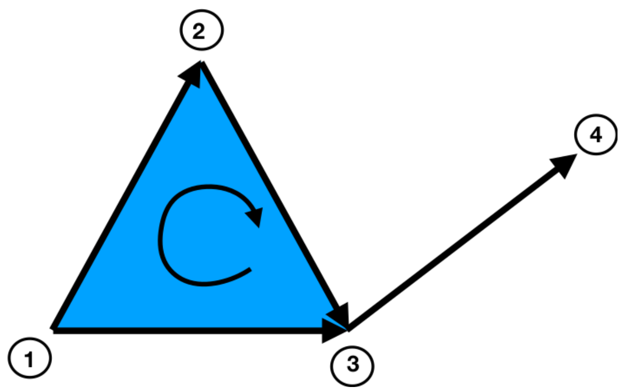

# Inference of higher-order interactions




**We can infer which higher-order interactions using higher-order communities and ground-truth community assignments**

**S. Khrisnagopal and GB (2021)**

# Boundary Operators




**Boundary operators**

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

**The boundary of the boundary is null**

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

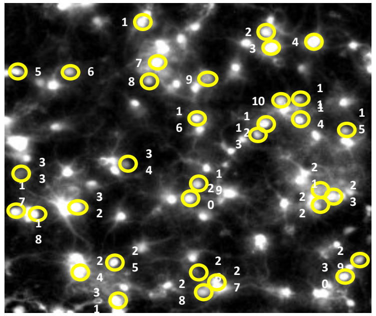
# Lesson IIb:
# Introduction to the Kuramoto model

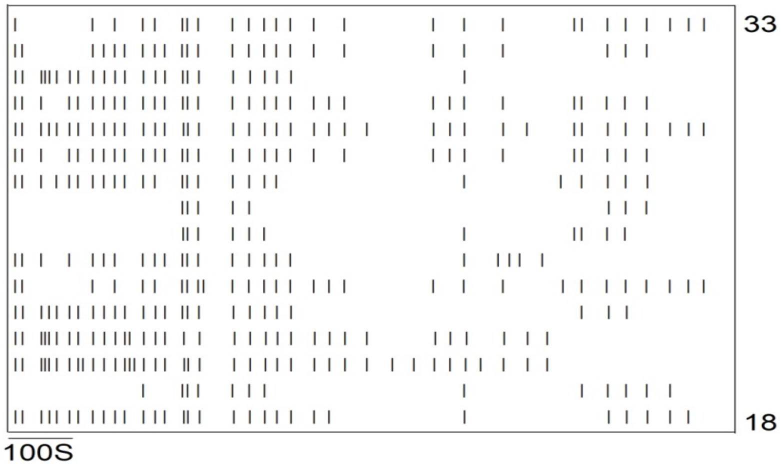
- **Kuramoto model on graphs**
  - **The phase transition**
  - **Gauge Invariance**
  - **Sketch of the solution on fully connected networks**
  - **Annealed approximation for solution on random graphs**

