Topology, geometry and dynamics of higher-order networks

An introduction to simplicial complexes
Lesson IV

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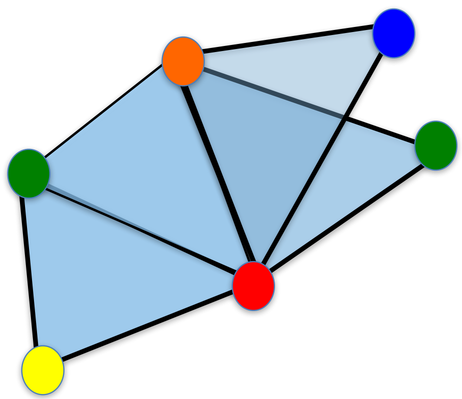
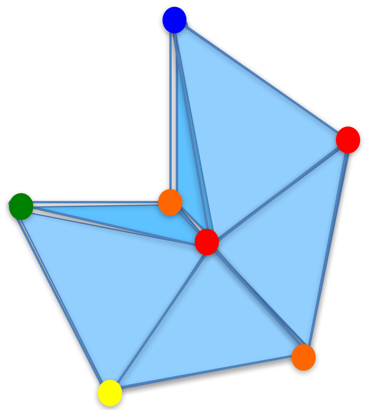
Higher-order networks

Higher-order networks are characterising the interactions between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.

