Topology, geometry and dynamics of higher-order networks

An introduction to simplicial complexes
Lesson III

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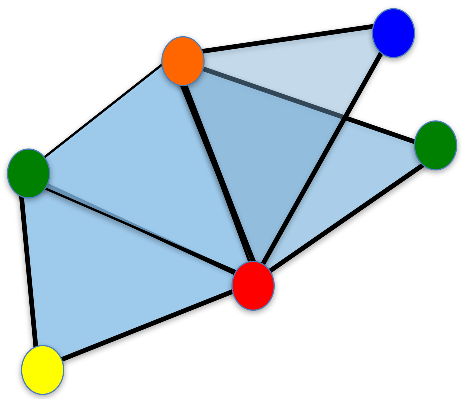
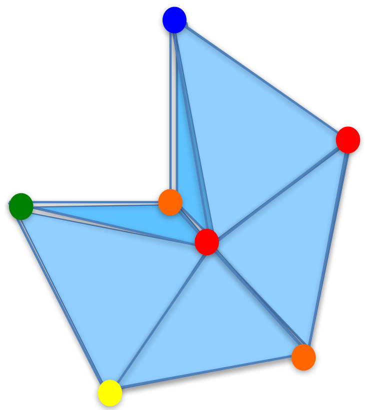
Higher-order networks

Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



d=2 simplicial complex

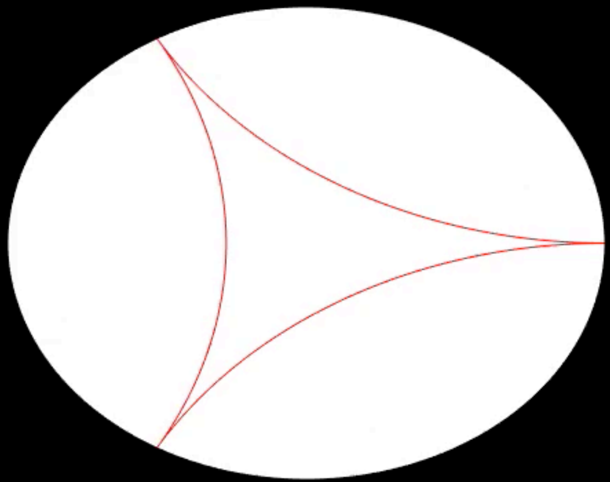


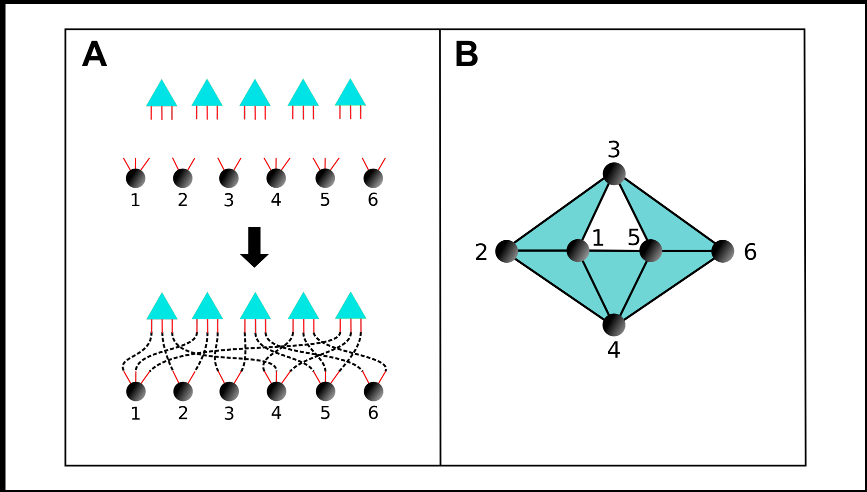
d=3 simplicial complex

Simplicial complex models

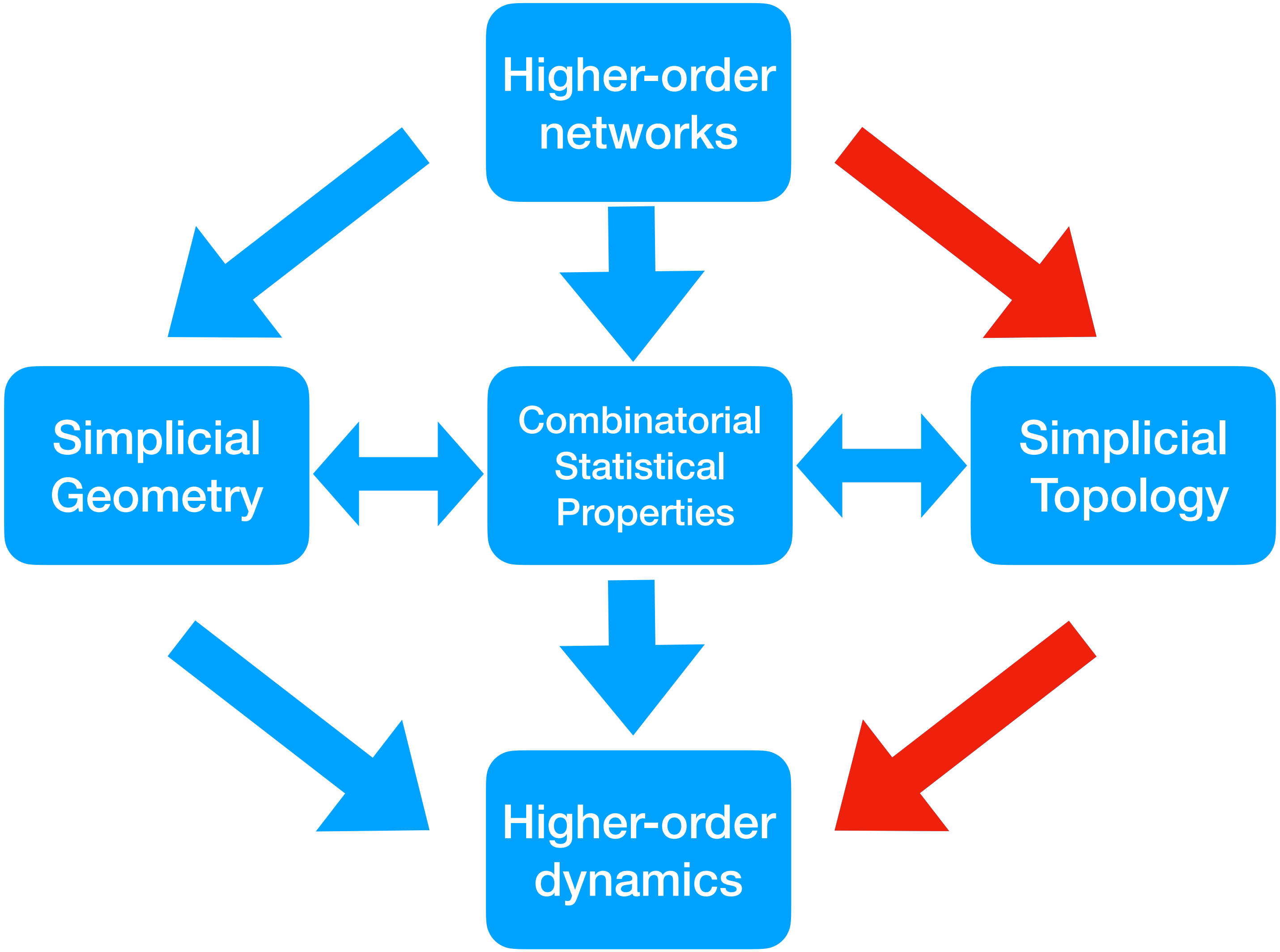
Emergent Geometry
Network Geometry with Flavor (NGF)
[Bianconi Rahmede ,2016 & 2017]

Maximum entropy model
Configuration model
of simplicial complexes
[Courtney Bianconi 2016]



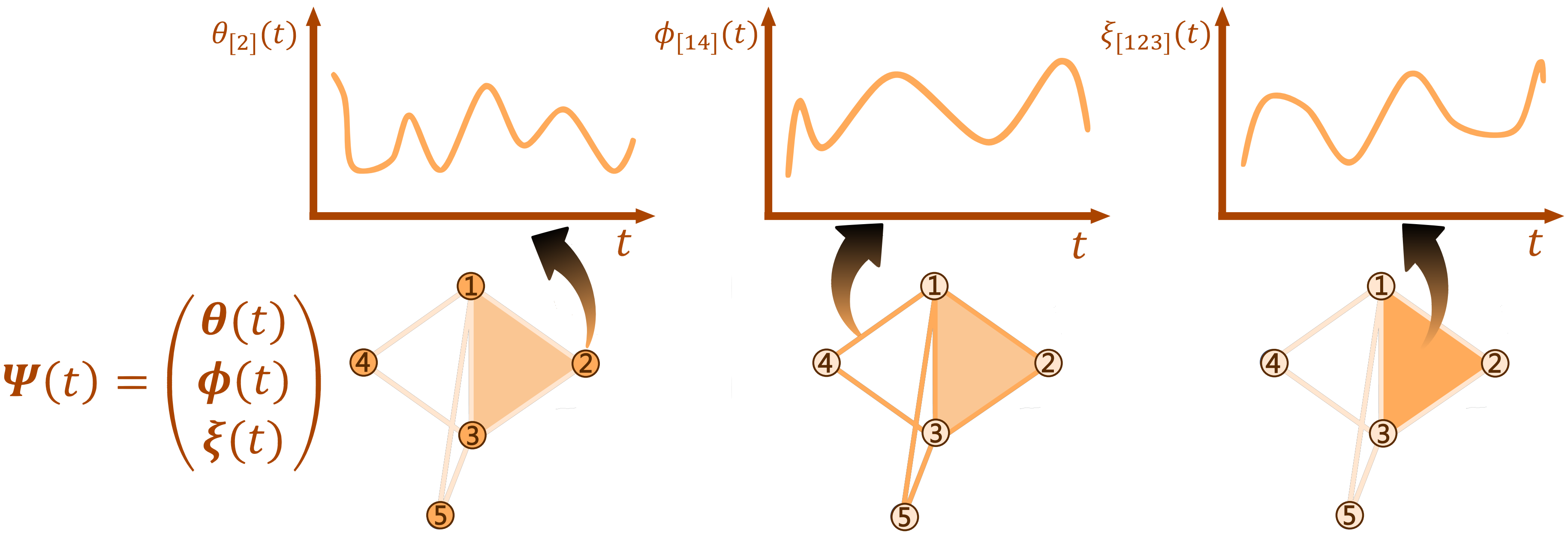


Higher-order structure and dynamics



Topological signals

Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals



Topological signals

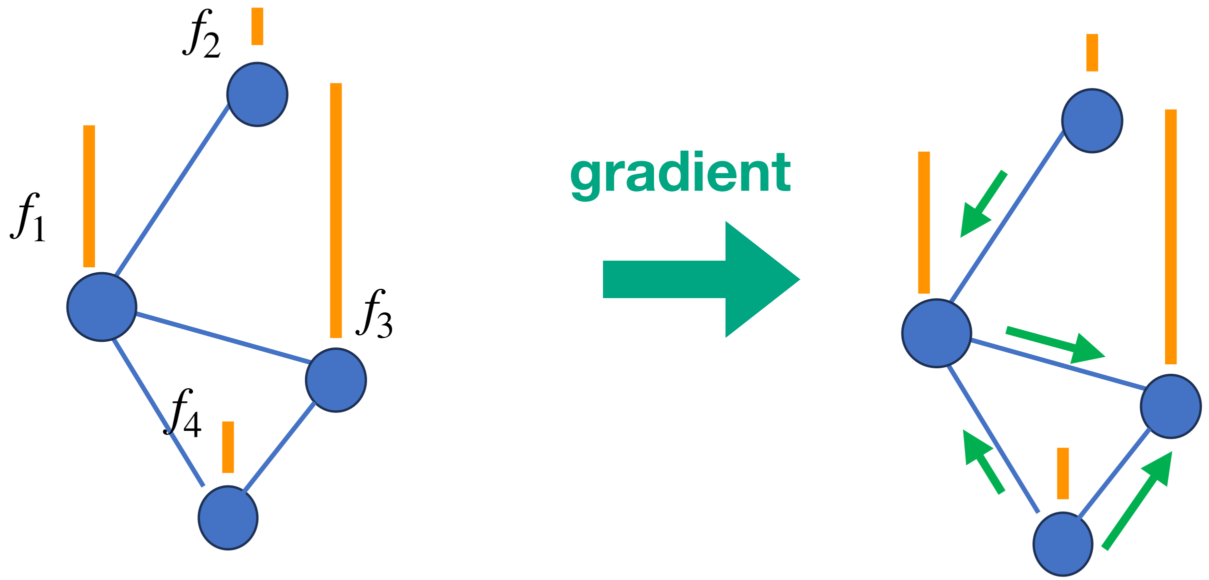
- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are cochains or vector fields