# Kuramoto model on a graph

# Synchronization is a fundamental dynamical process

**NEURONS**

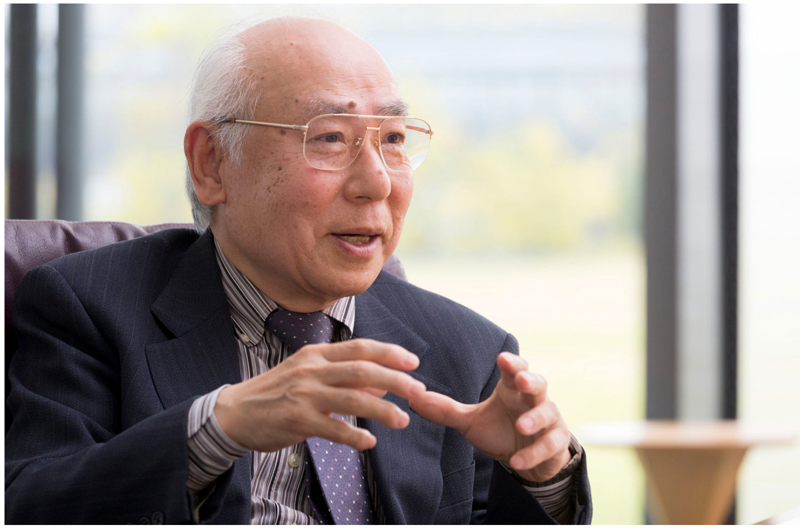





**FIREFLIES**

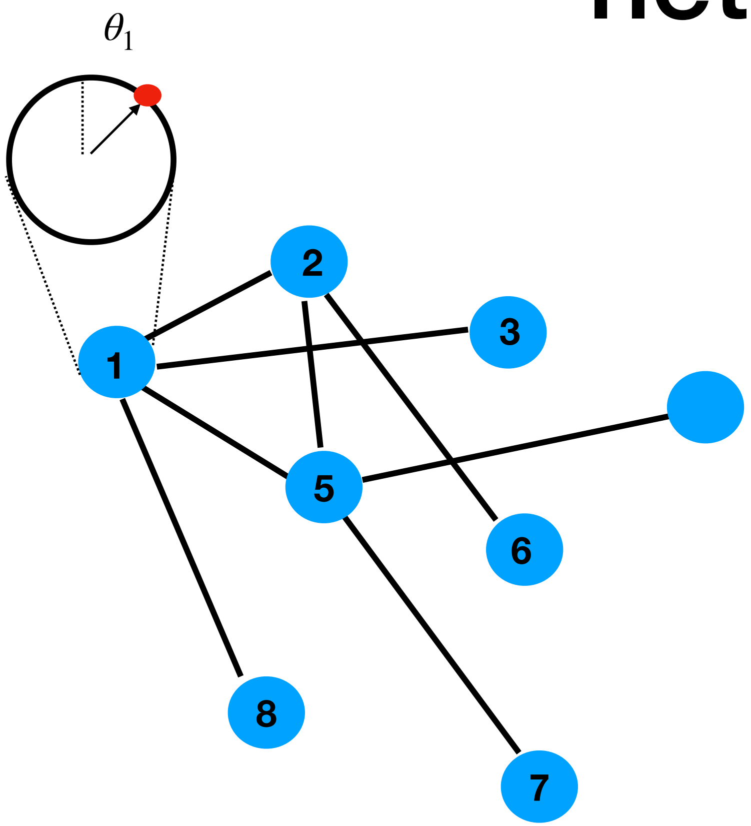
# Founding fathers of synchronisation

**Christiaan Huygens**



**Yoshiki Kuramoto**

# Kuramoto model on a network




**Given a network of N nodes defined by an adjacency matrix a we assign to each node a phase obeying**

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

**where the internal frequencies of the nodes are drawn randomly from**

$$\omega \sim \mathcal{N}(\Omega, 1)$$

**and the coupling constant is** $\sigma$

**The oscillators are non-identical**

# Order parameter for synchronization

We consider the global order parameter $R$

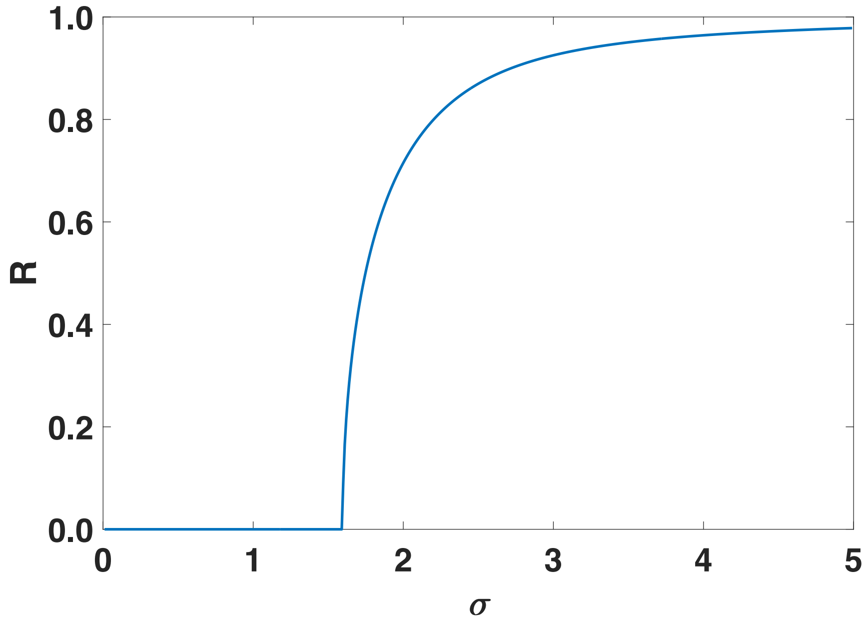
$$X = Re^{\mathrm{i}\hat{\Psi}} = \frac{1}{N}\sum_{r=1}^{N} e^{\mathrm{i}\theta_{\mathrm{r}}}$$

