Measure Theory Third Week

Theorem (1.4.10):

Let A be a Lebesgue measurable subset of $\mathbf{R}$ such that $\lambda(A) > 0$.

The set diff $(A) := \{x - y \mid x, y \in A\}$ contains an open interval containing 0.

Proof: By inner regularity, we can assume that A is compact.

With $\lambda(A) = r > 0$,

there is an open set B such that B contains A and $\lambda(B) < (1+\epsilon)r$. for any $\epsilon > 0$.

We require that ϵ be less than 1.

As $C := \mathbf{R}\backslash B$ is closed, disjoint from A

and thus has a positive distance d to the compact set A,

$A + \delta$ is contained in B for all δ satisfying $|\delta| < d$.

But if there were no overlap between the sets A and $A + \delta$ for any $|\delta| < d$,

then by translation invariance $A \cup (A + \delta)$ would be a Lebesgue measurable set of measure $2r$ inside of B,

which is impossible since $\lambda(B) < (1 + \epsilon)r$.

So for any given δ with $|\delta| < d$ there is an $a \in A \cap A + \delta$,

meaning that $a = a' + \delta$ for some other $a' \in A$ and $\delta = a - a'$. $\square$

We see that for every ϵ there is a d such that all but an ϵ fraction of the set A is used to get the difference set to include $(-d, d)$.

Theorem (1.4.11): Assuming the Axiom of Choice, there is a partition of $\mathbf{R}$ into two parts A, B,

meaning $A \cap B = \emptyset$ and $A \cup B = \mathbf{R}$,

such that for every finite interval I :

$\lambda^*(A \cap I) = \lambda^*(B \cap I) = \lambda^*(I)$ and

every Lebesgue measurable subset C either contained in either A or B has measure zero.

Note: The natural idea, a ring homomorphism from $\mathbf{R}$ to $\mathbf{Z}_2$ and letting $A = \phi^{-1}(0)$ and $B = \phi^{-1}(1)$, is not possible,

whenever $\phi(r) = 1$ then what should be $\phi(\frac{r}{2})$?

Need a group homomorphism.

Proof:

Let $W = \mathbf{Q} + \mathbf{Z}\sqrt{2}$,

$\phi : W \to \mathbf{Z}_2$ is defined by

$\phi(\frac{a}{b} + n\sqrt{2}) = n(\text{mod } 2)$.

Because $\sqrt{2}$ is irrational, ϕ is well defined and a group homomorphism:

$\frac{a}{b} + n\sqrt{2} = \frac{a'}{b'} + n'\sqrt{2} \Rightarrow \sqrt{2} = \frac{a'}{b'(n-n')} - \frac{a}{b(n-n')} \in \mathbf{Q}$, a contradiction when $n \neq n'$.

Also both $G_0 := \phi^{-1}(0) \subset W$ and $G_1 := \phi^{-1}(1) \subset W$ are dense in $\mathbf{R}$, (and this can be shown with the Euclidean algorithm on the pair 1 and $\sqrt{2}$ via smaller and smaller ways to write $\frac{a}{b} + n\sqrt{2}$ with both even and odd n).

Define an equivalence relation $\sim$ by

$r \sim s$ if and only if $r - s \in W$.

Let E be a set such that $|E \cap C| = 1$ (via Axiom of Choice) for every equivalence class C.

For every $r \in \mathbf{R}$, $\quad r = e + \frac{a}{b} + n\sqrt{2}$,

for some $e \in E$, $a, b \in \mathbf{Z}$, $n \in \mathbf{Z}$, and uniquely so.

A is the subset where n used is even and B is the subset where n used is odd.

A and B are well defined because r cannot equal $e' + \frac{a'}{b'} + n'\sqrt{2}$ for any other choices,

as then $e - e'$ would be in W and e and e' would belong to the same equivalence class.

Assume that either A or B contained a Lebesgue measurable set of positive measure.

Either $A - A$ or $B - B$ must contain an open interval and hence some member of the dense set G_1, in other words

$$\frac{a_0}{b_0}+n_0\sqrt{2} = e_1+\frac{a_1}{b_1}+n_1\sqrt{2}-e_2-\frac{a_2}{b_2}-n_2\sqrt{2}$$

with n_0 odd, both n_1 and n_2 either even or odd, and $e_1, e_2 \in E$.

As e_1 and e_2 must be equal (otherwise they would represent the same equivalence relation), $n_0 = n_1 - n_2$ would be a contradiction.

Now suppose that either $A \cap I$ or $B \cap I$ has an outer Lebesgue measure less than I for some finite interval I.

That means $A \cap I$ or $B \cap I$ can be covered by some open set of measure strictly less than I.

implying that either $I \setminus A = I \cap B$ or $I \setminus B = I \cap A$ contains a closed set of positive measure, which, by the above, neither does. $\square$

The same is true for three or more sets, but is much more difficult to show.