Discrete Gradient

If $f \in C^0$, then $g = \delta_0 f \in C^1$ indicates its discrete gradient

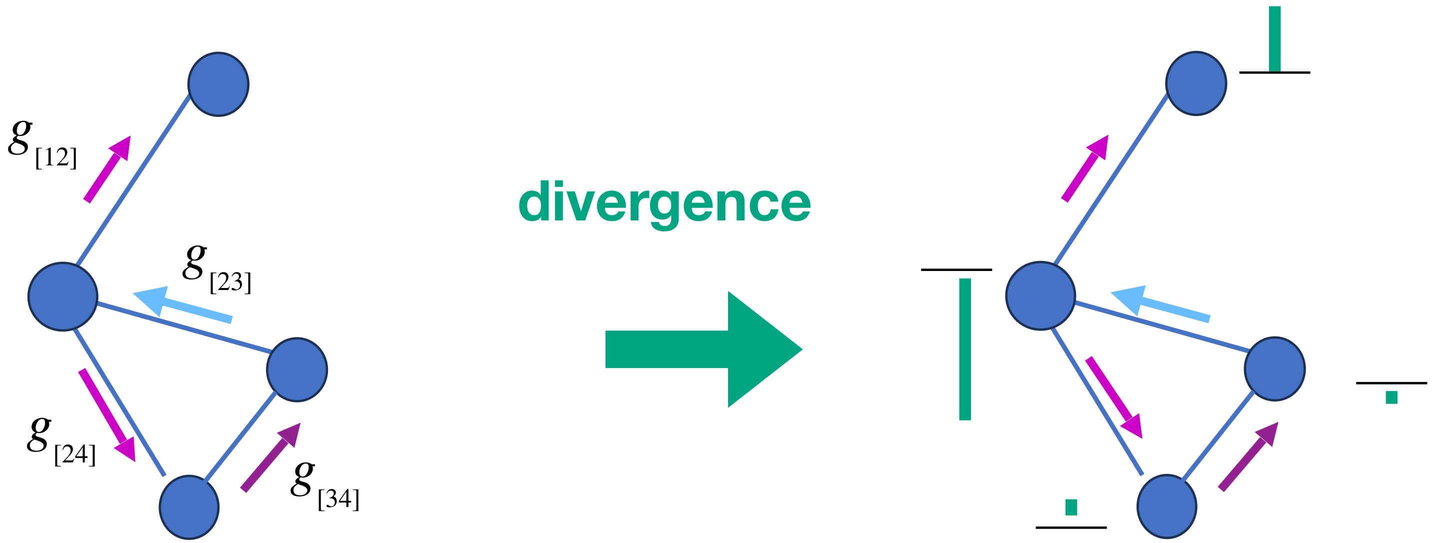
Indeed we have $\mathbf{g} = \mathbf{B}_{[1]}^{\top}\mathbf{f}$ which implies $g_{[r,s]} = f_s - f_r$



Discrete Divergence

If $g \in C^1$, then $f = \delta_0^* g \in C^0$ indicates its discrete divergence

Indeed we have $\mathbf{f} = \mathbf{B}_{[1]}\mathbf{g}$ which implies $f_r = \sum_s g_{[sr]} - \sum_s g_{[rs]}$

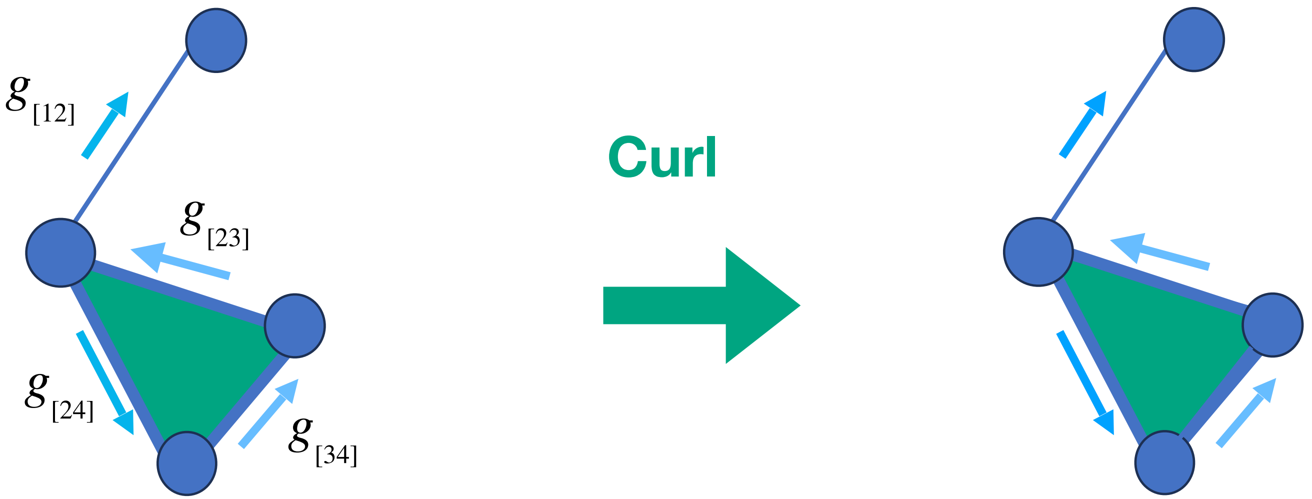


Discrete Curl

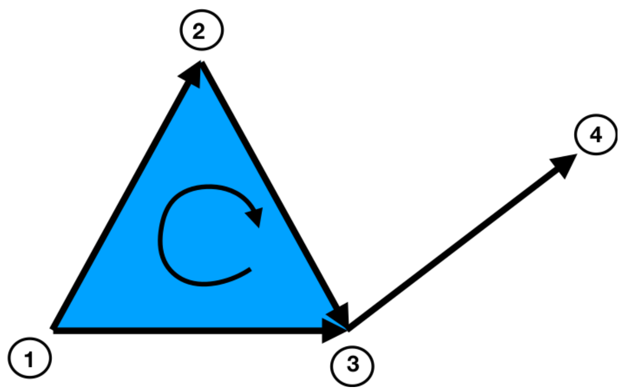
If $f \in C^1$, then $h = \delta_1 g \in C^2$ indicates its discrete curl

Indeed we have $\mathbf{g} = \mathbf{B}_{[2]}^{\top}\mathbf{f}$ which implies

$$h_{[r,s,q]} = g_{[r,s]} + g_{[s,q]} - g_{[r,q]}$$



Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.$$

$\mathbf{B}_{[1]}$ Discrete divergence

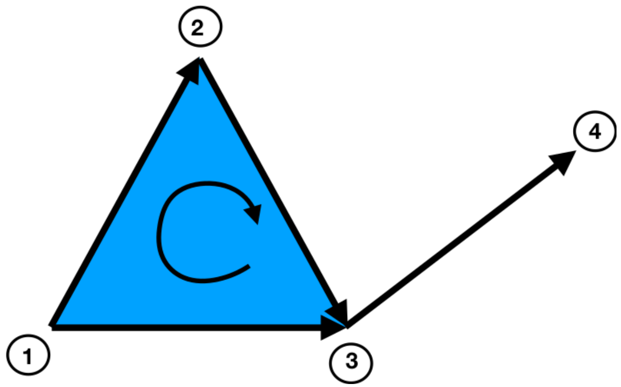
$\mathbf{B}_{[1]}^{\top}$ Discrete gradient

$\mathbf{B}_{[2]}^{\top}$ Discrete Curl

The boundary of the boundary is null

$$\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}, \quad \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n-1]}^{\top} = \mathbf{0}$$

Simplicial complexes and Hodge Laplacians



Hodge Laplacians

The Hodge Laplacians describe diffusion

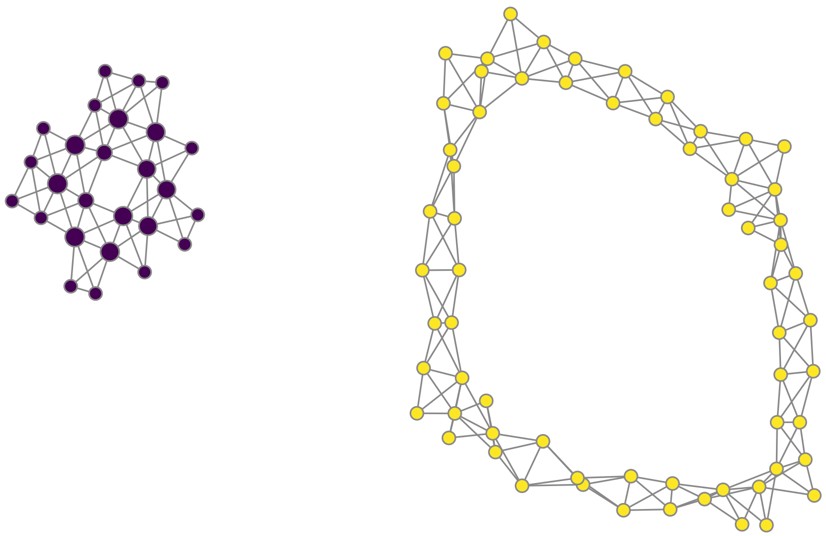
from n-simplices to n-simplices through (n-1) and (n+1) simplices

The Hodge Laplacian are semi-definite positive

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^{\top} \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^{\top}\mathbf{B}_{[2]}$$

The dimension of the kernel of the Hodge Laplacian $\mathrm{L}_{[m]}$ is given by the m Betti number β_m

Harmonic eigenvectors of the graph Laplacian



The quadratic form of the graph Laplacian reads

$$\mathbf{X}^\top \mathbf{L}_{[0]} \mathbf{X} = \frac{1}{2} \sum_{r,s} a_{rs} (X_r - X_s)^2$$

Therefore the harmonic eigenvectors of the graph Laplacian are constant on each connected component of the graph and zero everywhere else.

Harmonic eigenvectors of the Hodge Laplacian

The dimension of the kernel of the Hodge Laplacian

is given by the corresponding Betti number

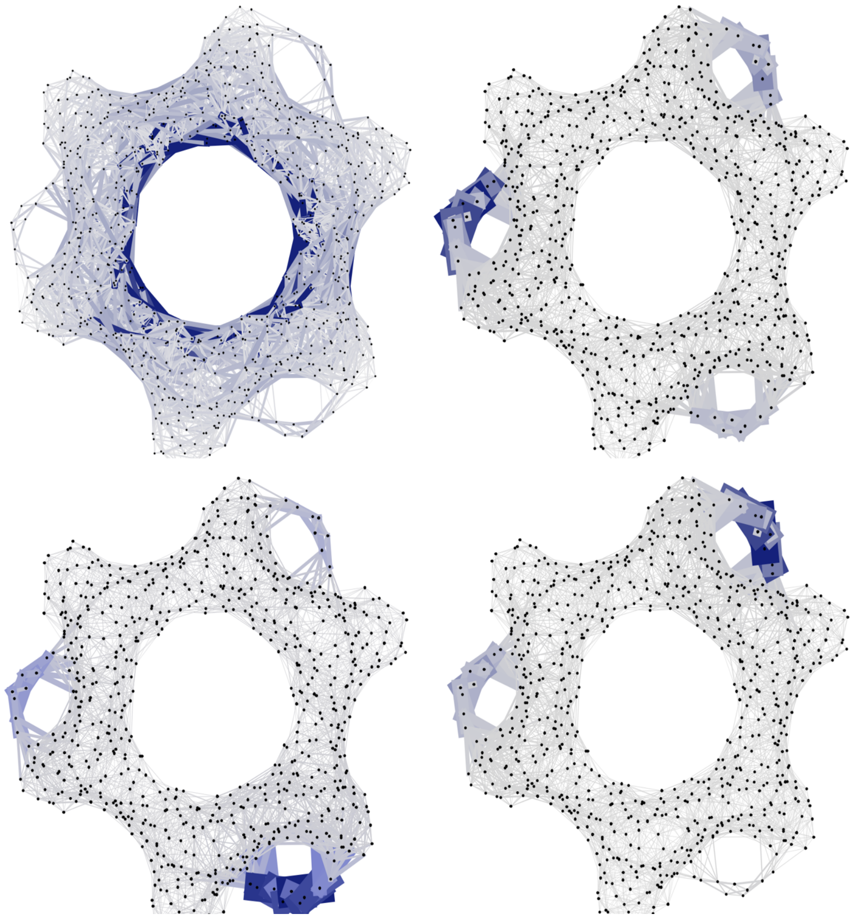
$$\textbf{dim ker}(\mathbf{L}_{[m]}) = \beta_m$$