which indicates the

synchronisation transition such that for

$$|\sigma - \sigma_c| \ll 1$$

$$R = \begin{cases} 0 & \textbf{for } \sigma < \sigma_c \\ c(\sigma - \sigma_c)^{1/2} & \textbf{for } \sigma \geq \sigma_c \end{cases}$$




**Kuramoto (1975)**

# Gauge invariance of the Kuramoto equation

Given the Kuramoto dynamics

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

If we perform the transformation

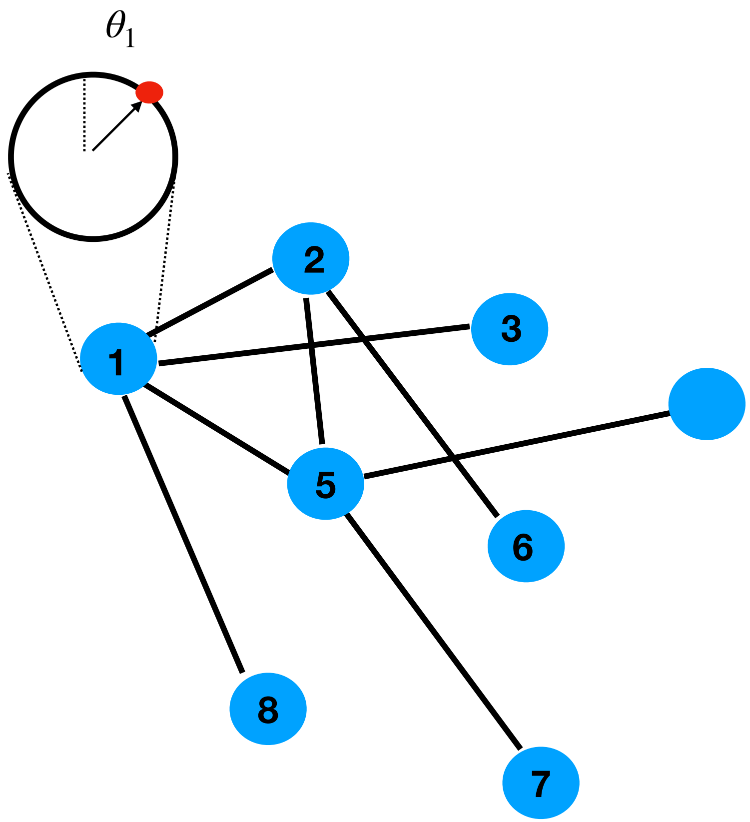
$$\theta_r \rightarrow \theta_r - \hat{\Omega} t$$

We obtain

$$\dot{\theta}_r = \omega_r - \hat{\Omega} + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right),$$

i.e. the dynamics is invariant under rescaling

of the average of the intrinsic frequencies , i.e. $\Omega \rightarrow \Omega - \hat{\Omega}$



# Solution of the Kuramoto model on a fully connected network

On a fully connected network the coupling constant  is rescaled as

$$\sigma \rightarrow \frac{\sigma}{N}$$

The Kuramoto equation

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

can be written in terms of the complex order parameter $X$ as

$$\dot{\theta}_r = \omega_r - \Omega + \sigma \mathrm{Im}(X e^{-\mathrm{i}\theta_{\mathrm{r}}})$$

Thanks to the gauge invariance we can study the dynamics in the rotating frame which reads