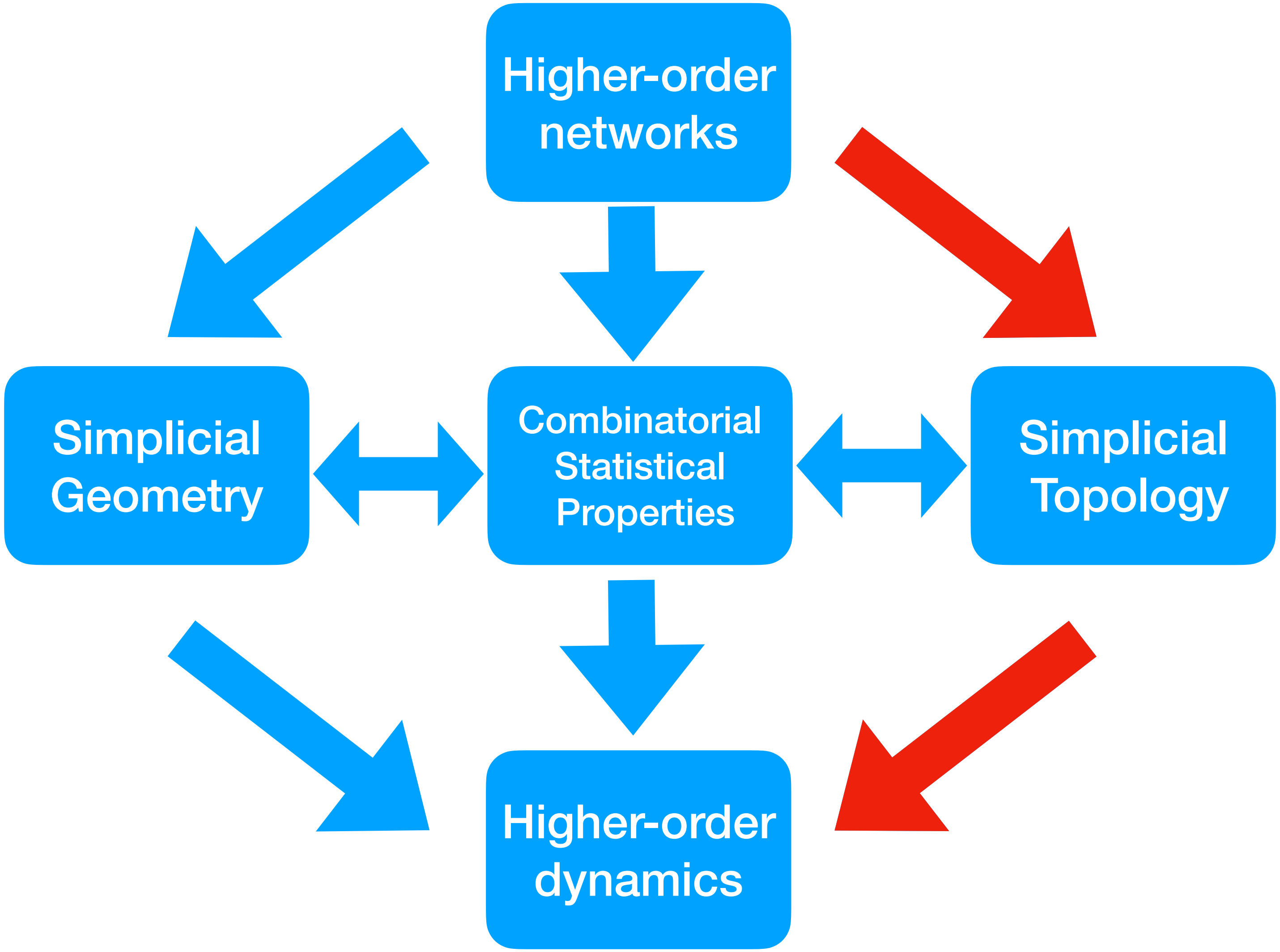


d=2 simplicial complex

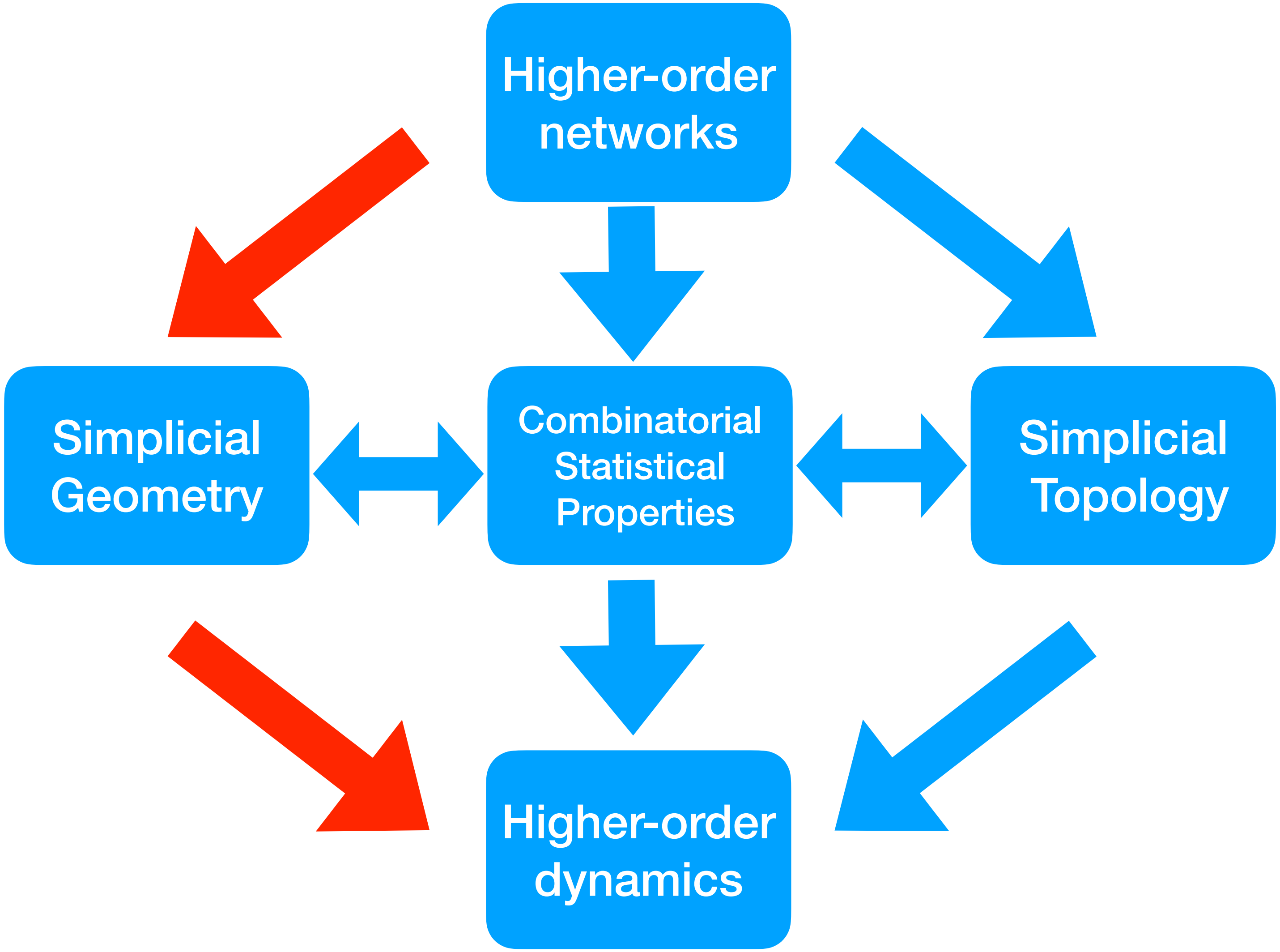


d=3 simplicial complex

Higher-order structure and dynamics

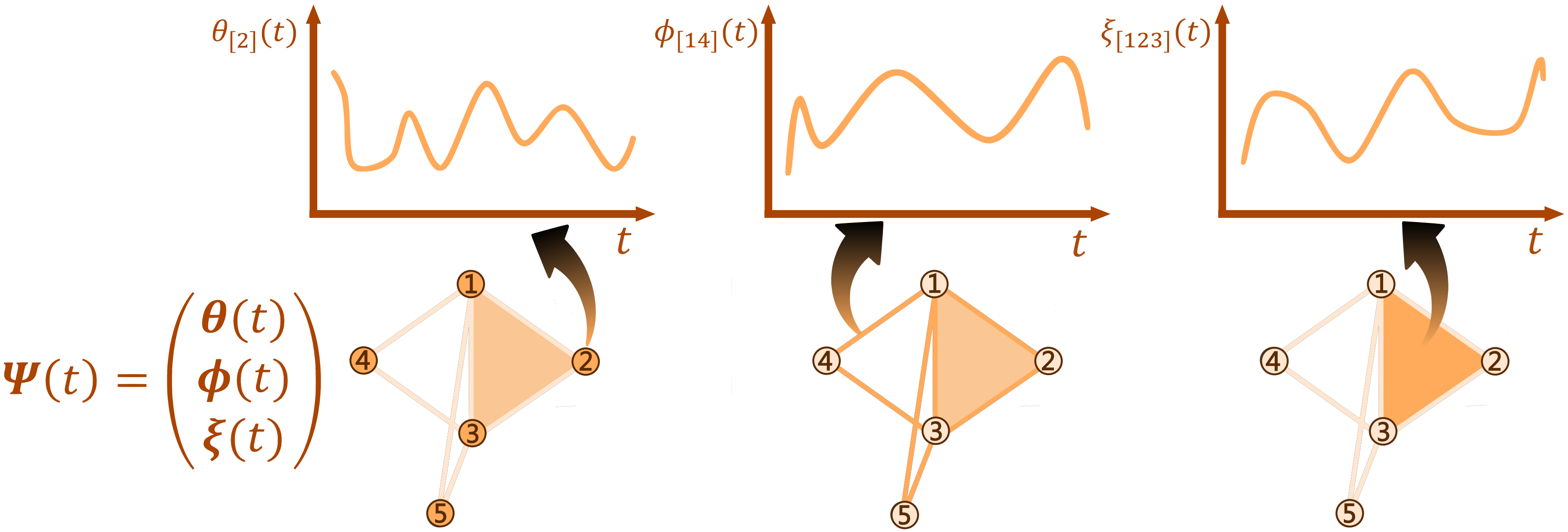


Higher-order structure and dynamics



Topological signals

Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals



Topological signals

- Citations in a collaboration network
- Speed of wind at given locations
- Currents at given locations in the ocean
- Fluxes in biological transportation networks
- Synaptic signal
- Edge signals in the brain