The harmonic eigenvectors

are associated to the generators of the homology

They are in general non-uniform over the m-simplices of the simplicial complex

Harmonic eigenvectors



Wee et al. (2023)

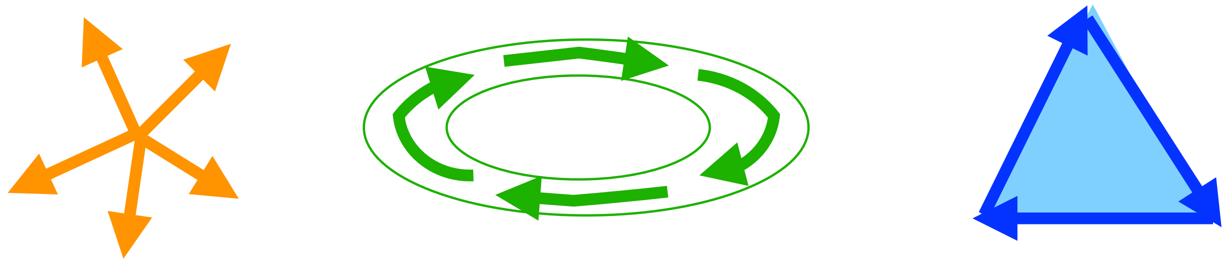
Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

in a irrotational, harmonic and solenoidal components

$$\boxed{\mathbb{R}^{D_m} = \textbf{im}(\mathbf{B}_{[m]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})}$$

which in the case of topological signals of the links can be sketched as



Irrotational component
Gradient Flow

Harmonic component

Solenoidal component
Curl Flow

Lesson III:
The Topological Kuramoto model

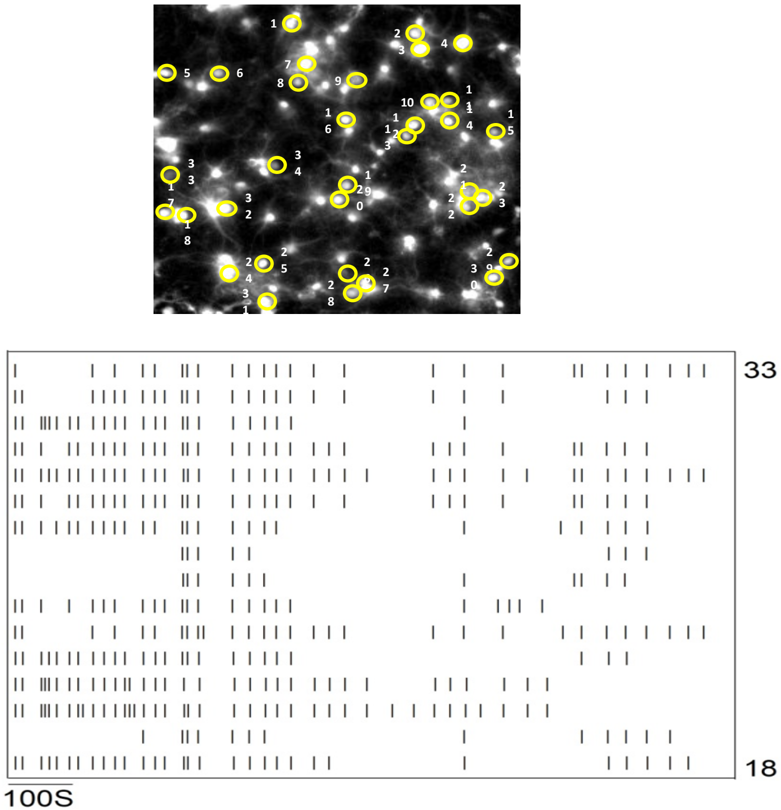
- **Brief recap of the Kuramoto model on graphs**
- **The Kuramoto model on graph and its relation to topology**
  - **Gauge invariance**
  - **Spatial distribution of synchronized dynamics (harmonic component)**

- **The higher-order Topological Kuramoto model**
  - **Gauge invariance**
  - **The Topological Kuramoto model as a gradient flow**
  - **Spatial distribution of synchronized dynamics (harmonic component)**
  - **Signal projected one dimension up or down(solenoidal and irrotational component)**
  - **Solution on a fully connected network**

Kuramoto model on a graph

Synchronization is a fundamental dynamical process

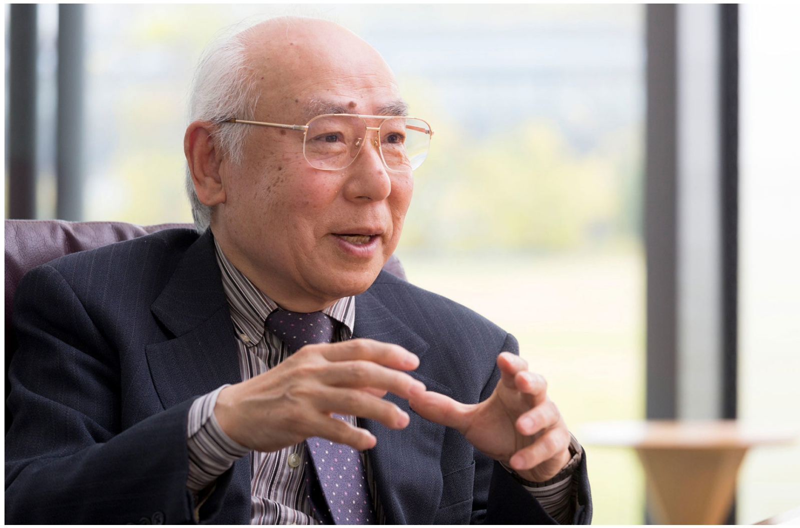
NEURONS



FIREFLIES

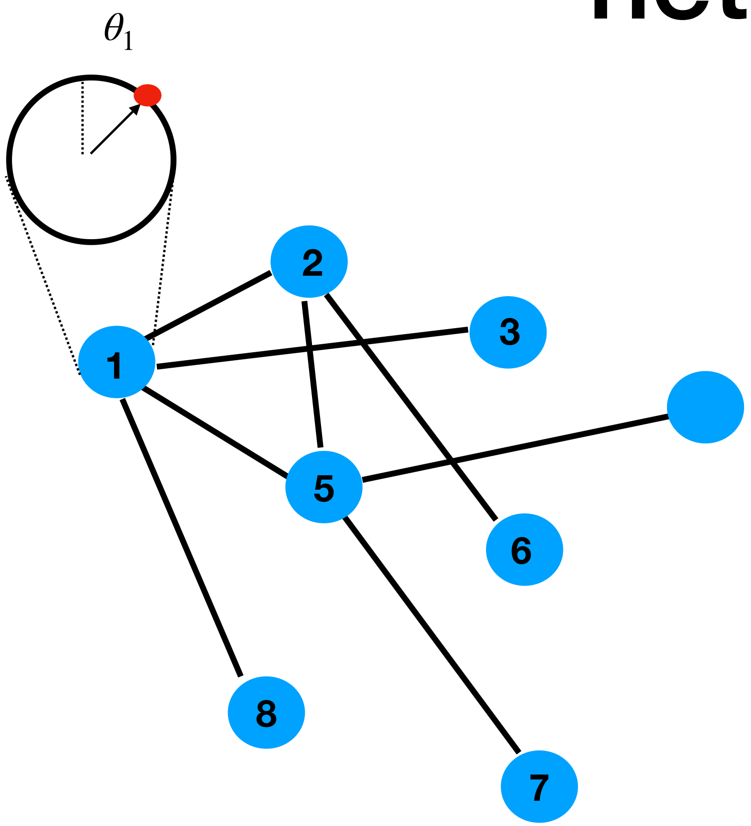
Founding fathers of synchronisation

Christiaan Huygens



Yoshiki Kuramoto

Kuramoto model on a network



Given a network of N nodes defined by an adjacency matrix a we assign to each node a phase obeying

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

where the internal frequencies of the nodes are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

The oscillators are non-identical

Order parameter for synchronization

We consider the global order parameter R

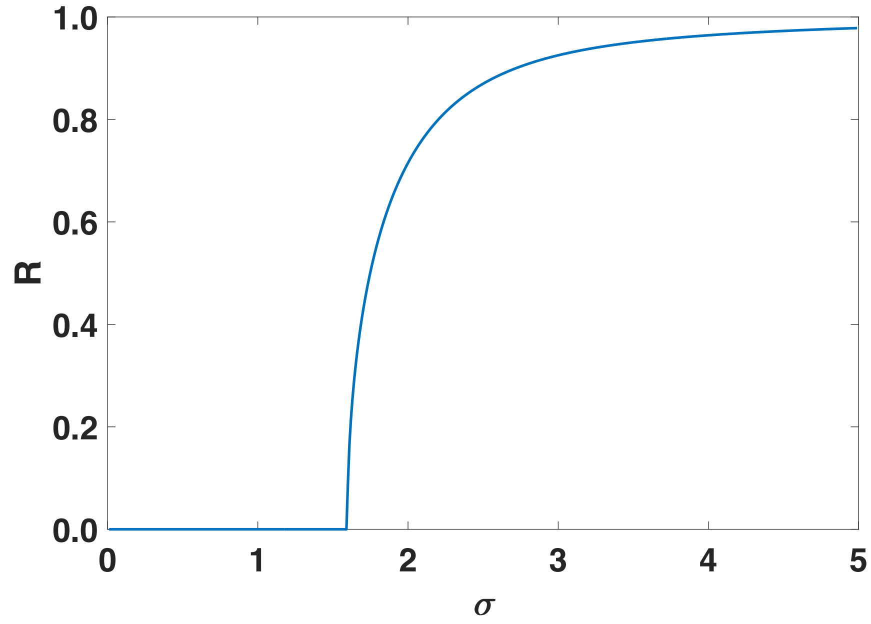
$$X = Re^{\mathrm{i}\hat{\Psi}} = \frac{1}{N}\sum_{r=1}^{N} e^{\mathrm{i}\theta_\mathrm{r}}$$

which indicates the

synchronisation transition such that for

$$|\sigma - \sigma_c| \ll 1$$

$$R = \begin{cases} 0 & \textbf{for } \sigma < \sigma_c \\ c(\sigma - \sigma_c)^{1/2} & \textbf{for } \sigma \geq \sigma_c \end{cases}$$



Kuramoto (1975)

Gauge invariance of the Kuramoto equation

Given the Kuramoto dynamics

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

If we perform the transformation

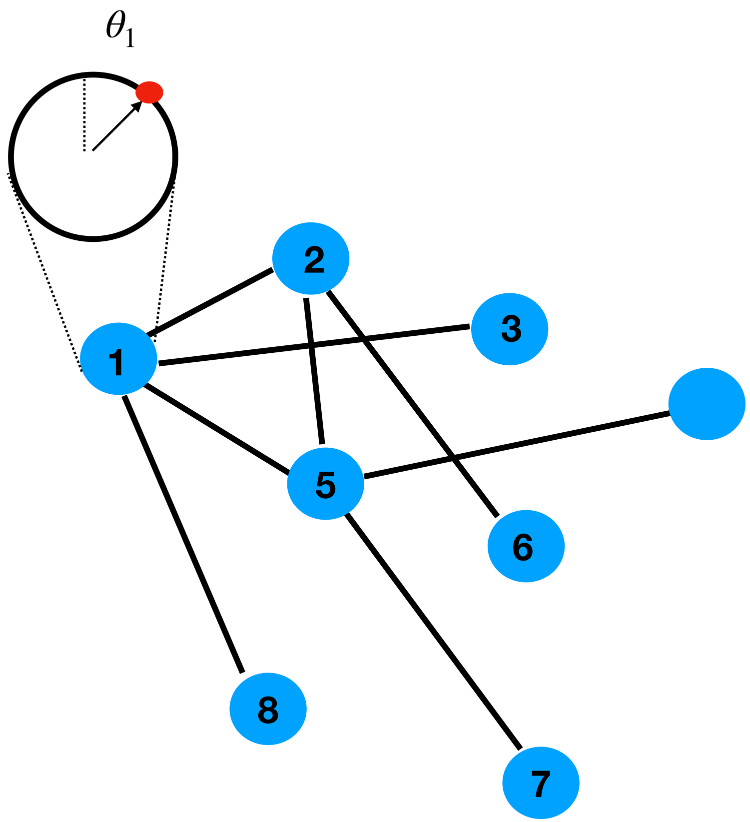
$$\theta_r \rightarrow \theta_r - \hat{\Omega} t$$