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

# Solution of the Kuramoto model on a fully connected network

Looking for the stationary states $\dot{\theta}_r = 0$ of

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

We obtain $\sin(\theta_r) = \dfrac{\omega_r - \Omega}{\sigma R}$ only valid for nodes such that

$$\left| \frac{\omega_r - \Omega}{\sigma R} \right| \leq 1$$

(frozen nodes)

# Solution of the Kuramoto model on a fully connected network

Assuming that only the frozen nodes contribute to the order parameter, since $X = R$ in the rotating frame, we obtain the self-consistent equation for the order parameter

$$R = \frac{1}{N} \sum_{r|r\ are\ frozen} \cos\theta_r = \frac{1}{N} \sum_{r|r\ are\ frozen} \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R}\right)^2}$$

Or, equivalently considering the probability density distribution $g(\omega)$ for the intrinsic frequencies,

$$R = \int_{\left|\frac{\omega-\Omega}{\sigma R}\right| \le 1} g(\omega) \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R}\right)^2} d\omega$$

# Synchronization threshold on a fully connected network

Given the self consistent equation for the order parameter,

$$R = \int_{\left|\frac{\omega-\Omega}{\sigma R}\right|\leq 1} g(\omega)\sqrt{1-\left(\frac{\omega-\Omega}{\sigma R}\right)^2}\,d\omega$$

We derive the synchronization threshold.

Change variable $x = (\omega - \Omega)/(\sigma R)$, $\sigma R dx = d\omega$

$$1 = \sigma \int_{|x|\leq 1} g(\sigma R x + \Omega)\sqrt{1-x^2}\,dx$$

Now we develop $g(y) \simeq g(\Omega) + g''(\Omega)y^2/2$ with $g''(\Omega) < 0$ getting

$$1 = \sigma g(\Omega)\pi/2 + g''(\Omega)\sigma^2 R^2/16$$

# Synchronization threshold on a fully connected network

Starting form

$$1 = \sigma g(\Omega)\pi/2 + g''(\Omega)\sigma^2 R^2/16$$

We found that the synchronization threshold is

$$\sigma_c = \frac{2}{\pi g(\Omega)}$$

and for $|\sigma - \sigma_c| \ll 1$

$$R \simeq \frac{4}{\sigma_c^2 \sqrt{-g''(\Omega)\pi}} \sqrt{\sigma - \sigma_c}$$

# Solution of the Kuramoto model in the annealed approximation

The e Kuramoto model on a random graph with given degree distribution can be studied within the annealed approximation obtained by making the substitution

$$a_{rs} \to p_{rs} = \frac{k_r k_s}{\langle k \rangle N}$$

Therefore the Kuramoto model becomes

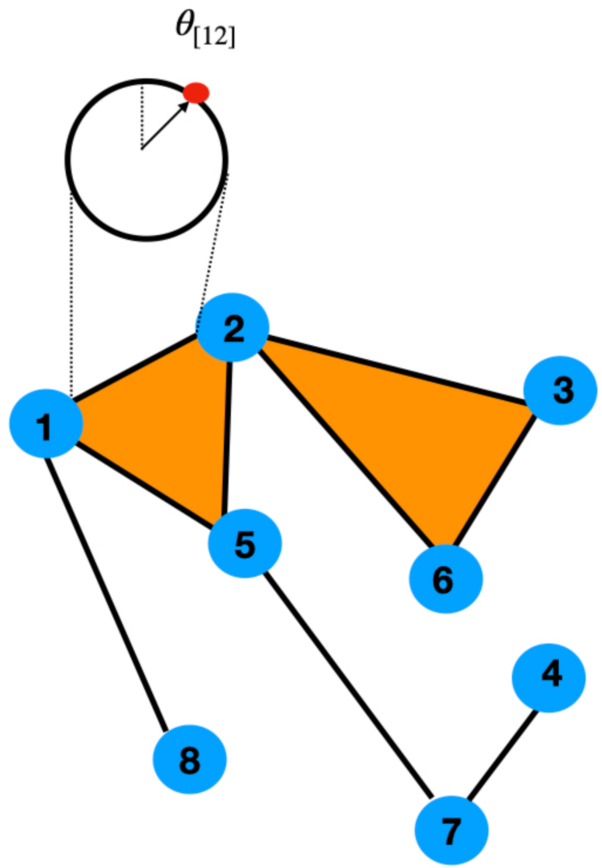
$$\dot{\theta}_r = \omega_r - \sigma \sum_s \frac{k_r k_s}{\langle k \rangle N} \sin(\theta_s - \theta_r)$$