Topological signals
are cochains or vector fields

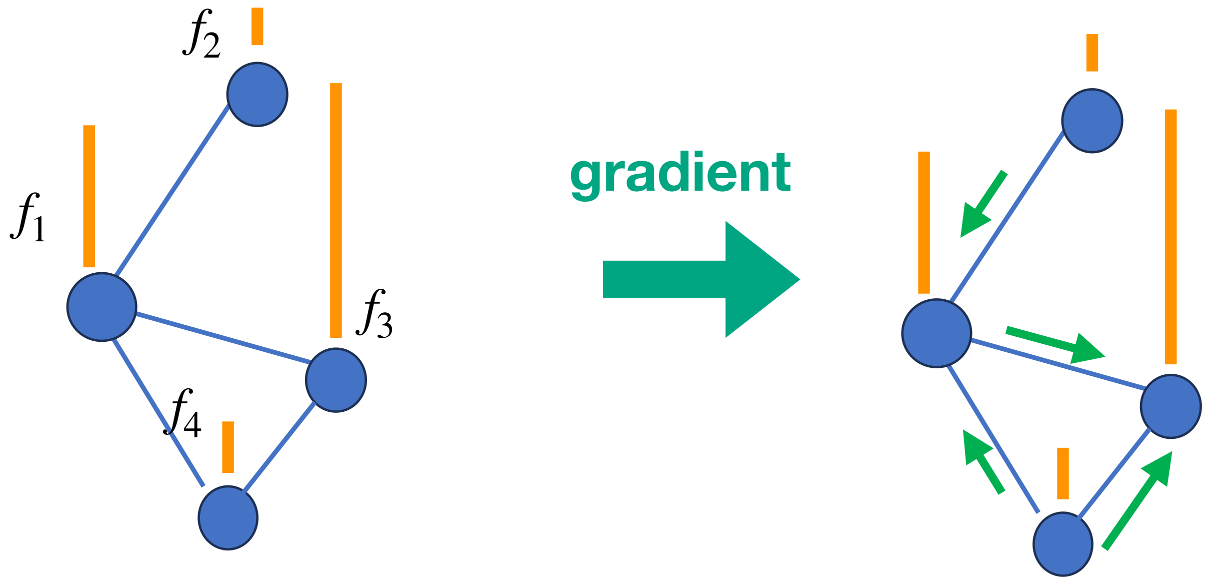
Recap of Topology with further Topological, Geometrical and Combinatorial properties

of higher-order networks

Discrete Gradient

If $f \in C^0$, then $g = \delta_0 f \in C^1$ indicates its discrete gradient

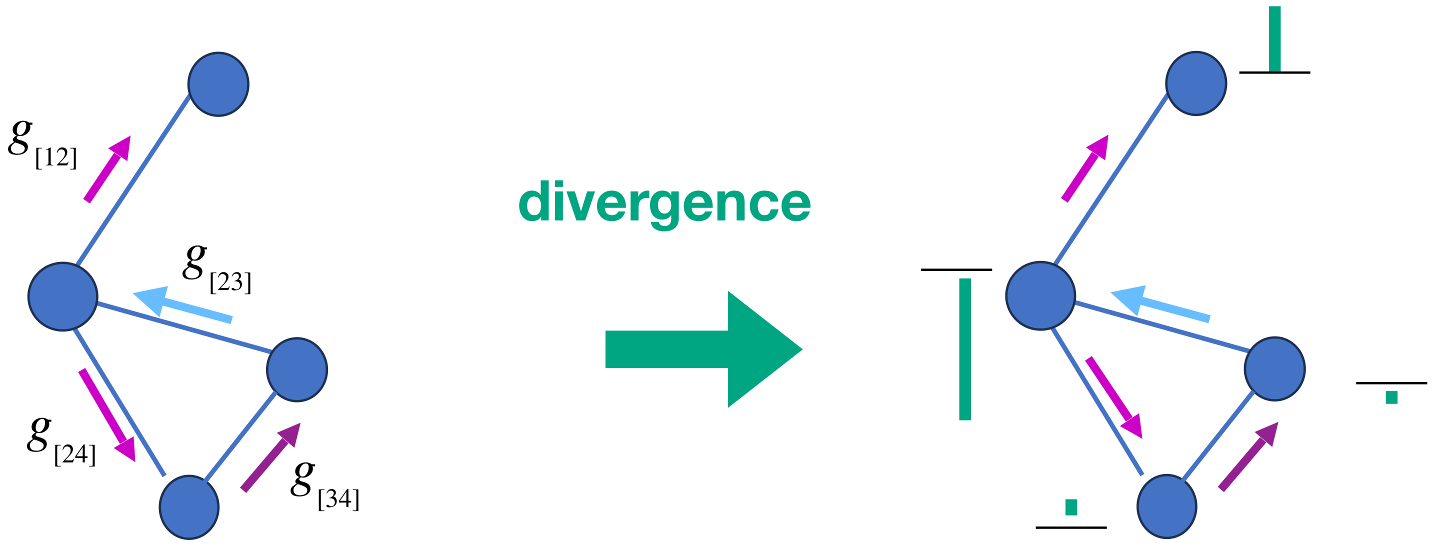
Indeed we have $\mathbf{g} = \mathbf{B}_{[1]}^\top \mathbf{f}$ which implies $g_{[r,s]} = f_s - f_r$