We obtain

$$\dot{\theta}_r = \omega_r - \hat{\Omega} + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right),$$

i.e. the dynamics is invariant under rescaling

of the average of the intrinsic frequencies , i.e. $\Omega \rightarrow \Omega - \hat{\Omega}$



The standard Kuramoto model
Under the lens of Topology

Standard Kuramoto model in terms of boundary matrices

The standard Kuramoto model, can be expressed in terms

of the boundary matrix $\mathbf{B}_{[1]}$ as

$$\boxed{\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}}$$

where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

The standard Kuramoto model in terms of boundary matrices

Let us show that the Kuramoto equations

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

can be also written in matrix form as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using the explicit expression of the elements of the boundary matrix $\mathbf{B}_{[1]}$

$$[B_{[1]}]_{r\ell} = \begin{cases} -1 & \text{if } \ell = [r, s] \\ 1 & \text{if } \ell = [s, r] \\ 0 & \text{otherwise} \end{cases}$$

Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_r = \omega_r - \sigma \sum_{\ell} [B_{[1]}]_{r\ell} \sin \left(\sum_{s'} [B_{[1]}]_{\ell s'} \theta_{s'} \right)$$

For the link $\ell = [r, s]$ we obtain

$$[B_{[1]}]_{r\ell} \sin \left(\sum_{s'} [B_{[1]}]_{\ell s'} \theta_{s'} \right) = -a_{rs} \sin(\theta_s - \theta_r)$$

Proof

To prove the above statement we write element wise the equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

obtaining

$$\theta_r = \omega_r - \sigma \sum_{\ell} [B_{[1]}]_{r\ell} \sin\left(\sum_{s} [B_{[1]}]_{\ell s} \theta_s \right)$$

For the link $\ell = [s, r]$ we obtain

$$[B_{[1]}]_{r\ell} \sin\left(\sum_{s'} [B_{[1]}]_{\ell s'} \theta_{s'} \right) = a_{rs} \sin(\theta_r - \theta_s) = -a_{rs} \sin(\theta_s - \theta_r)$$

Gauge invariance of the Kuramoto equation

Given the Kuramoto dynamics

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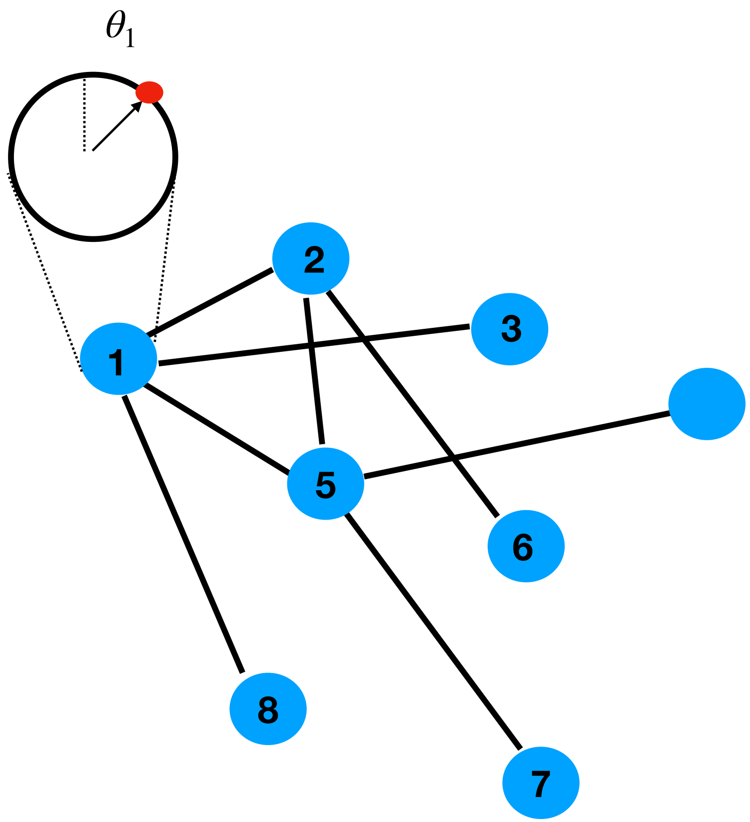
i.e. the dynamics is invariant under rescaling

of the average of the intrinsic frequencies , i.e. $\Omega \to \Omega - \hat{\Omega}$



Gauge invariance of the Kuramoto equation

Given the Kuramoto dynamics

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + \sigma \mathbf{B}_{[1]} \sin(\mathbf{B}_{[1]}^\top \boldsymbol{\theta})$$

If we perform the transformation

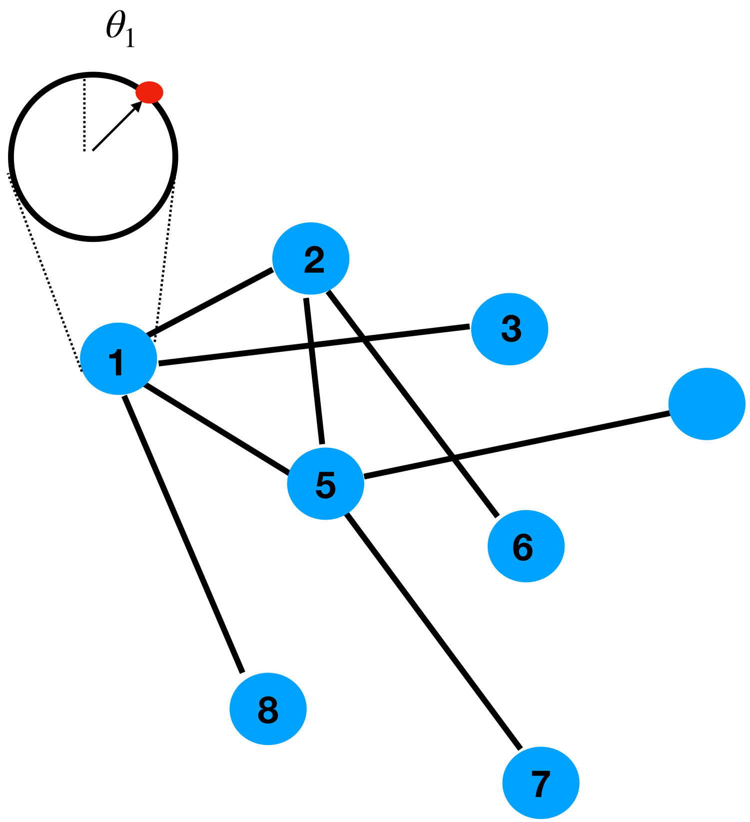
$$\boldsymbol{\theta} \to \boldsymbol{\theta} - \hat{\Omega} t \mathbf{1}$$

We obtain

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \hat{\Omega}\mathbf{1} + \sigma \mathbf{B}_{[1]} \sin(\mathbf{B}_{[1]}^\top \boldsymbol{\theta})$$

i.e. the dynamics is invariant under rescaling

$$\boldsymbol{\omega} \to \boldsymbol{\omega} - \hat{\Omega}\mathbf{1}$$



Dynamics learns topology

The Kuramoto model

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta},$$

In the Kuramoto model the free dynamics is localised on the constant (Harmonic) eigenvector

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\theta} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is constant in each connected component

Standard Kuramoto model

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

In the Standard Kuramoto model the free dynamics is uniform over the whole (connected) network

Linearised dynamics

Let us investigate the linearisation of the Kuramoto dynamics.

Let us start from the nonlinear system

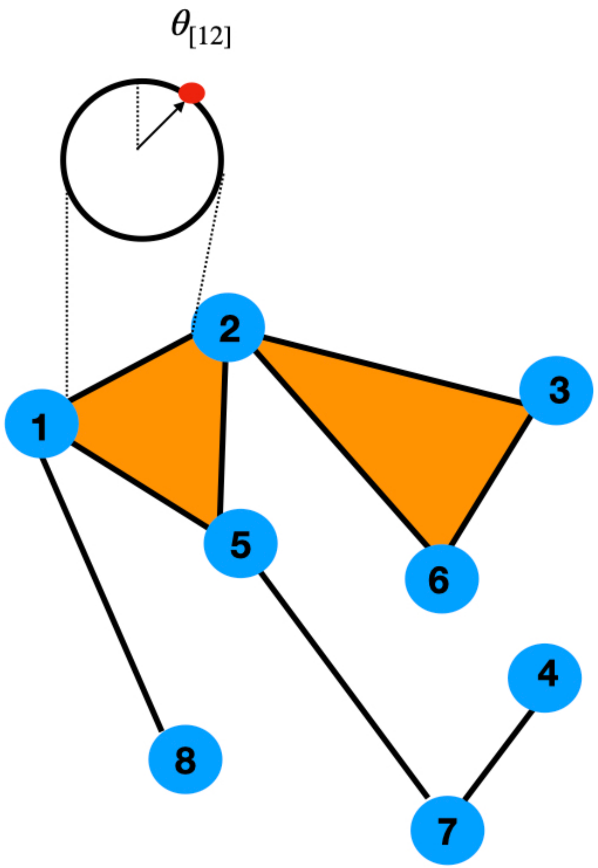
$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^{\top} \boldsymbol{\theta}$$

Using $\sin \mathbf{x} \simeq \mathbf{x}$ we get the linearised dynamics

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma \mathbf{L}_{[0]} \boldsymbol{\theta}$$

Topological Kuramoto model

The higher-order simplicial Kuramoto model



How to define
the higher-order Kuramoto model
coupling higher dimensional
topological signals?

A. P. Millán, J. J. Torres, and G.Bianconi,
Physical Review Letters, *124*, 218301 (2020)

Topological signals

We associate to each

m-dimensional simplex α a phase ϕ_α

For instance for m=1 we might associate to each link a oscillating flux

The vector of phases is indicated by

$$\boxed{\boldsymbol{\phi} = (\ldots, \phi_\alpha \ldots)^\top}$$

Standard Kuramoto model in terms of boundary matrices

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where we have defined the vectors

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i \ldots)^{\top}$$
$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_i \ldots)^{\top}$$

and we use the notation $\sin \mathbf{x}$

to indicates the column vector where the sine function is taken element wise

Topological synchronisation

We propose to study the higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},}$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

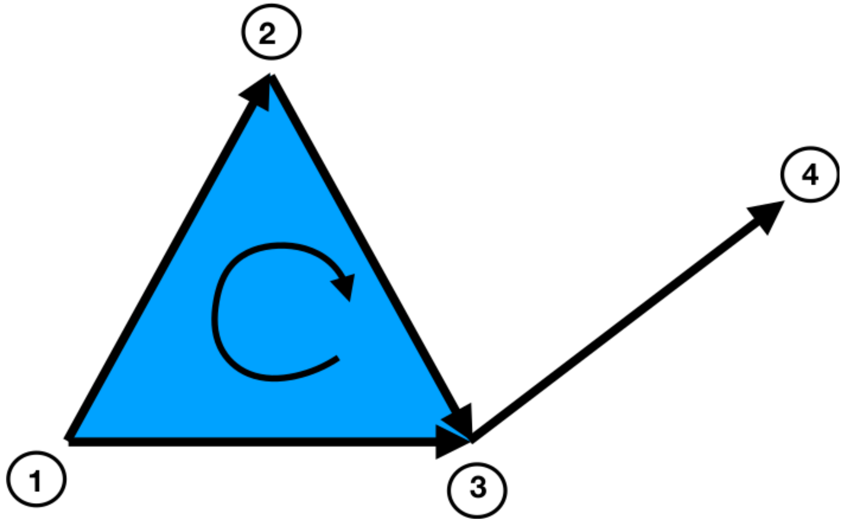
$$\boldsymbol{\phi} = (\ldots, \theta_{\alpha} \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_{\alpha} \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_{\alpha} \sim \mathcal{N}(\Omega, 1)$$