A measure μ of a measure space $(X, \mathcal{A}, \mu)$ is *complete*

if $A \in \mathcal{A}$, $\mu(A) = 0$ and $B \subseteq A$ imply that $B \in \mathcal{A}$.

With $(X, \mathcal{A}, \mu)$ a measure space,

the completion $\mathcal{A}_\mu$ is the collection of subsets A

for which there are sets $E, F \in \mathcal{A}$

with $E \subseteq A \subseteq F$ and $\mu(F \backslash E) = 0$.

The completion $\overline{\mu}$ is the measure defined on $\mathcal{A}_\mu$

such that $\overline{\mu}(A) = \mu(E) = \mu(F)$.

This is well defined as there cannot be two such levels (otherwise monotonicity is violated):

Suppose $\mu(F') = \mu(E') > \mu(E) = \mu(F)$ with $E' \subseteq A \subseteq F'$, $E \subseteq A \subseteq F$, $\mu(F' \backslash E') = \mu(F \backslash E) = 0$ with all four sets in $\mathcal{A}$.

It follows that $E' \subseteq A \subseteq F$ and by comtainment $\mu(E') \leq \mu(F)$, a contradiction.

Lemma (1.5.1): Let $(X, \mathcal{A}, \mu)$ be a measure space.

$\mathcal{A}_\mu$ is a σ-algebra on X that includes $\mathcal{A}$

and $\overline{\mu}$ is a measure defined on $\mathcal{A}_\mu$ that is complete.

Proof: Containment of $\mathcal{A}$ in $\mathcal{A}_\mu$ and closure by complementation are trivial.

If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{A}_\mu$

and E_i and F_i are sequences in $\mathcal{A}$

with $\forall\ i \;\; E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \backslash E_i) = 0$

then by countable additivity

$0 = \sum_{i=1}^{\infty} \mu(F_i \backslash E_i) \geq \mu(\cup_{i=1}^{\infty}(F_i \backslash E_i)) \geq \mu(\cup_{i=1}^{\infty} F_i \backslash \cup_{i=1}^{\infty} E_i) \geq 0,$

implying that $\cup_{i=1}^{\infty} A_i \in \mathcal{A}_\mu$.

And if the $A_1, A_2, \ldots$ are disjoint the same pairs E_i and F_i of sequences show that

$\sum_{i=1}^{\infty} \mu(F_i) = \sum_{i=1}^{\infty} \mu(E_i) \leq \overline{\mu}(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(F_i),$

hence equality and countable additivity. $\square$

Let $(X, \mathcal{A}, \mu)$ be a measure space,

and A any subset of X.

$\mu^*(A) = \inf\{\mu(B) \mid A \subseteq B,\ B \in \mathcal{A}\}$ and

$\mu_*(A) = \sup\{\mu(B) \mid A \supseteq B,\ B \in \mathcal{A}\}$.

$\mu^*(A)$ is the outer measure and $\mu_*(A)$ is the inner measure.

Lemma: μ^* is an outer measure.

Proof: $\mu^*(\emptyset) = 0$ and monotonicity are trivial.

Let $A_1, A_2, \ldots$ be a sequence of sets.

Suppose that $\sum_{i=1}^{\infty} \mu^*(A_i) < \infty$:

For every $i = 1, 2, \ldots$ let B_i be a set in $\mathcal{A}$ containing A_i

such that $\mu(B_i) \leq \mu^*(A_i) + \frac{\epsilon}{2^i}$.

$B = \cup_{i=1}^{\infty} B_i$ includes $A = \cup_{i=1}^{\infty} A_i$ and

$\sum_{i=1}^{\infty} \mu^*(A_i) \geq \sum_{i=1}^{\infty} \mu(B_i) - \epsilon \geq \mu(B) - \epsilon \geq \mu^*(A) - \epsilon$.

True for every ϵ implies the inequality. $\square$

Lemma (1.5.5) Given that $\mu^*(A) < \infty$, A belongs to $\mathcal{A}_\mu$ if and only if $\mu_*(A) = \mu^*(A)$.

Proof: $\Rightarrow$ If A belongs to $\mathcal{A}_\mu$ then there are sets $E, F \in \mathcal{A}$ such that $E \subseteq A \subseteq F$ and $\mu(F \backslash E) = 0$.

From $\mu(E) \leq \mu_*(A) \leq \mu^*(A) \leq \mu(F)$

all are equal.

$\Leftarrow$ On the other hand, if $\mu_*(A) = \mu^*(A) < \infty$

there are sequences of sets $E_1, E_2, \ldots$ and $F_1, F_2, \ldots$

with $E_i \subseteq A$ and $A \subseteq F_i$ and

$\mu(E_i) \geq \mu_*(A) - \frac{1}{2^i}$ and $\mu^*(A) + \frac{1}{2^i} \geq \mu(F_i)$.

The sets $E = \cup_i^\infty E_i$ and $F = \cap_i^\infty F_i$

are both in $\mathcal{A}$ and have the same common measure size $\mu^*(A) = \mu_*(A)$, with $E \subseteq A \subseteq F$.