Discrete Divergence

If $g \in C^1$, then $f = \delta_0^* g \in C^0$ indicates its discrete divergence

Indeed we have $\mathbf{f} = \mathbf{B}_{[1]}\mathbf{g}$ which implies $f_r = \sum_s g_{[sr]} - \sum_s g_{[rs]}$

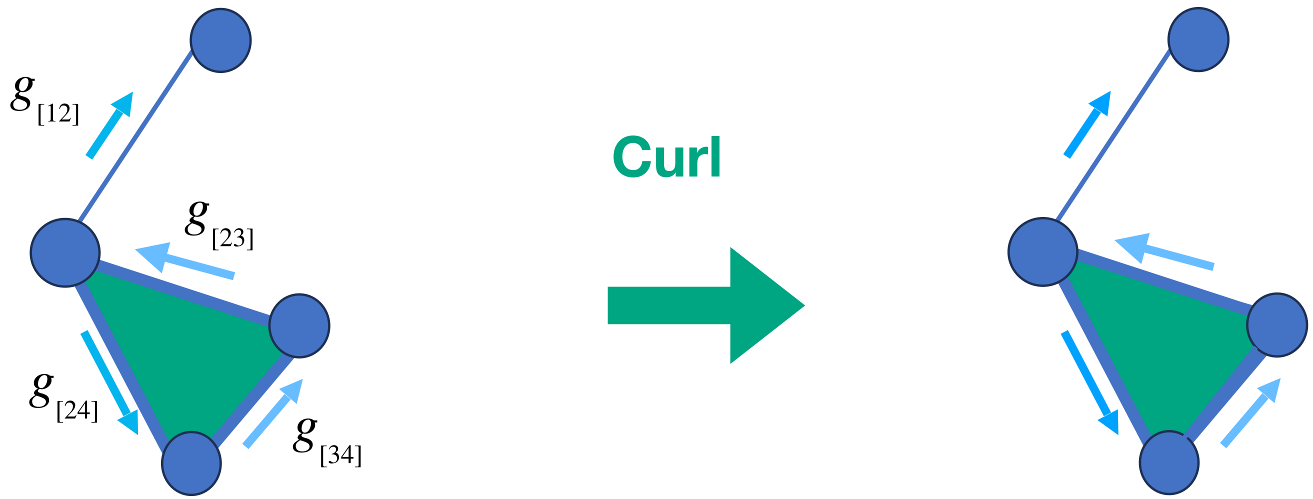


Discrete Curl

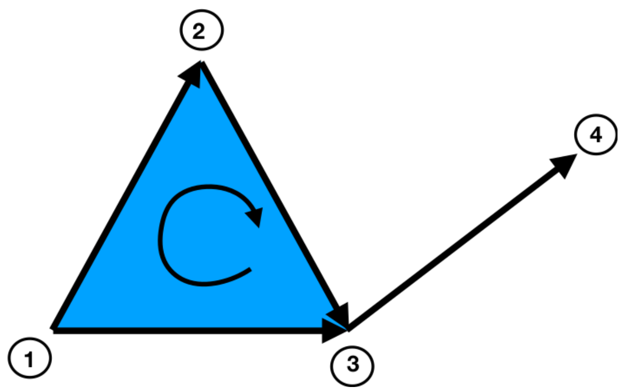
If $f \in C^1$, then $h = \delta_1 g \in C^2$ indicates its discrete curl