Topologically induced many-body interactions



$$\dot{\phi}_{[12]} = \hat{\omega}_{[12]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[12]} - \phi_{[23]}) + \sin(\phi_{[13]} + \phi_{[12]})\right],$$

$$\dot{\phi}_{[13]} = \hat{\omega}_{[13]} + \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[13]} + \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[23]} = \hat{\omega}_{[23]} - \sigma \sin(\phi_{[23]} - \phi_{[13]} + \phi_{[12]}) - \sigma \left[\sin(\phi_{[23]} - \phi_{[12]}) + \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

$$\dot{\phi}_{[34]} = \hat{\omega}_{[34]} - \sigma \left[\sin(\phi_{[34]}) - \sin(\phi_{[13]} + \phi_{[23]} - \phi_{[34]})\right],$$

Hamiltonian of the Topological Kuramoto model

The Topological Kuramoto is an Hamiltonian gradient flow

Hamiltonian of the Standard Kuramoto model (XY model)

$$H = -\boldsymbol{\omega}^\top \boldsymbol{\theta} - \sigma \mathbf{1}^\top \cos(\mathbf{B}_{[1]}^\top \boldsymbol{\theta})$$

$$= -\sum_{i=1}^{N} \omega_i \theta_i - \sigma \sum_{<i,j>} \cos(\theta_j - \theta_i)$$

Hamiltonian of the Topological Kuramoto model

$$H = -\hat{\boldsymbol{\omega}}^\top \boldsymbol{\phi} - \sigma \mathbf{1}^\top \cos(\mathbf{B}_{[m]} \boldsymbol{\phi}) - \sigma \mathbf{1}^\top \cos(\mathbf{B}_{[m+1]}^\top \boldsymbol{\phi})$$

Dynamics learns topology

Topological Higher-order Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^\top \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^\top \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

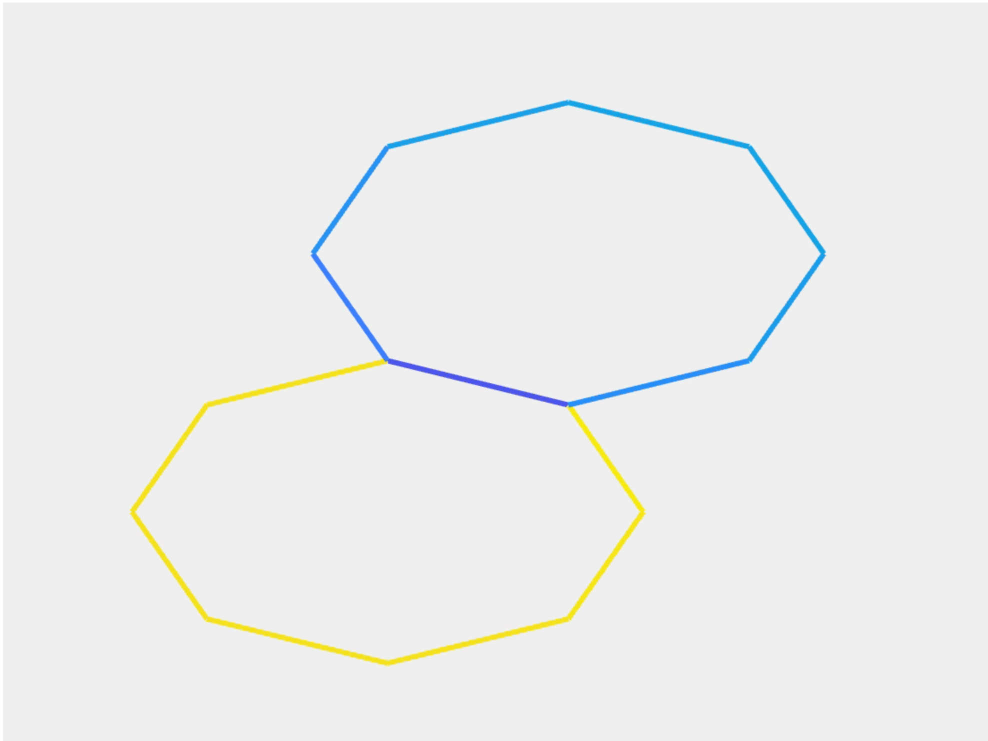
In the Topological Kuramoto model the free dynamics is localised on the m-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

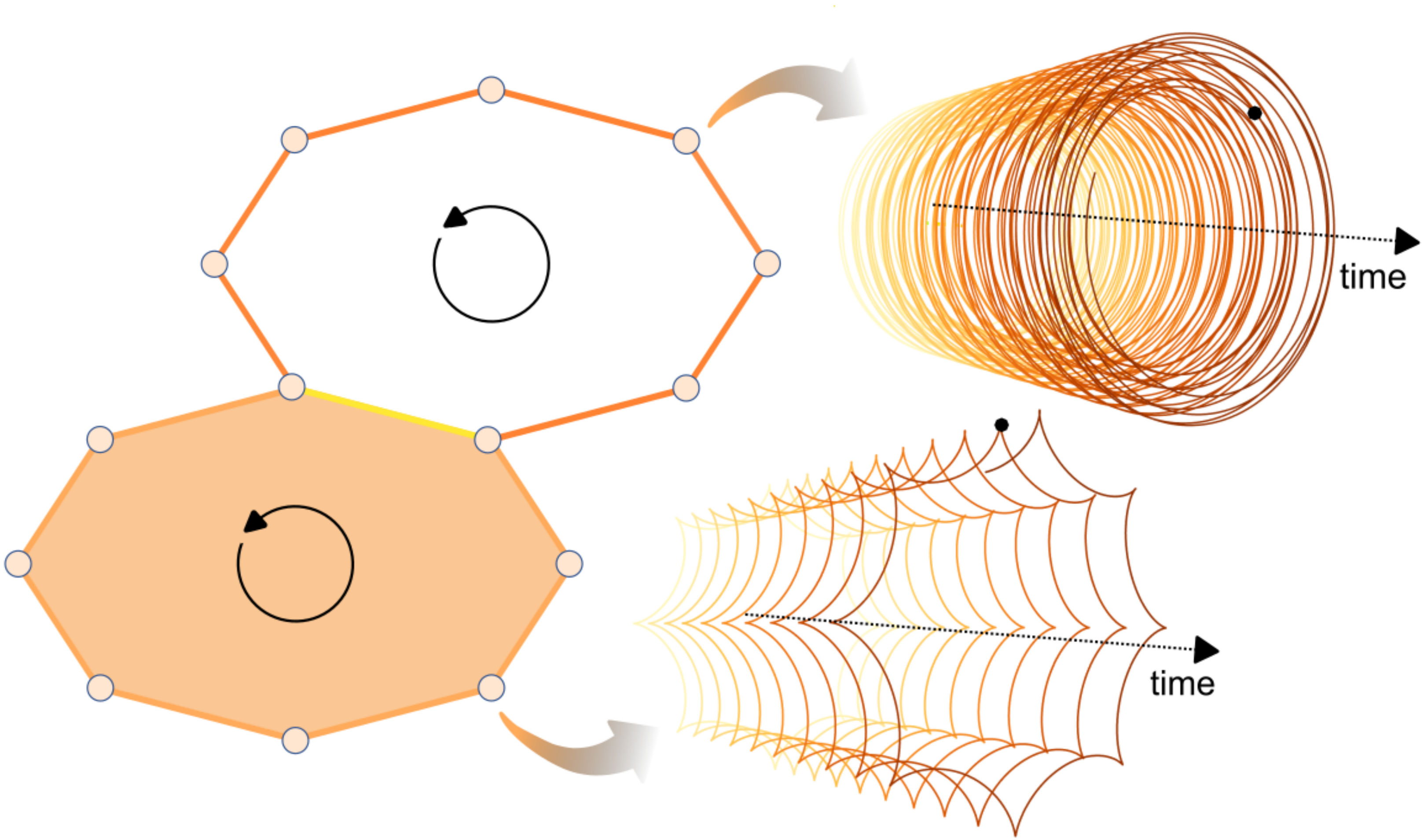
The free dynamics is localised on harmonic components

Topological Synchronisation

The dynamical ordered state has many minima
Each corresponding to a single homology class of the simplicial complex (hole)



Topological Synchronisation



In the Topological Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

the dynamics of the synchronised state is localised on the m-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

The harmonic mode of the non-linear Kuramoto model

Let us now study the full nonlinear Topological Kuramoto equation

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi}, \quad (2)$$

Let us consider any harmonic eigenvector $\mathbf{u}_{harm}^{\top}$ of the Hodge Laplacian $\mathbf{L}_{[m]} = \mathbf{B}_{[m+1]} \mathbf{B}_{[m+1]}^{\top} + \mathbf{B}_{[m]}^{\top} \mathbf{B}_{[m]}$.

Since Hodge decomposition applies $\mathbf{u}_{harm}^{\top} \mathbf{B}_{[m+1]} = \mathbf{u}_{harm}^{\top} \mathbf{B}_{[m]}^{\top} = \mathbf{0}$

By multiplying (2) by $\mathbf{u}_{harm}^{\top}$ we obtain $\dfrac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$

Therefore the harmonic modes oscillate at constant frequency also in the nonlinear Topological Kuramoto model.

Gauge invariance of the Topological Kuramoto equation

Given the Topological Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

If we perform the transformation

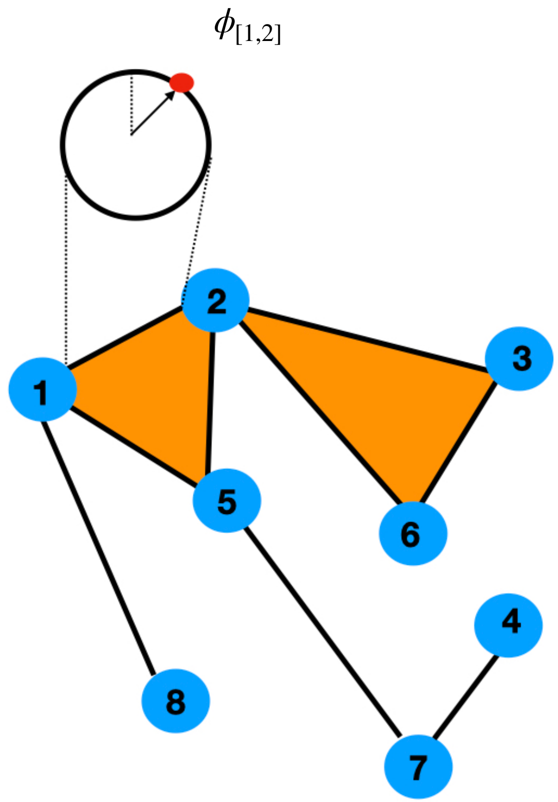
$$\boldsymbol{\phi} \rightarrow \boldsymbol{\phi} - \hat{\Omega} t \mathbf{u}^{harm}$$

We obtain

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} - \hat{\Omega} \mathbf{u}^{harm} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi}, ,$$

i.e. the dynamics is invariant under rescaling

$$\boldsymbol{\omega} \rightarrow \boldsymbol{\omega} - \hat{\Omega} \mathbf{u}^{harm}$$



Linearised Dynamics

The linearised dynamics is dictated by the Hodge-Laplacian

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m]} \boldsymbol{\phi},$$

The harmonic component of the signal oscillates freely

The other modes freeze asymptotically in time

In the Topological Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

the dynamics of the synchronised state is localised on the m-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

If we define a higher-order Kuramoto model on

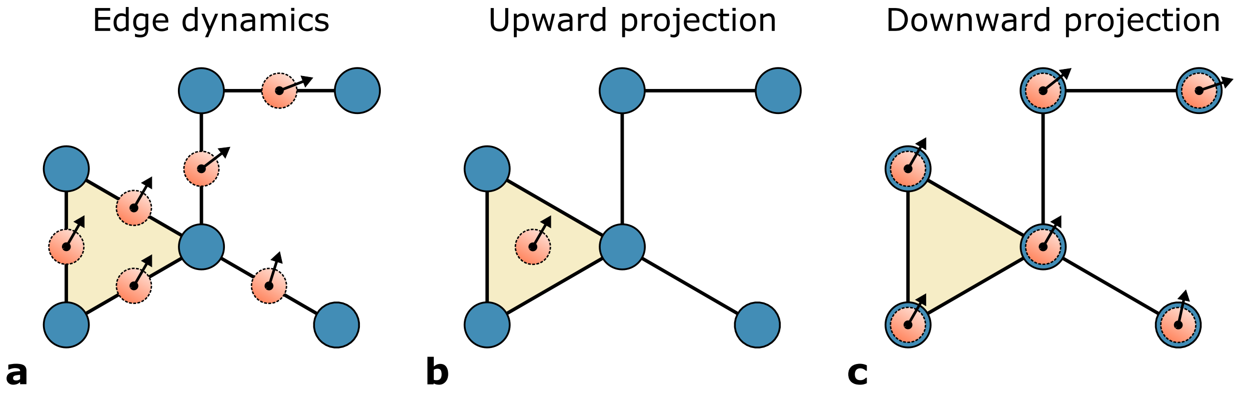
m-simplices,

(let us say links, m=1) a key question is:

What is the dynamics induced

on (m-1) faces and (m+1) faces?

i.e. what is the dynamics induced on nodes and triangles?



