Measure Theory: Exercises 3

1. Show that there exists a Borel set $A \subseteq \mathbf{R}$ such that $0 < \lambda(I \cap A) < \lambda(I)$ for every open interval I of finite length.

2. Show that if a set $B \subseteq \mathbf{R}$ satisfies $\lambda^*(B) > 0$ then B includes a set that is not Lebesgue measurable.

3. Let μ and ν be finite measures on some measurable space $(X, \mathcal{A})$. Show that $\mathcal{A}_\mu$ and $\mathcal{A}_\nu$ need not be equal. Prove or disprove: $\mathcal{A}_\mu = \mathcal{A}_\nu$ if and only if μ and ν have exactly the same sets of measure zero.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that for every subset A of X that $\mu^*(A) + \mu_*(X \backslash A) = \mu(X)$.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that if $E_i \subseteq A \subseteq F_i$ and $\mu(F_i \backslash E_i) = 0$ for $i = 1, 2$ then $\mu(E_1) = \mu(E_2) = \mu(F_1) = \mu(F_2)$.

6. Let μ_1 and μ_2 be two complete and finite measures on a space X defined on the same sigma-algebra $\mathcal{A}$ of subsets of X. True or false: $\mu_1 = \mu_2$ if and only if they have the same subsets of measure zero.

7.