Indeed we have $\mathbf{g} = \mathbf{B}_{[2]}^\top \mathbf{f}$ which implies

$$h_{[r,s,q]} = g_{[r,s]} + g_{[s,q]} - g_{[r,q]}$$



Boundary Operators



Boundary operators

$$
\mathbf{B}_{[1]} = \begin{array}{c} \\ {[1]} \\ {[2]} \\ {[3]} \\ {[4]} \end{array} \begin{array}{cccc} {[1,2]} & {[1,3]} & {[2,3]} & {[3,4]} \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ {[1,2]} \\ {[1,3]} \\ {[2,3]} \\ {[3,4]} \end{array} \begin{array}{c} {[1,2,3]} \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.
$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

$\mathbf{B}_{[1]}^\top$ **Discrete gradient**

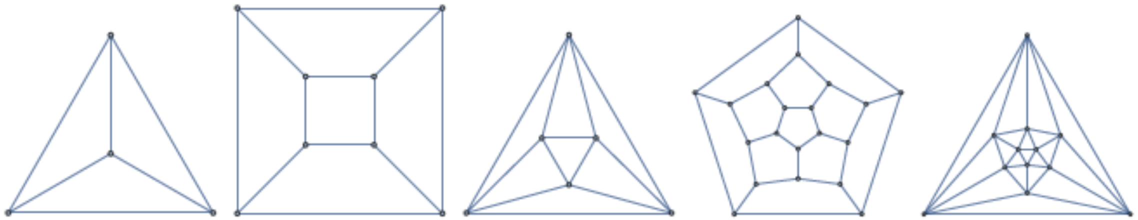
$\mathbf{B}_{[2]}^\top$ **Discrete Curl**

The boundary of the boundary is null

$$
\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}, \quad \mathbf{B}_{[n]}^\top \mathbf{B}_{[n-1]}^\top = \mathbf{0}
$$

Cell complexes

(a) (b) (c) (d) (e)



(a) (b) (c) (d) (e)

A cell complex $\hat{\mathcal{K}}$ has the following two properties:

(a) it is formed by a set of cells that is closure-finite, meaning that every cell is covered by a finite union of open cells;

(b) given two cells of the cell complex $\alpha \in \hat{\mathcal{K}}$ and $\alpha' \in \hat{\mathcal{K}}$ then either their intersection belongs to the cell complex, i.e. $\alpha \cap \alpha' \in \hat{\mathcal{K}}$ or their intersection is a null set, i.e. $\alpha \cap \alpha' = \emptyset$.

Regular polytopes

d = 1
link
d = 2
p-polygon
d = 3
tetrahedron
cube
octahedron
dodecahedron
icosahedron
d = 4
pentachoron
tesseract
hexadecachoron
24-cell
120-cell
600-cell
d > 4
simplex
cube
orthoplex

Regular polytopes

- Fun mathematical facts
- The regular polytopes in dimension d>4 are only three: d-simplices, d-hypercubes, and d-orthoplexes
- In d=4 dimensions there "monsters" such as the 120-cell and the 600-cell.