Projected dynamics on m-1 and m+1 faces

A natural way to project the dynamics is to use the incidence matrices obtaining

$$\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[m+1]}^{\top}\boldsymbol{\phi}$$ **Discrete curl**

$$\boldsymbol{\phi}^{[-]} = \mathbf{B}_{[m]}\boldsymbol{\phi}$$ **Discrete divergence**

Projected dynamics on m-1 and m+1 faces

Thanks to Hodge decomposition,

the projected dynamics

on the (m-1) and (m+1) faces

decouple

$$\boxed{\begin{aligned} \dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[m+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m+1]}^{[down]} \sin(\boldsymbol{\phi}^{[+]}) \\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[m]}\hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m-1]}^{[up]} \sin(\boldsymbol{\phi}^{[-]}) \end{aligned}}$$

Proof

Starting from the Topological Kuramoto dynamics

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi}$$

We apply $\mathbf{B}_{[m+1]}^{\top}$ to both sides of the equations getting for $\boldsymbol{\phi}^{[+]} = \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi}$

$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[m+1]}^{\top} \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]}^{\top} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m+1]}^{\top} \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

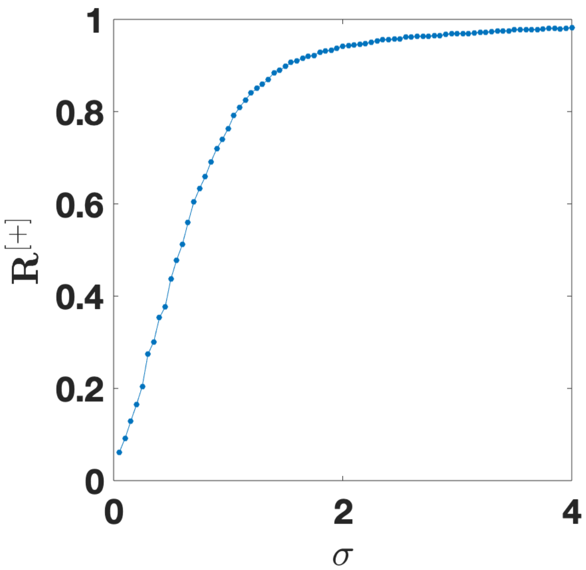
Using $\mathbf{B}_{[m+1]}^{\top} \mathbf{B}_{[m+1]} = \mathbf{L}_{m+1}^{down}, \mathbf{B}_{[m+1]}^{\top} \mathbf{B}_{[m]}^{\top} = \mathbf{0}$ we get

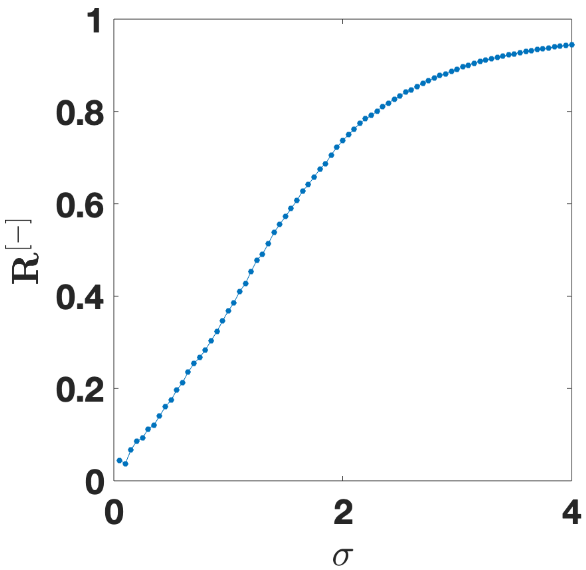
$$\dot{\boldsymbol{\phi}}^{[+]} = \mathbf{B}_{[m+1]}^{\top} \hat{\boldsymbol{\omega}} - \sigma \mathbf{L}_{[m+1]}^{down} \sin \boldsymbol{\phi}^{[+]}$$

A similar derivation holds for getting the equation for $\boldsymbol{\phi}^{[-]}$

Simplicial Synchronization transition

$$R^{[+]} = \frac{1}{N_{m+1}} \left| \sum_{\alpha=1}^{N_{m+1}} e^{\mathrm{i}\phi_\alpha^{[+]}} \right| \qquad R^{[-]} = \frac{1}{N_{m-1}} \left| \sum_{\alpha=1}^{N_{m-1}} e^{\mathrm{i}\phi_\alpha^{[-]}} \right|$$





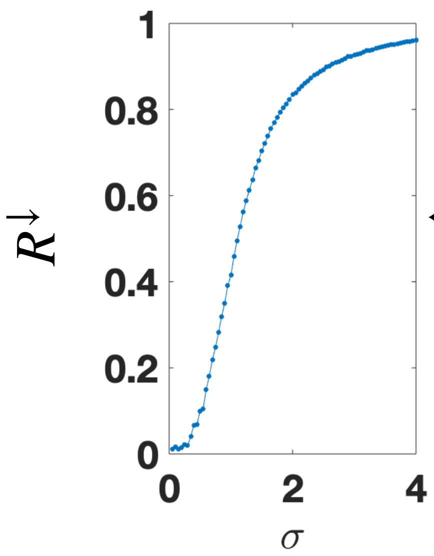
Order parameters using the n-dimensional phases

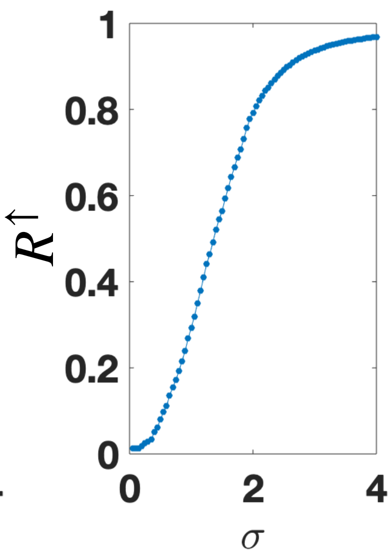
$$\boldsymbol{\phi}^{\downarrow} = \mathbf{L}^{down}_{[n]}\boldsymbol{\phi}$$

$$\boxed{R^{\downarrow} = \frac{1}{N_m}\left|\sum_{\alpha=1}^{N_m} e^{\mathrm{i}\phi^{\downarrow}_{\alpha}}\right|}$$

$$\boldsymbol{\phi}^{\uparrow} = \mathbf{L}^{up}_{[n]}\boldsymbol{\phi}$$

$$\boxed{R^{\uparrow} = \frac{1}{N_m}\left|\sum_{\alpha=1}^{N_m} e^{\mathrm{i}\phi^{\uparrow}_{\alpha}}\right|}$$





Only if we perform

the correct topological filtering

of the topological signal

we can reveal higher-order topological synchronisation

Explosive topological synchronisation

We propose the Explosive Topological Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R^{[-]} \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma R^{[+]} \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi}}$$