Which can be written as

$$\dot{\theta}_r = \omega_r - \sigma k_r \mathrm{Im} \hat{X} e^{-\mathrm{i}\theta_r} \text{ with } \hat{X} = \frac{1}{\langle k \rangle N} \sum_s k_s e^{-\mathrm{i}\theta_s}$$

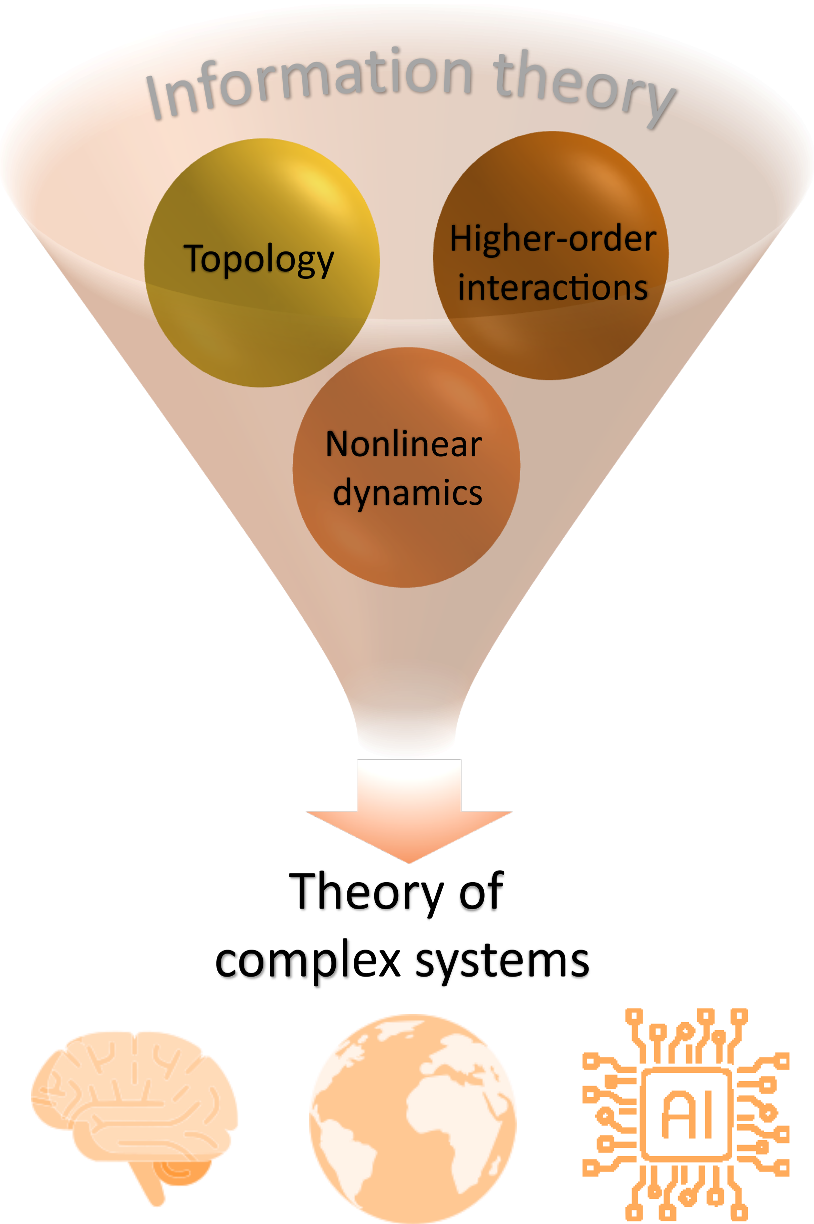
which can be studied following similar steps detailed for the fully connected case.

# The higher-order simplicial Kuramoto model




**How to define the higher-order Kuramoto model coupling higher dimensional topological signals?**

**A. P. Millán, J. J. Torres, and G.Bianconi,**
***Physical Review Letters*, *124*, 218301 (2020)**




# Complexity challenge