Projected dynamics

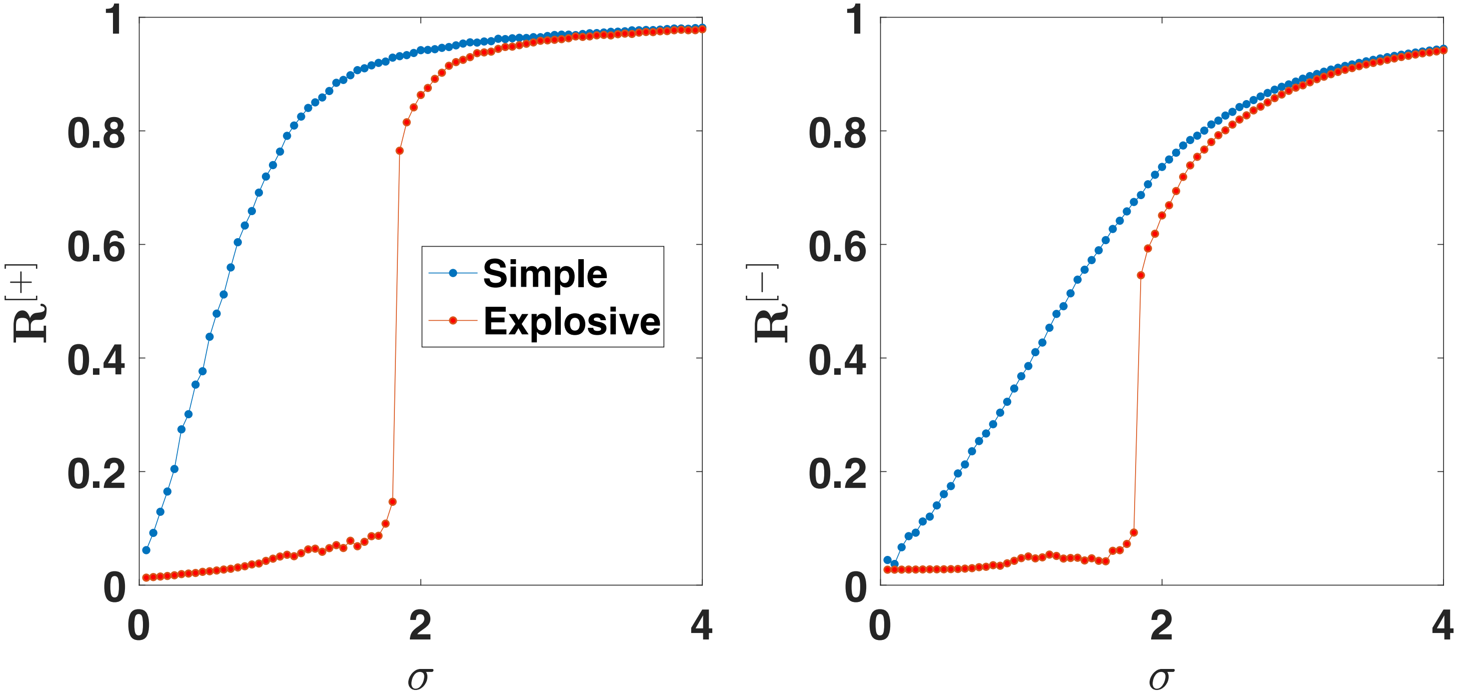
The projected dynamics on

(m+1) and (m-1) are now coupled

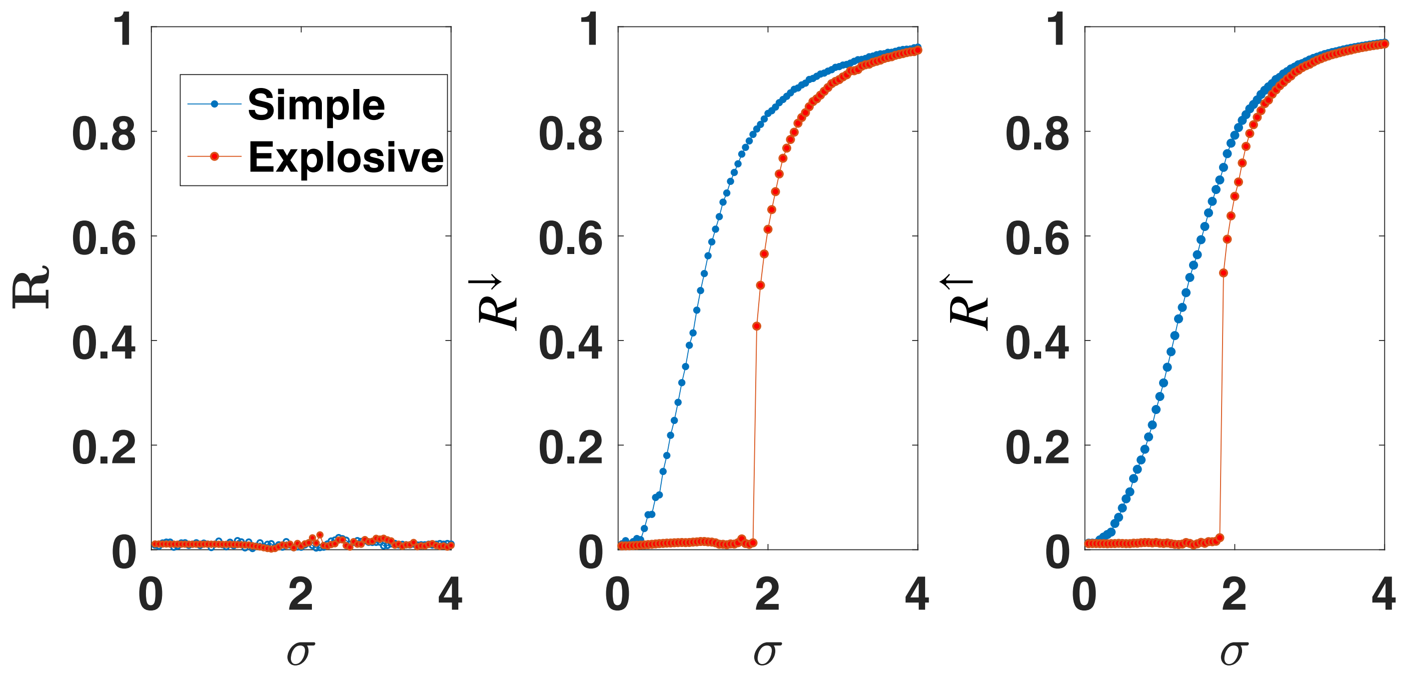
by their order parameters

$$\boxed{\begin{aligned} \dot{\boldsymbol{\phi}}^{[+]} &= \mathbf{B}_{[m+1]}^{\top}\hat{\boldsymbol{\omega}} - \sigma R^{[-]}\mathbf{L}_{[m+1]}^{[down]}\sin(\boldsymbol{\phi}^{[+]}) \\ \dot{\boldsymbol{\phi}}^{[-]} &= \mathbf{B}_{[m]}\hat{\boldsymbol{\omega}} - \sigma R^{[+]}\mathbf{L}_{[m-1]}^{[up]}\sin(\boldsymbol{\phi}^{[-]}) \end{aligned}}$$

The explosive simplicial synchronisation transition

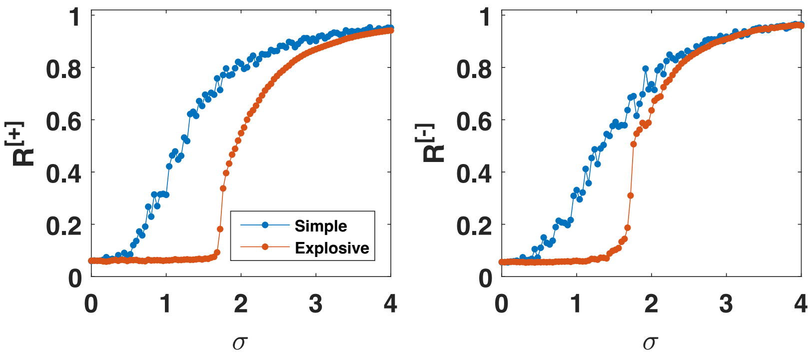


Order parameters associated to n-faces

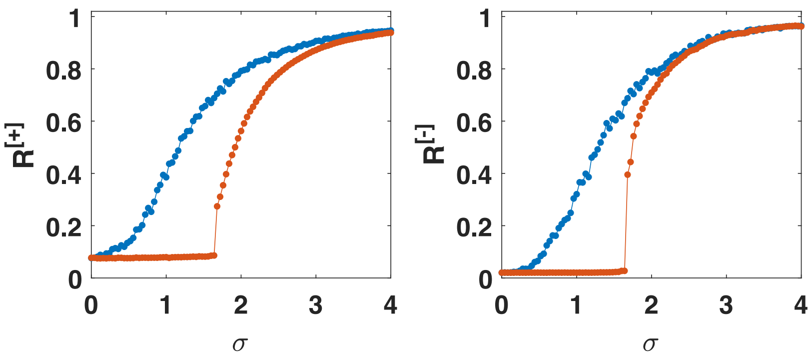


Higher-order synchronisation on real Connectomes

Homo sapiens Connectome

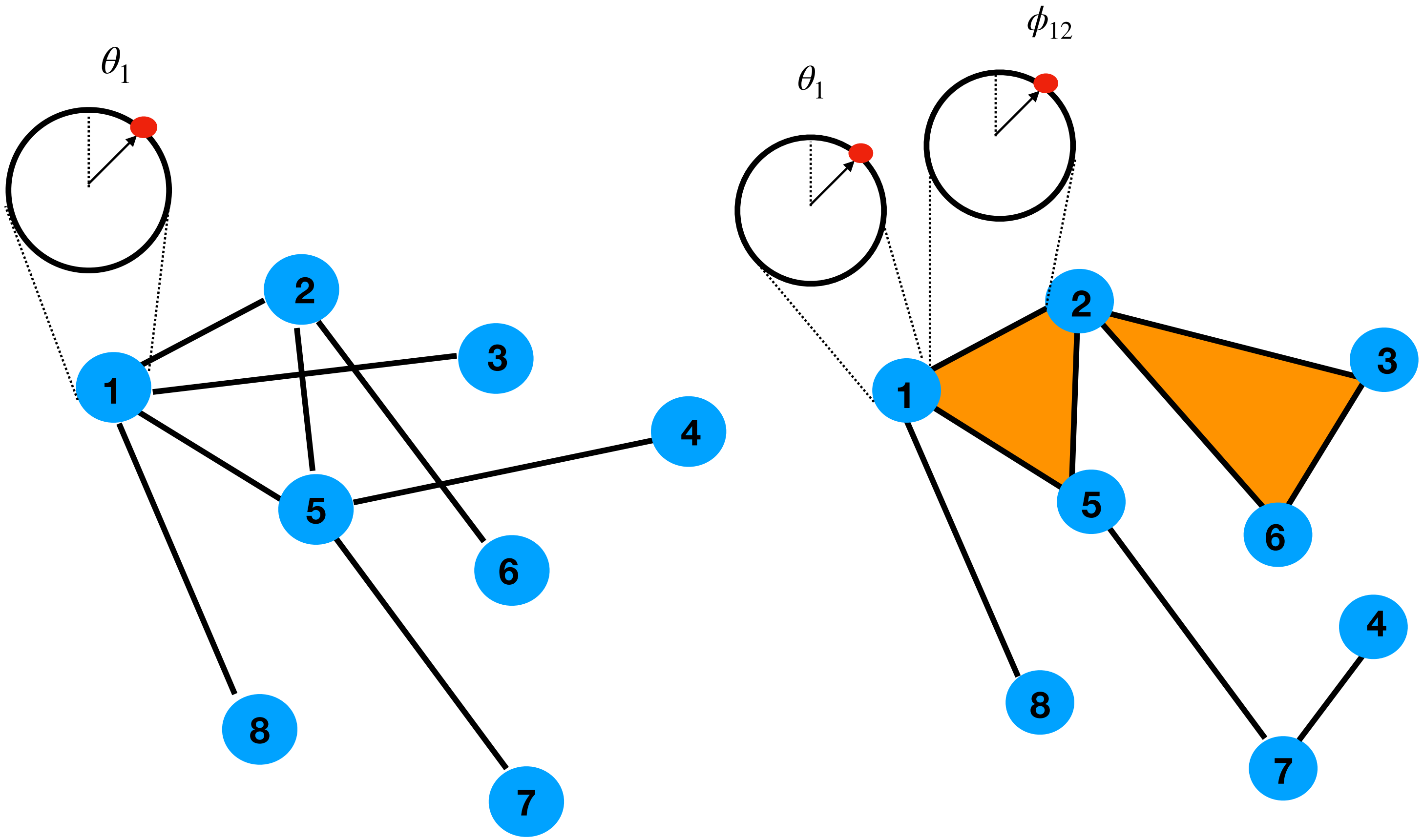


C.elegans Connectome



Non examinable material

Coupling topological signals of different dimension

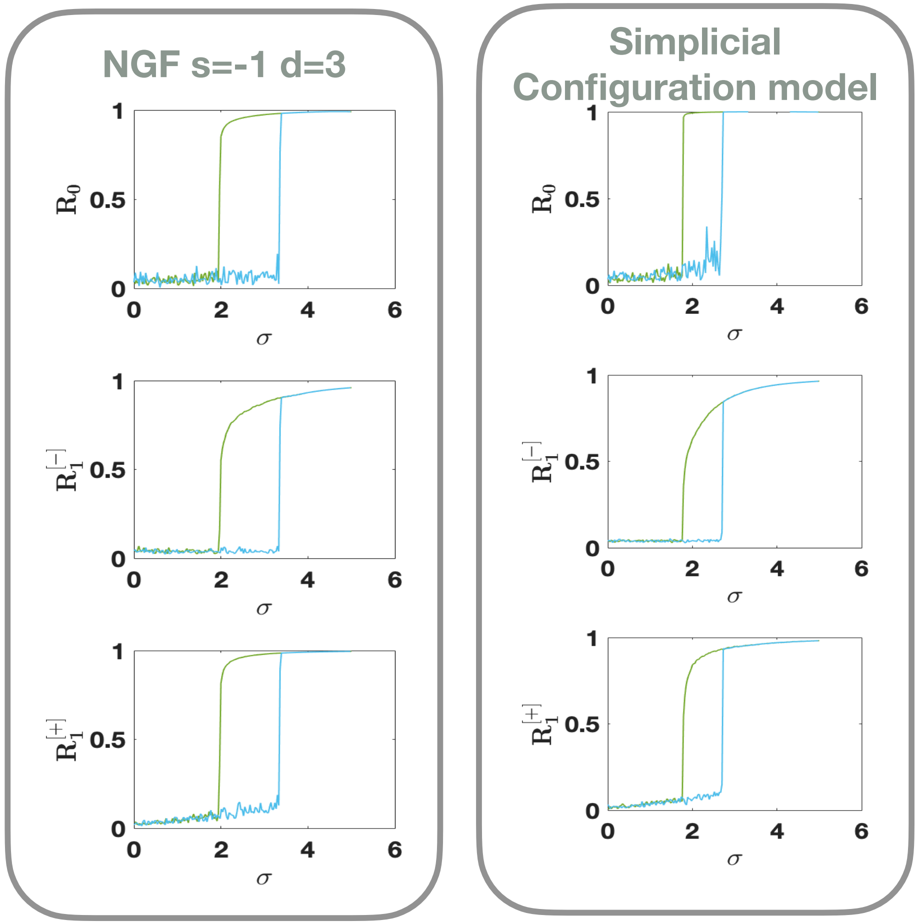


R. Ghorbanchian, J. Restrepo, J.J. Torres and G. Bianconi (2020)

Explosive synchronisation of globally coupled topological signals

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R_1^{[-]} \mathbf{B}_{[1]} \sin \mathbf{B}_{[1]}^\top \boldsymbol{\theta}$$

$$\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma R_1^{[+]} R_0 \mathbf{B}_{[1]}^\top \sin \mathbf{B}_{[1]} \boldsymbol{\phi}$$
$$-\sigma R_1^{[-]} \mathbf{B}_{[2]} \sin \mathbf{B}_{[2]}^\top \boldsymbol{\phi}$$



Coupled node and link topological signals on networks

$$\boxed{\begin{aligned}\dot{\boldsymbol{\theta}} &= \boldsymbol{\omega} - \sigma R_1^{[-]}\mathbf{B}_{[1]}\sin\mathbf{B}_{[1]}^{\top}\boldsymbol{\theta}\\ \dot{\boldsymbol{\phi}} &= \hat{\boldsymbol{\omega}} - \sigma R_0\mathbf{B}_{[1]}^{\top}\sin\mathbf{B}_{[1]}\boldsymbol{\phi}\end{aligned}}$$

$$\omega_i \sim \mathcal{N}(\Omega_0, 1/\tau_0)$$
$$\hat{\omega}_i \sim \mathcal{N}(\Omega_1, 1/\tau_1)$$

Dynamics projected on the nodes

$$\boldsymbol{\psi} = \mathbf{B}_{[n]}\boldsymbol{\phi}$$

$$\tilde{\boldsymbol{\omega}} = \mathbf{B}_{[1]}\hat{\boldsymbol{\omega}}$$

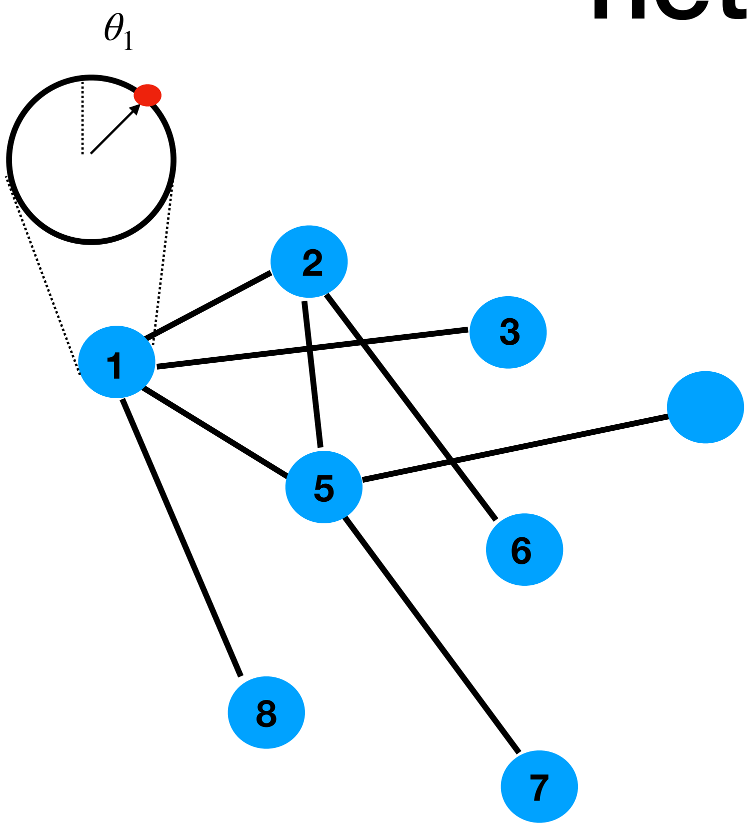
$$\boxed{\begin{aligned}\dot{\boldsymbol{\theta}} &= \boldsymbol{\omega} - \sigma R_1^{[-]}\mathbf{B}_{[1]}\sin\mathbf{B}_{[1]}^{\top}\boldsymbol{\theta}\\ \dot{\boldsymbol{\psi}} &= \tilde{\boldsymbol{\omega}} - \sigma R_0\mathbf{L}_{[0]}\sin\boldsymbol{\psi}\end{aligned}}$$

Correlation of projected frequencies

$$\langle\tilde{\omega}_r\rangle = \left[\sum_{s<r} a_{rs} - \sum_{s>r} a_{rs}\right]\Omega_1$$

$$\langle\tilde{\omega}_r\tilde{\omega}_s\rangle - \langle\tilde{\omega}_r\rangle\langle\tilde{\omega}_s\rangle = [\mathbf{L}_{[0]}]_{rs}\frac{1}{\tau_1^2}$$

Kuramoto model on a network



Given a network of N nodes
defined by an adjacency matrix a
we assign to each node a phase obeying

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

where the internal frequencies of the nodes
are drawn randomly from

$$\omega \sim \mathcal{N}(\Omega, 1)$$

and the coupling constant is σ

The oscillators are non-identical

Solution of the Kuramoto model on a fully connected network

On a fully connected network the coupling constant is rescaled as

$$\sigma \to \frac{\sigma}{N}$$

The Kuramoto equation

$$\dot{\theta}_r = \omega_r + \sigma \sum_{s=1}^{N} a_{rs} \sin\left(\theta_s - \theta_r\right)$$

can be written in terms of the complex order parameter X as

$$\dot{\theta}_r = \omega_r - \Omega + \sigma \mathrm{Im}(X e^{-\mathrm{i}\theta_\mathrm{r}})$$

Thanks to the gauge invariance we can study the dynamics in the rotating frame which reads

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

Solution of the Kuramoto model on a fully connected network

Looking for the stationary states $\dot{\theta}_r = 0$ of

$$\dot{\theta}_r = \omega_r - \Omega - \sigma R \sin(\theta_r)$$

We obtain $\sin(\theta_r) = \dfrac{\omega_r - \Omega}{\sigma R}$ only valid for nodes such that

$$\left| \frac{\omega_r - \Omega}{\sigma R} \right| \leq 1$$

(frozen nodes)

Solution of the Kuramoto model on a fully connected network

Assuming that only the frozen nodes contribute to the order parameter, since $X = R$ in the rotating frame, we obtain the self-consistent equation for the order parameter

$$R = \frac{1}{N} \sum_{r|r \ are \ frozen} \cos \theta_r = \frac{1}{N} \sum_{r|r \ are \ frozen} \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R} \right)^2}$$

Or, equivalently considering the probability density distribution $g(\omega)$ for the intrinsic frequencies,

$$R = \int_{\left| \frac{\omega - \Omega}{\sigma R} \right| \leq 1} g(\omega) \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R} \right)^2} d\omega$$

Coupled node and link topological signals on networks

$$\boxed{\begin{aligned}\dot{\boldsymbol{\theta}} &= \boldsymbol{\omega} - \sigma R_1^{[-]}\mathbf{B}_{[1]}\sin\mathbf{B}_{[1]}^\top\boldsymbol{\theta}\\ \dot{\boldsymbol{\phi}} &= \hat{\boldsymbol{\omega}} - \sigma R_0\mathbf{B}_{[1]}^\top\sin\mathbf{B}_{[1]}\boldsymbol{\phi}\end{aligned}}$$

$$\omega_i \sim \mathcal{N}(\Omega_0, 1/\tau_0)$$
$$\hat{\omega}_i \sim \mathcal{N}(\Omega_1, 1/\tau_1)$$

Dynamics projected on the nodes

$$\boldsymbol{\psi} = \mathbf{B}_{[n]}\boldsymbol{\phi}$$

$$\tilde{\boldsymbol{\omega}} = \mathbf{B}_{[1]}\hat{\boldsymbol{\omega}}$$

$$\boxed{\begin{aligned}\dot{\boldsymbol{\theta}} &= \boldsymbol{\omega} - \sigma R_1^{[-]}\mathbf{B}_{[1]}\sin\mathbf{B}_{[1]}^\top\boldsymbol{\theta}\\ \dot{\boldsymbol{\psi}} &= \tilde{\boldsymbol{\omega}} - \sigma R_0\mathbf{L}_{[0]}\sin\boldsymbol{\psi}\end{aligned}}$$

Correlation of projected frequencies

$$\langle\tilde{\omega}_r\rangle = \left[\sum_{s<r} a_{rs} - \sum_{s>r} a_{rs}\right]\Omega_1$$

$$\langle\tilde{\omega}_r\tilde{\omega}_s\rangle - \langle\tilde{\omega}_r\rangle\langle\tilde{\omega}_s\rangle = [\mathbf{L}_{[0]}]_{rs}\frac{1}{\tau_1^2}$$

The node order parameter (coupled to edge dynamics) on a fully connected network

The node order parameter can be obtained similarly as for the standard Kuramoto model

$$R_0 = \frac{1}{N} \sum_{r|r \text{ are frozen}} \cos\theta_r = \frac{1}{N} \sum_{r|r \text{ are frozen}} \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R_0 R_1^{[-]}}\right)^2}$$

Or, equivalently considering the probability density distribution $g(\omega)$ for the intrinsic frequencies,

$$R_0 = \int_{\left|\frac{\omega - \Omega}{\sigma R_0 R_1^{[-]}}\right| \le 1} g(\omega) \sqrt{1 - \left(\frac{\omega - \Omega}{\sigma R_0 R_1^{[-]}}\right)^2} d\omega$$

The edge order parameter on a fully connected network

The edge order parameter can be obtained following similar steps obtaining

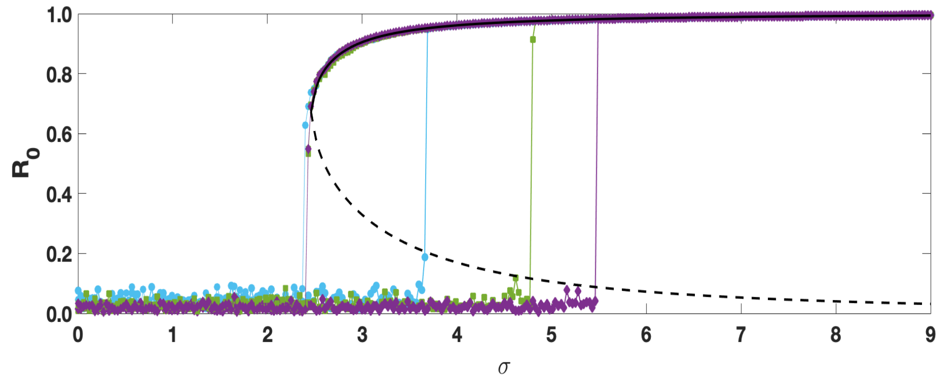
$$R_1^{[-]} = \frac{1}{N} \sum_{r|r\ are\ frozen} \cos\psi_r = \frac{1}{N} \sum_{r|r\ are\ frozen} \sqrt{1 - \left(\frac{\tilde{\omega}}{\sigma R_0}\right)^2}$$

Or, equivalently considering the probability density distribution $\tilde{g}(\tilde{\omega})$ for the intrinsic frequencies,

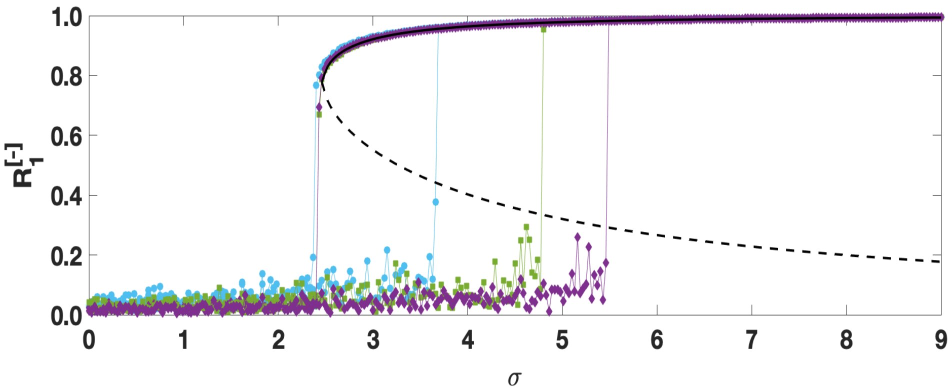
$$R_1^{[-]} = \int_{\left|\frac{\tilde{\omega}}{\sigma R_0}\right| \le 1} \tilde{g}(\tilde{\omega}) \sqrt{1 - \left(\frac{\tilde{\omega}}{\sigma R_0}\right)^2} d\tilde{\omega}$$

Solution on a fully connected network

Fully connected networks undergo a discontinuous synchronisation transition of topological signals defined on nodes and links



The hysteresis loop is not closed in the infinite network limit and on finite size networks is driven by finite size effects



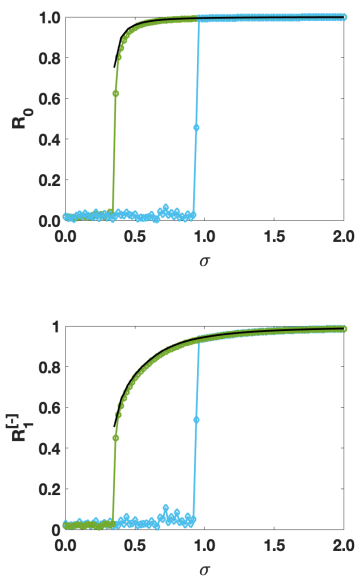
Annealed solution on random networks

The annealed solution captures the backward transition

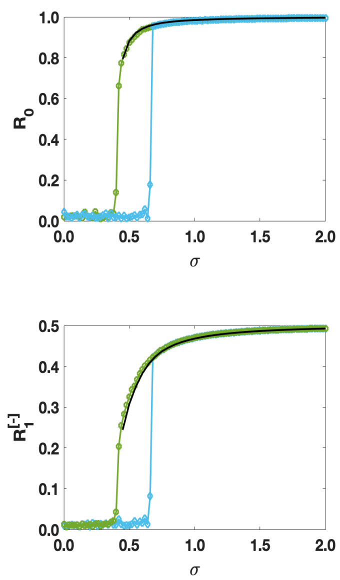
Reveals that the transition is discontinuous

Gives very reliable results for connected networks that are not too sparse

Poisson network



Power-law network



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Higher-order Topological Kuramoto model

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