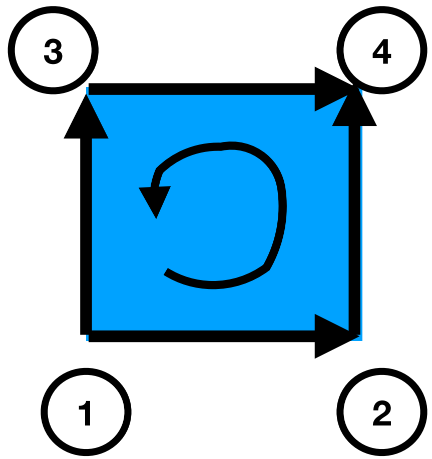
Boundary matrix of a cell complex

The boundary matrix of a cell complex has matrix elements

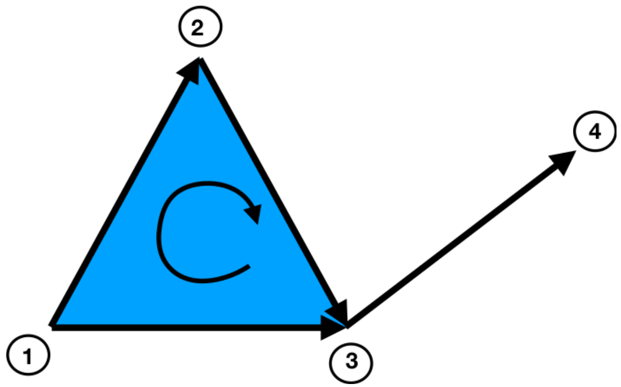
$$B_{[m]}(\alpha_r^{m-1},\alpha_s^m)=\begin{cases}1 & \textbf{if } \alpha_r^{m-1}\sim\alpha_s^m\\ -1 & \textbf{if } \alpha_r^{m-1}\nsim\alpha_s^m\\ 0 & \textbf{otherwise}\end{cases}$$

Example

$$\mathbf{B}_{[1]}=\begin{array}{c|cccc} & [1,2] & [1,3] & [3,4] & [2,4]\\ \hline [1] & -1 & -1 & 0 & 0\\ [2] & 1 & 0 & 0 & -1\\ [3] & 0 & 1 & -1 & 0\\ [4] & 0 & 0 & 1 & 1\end{array},\quad \mathbf{B}_{[2]}=\begin{array}{c|c} & [1,2,4,3]\\ \hline [1,2] & 1\\ [1,3] & -1\\ [3,4] & -1\\ [2,4] & 1\end{array}$$



Simplicial complexes and Hodge Laplacians



Hodge Laplacians

The Hodge Laplacians describe diffusion

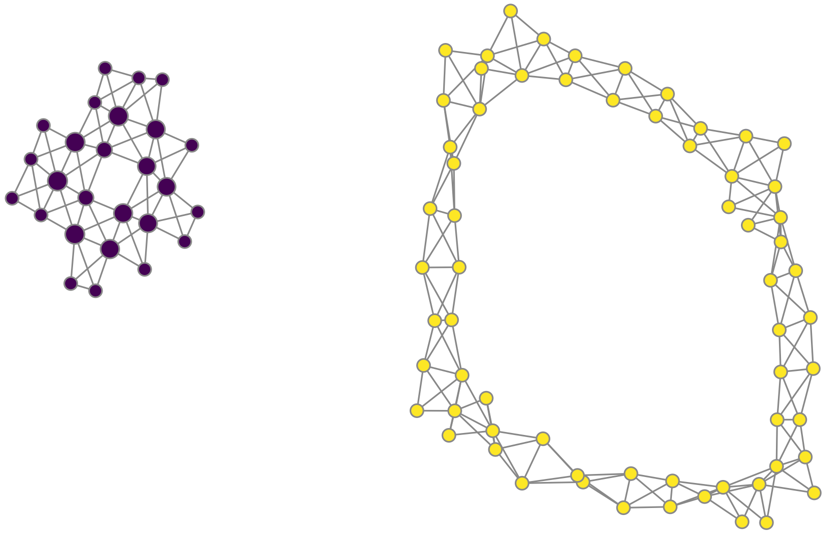
from n-simplices to n-simplices through (n-1) and (n+1) simplices

The Hodge Laplacian are semi-definite positive

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^{\top} \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^{\top}\mathbf{B}_{[2]}$$

The dimension of the kernel of the Hodge Laplacian $\mathrm{L}_{[m]}$ is given by the m Betti number β_m

Harmonic eigenvectors of the graph Laplacian



The quadratic form of the graph Laplacian reads

$$\mathbf{X}^\top \mathbf{L}_{[0]} \mathbf{X} = \frac{1}{2} \sum_{r,s} a_{rs} (X_r - X_s)^2$$

Therefore the harmonic eigenvectors of the graph Laplacian are constant on each connected component of the graph and zero everywhere else.

Harmonic eigenvectors of the Hodge Laplacian

The dimension of the kernel of the Hodge Laplacian

is given by the corresponding Betti number

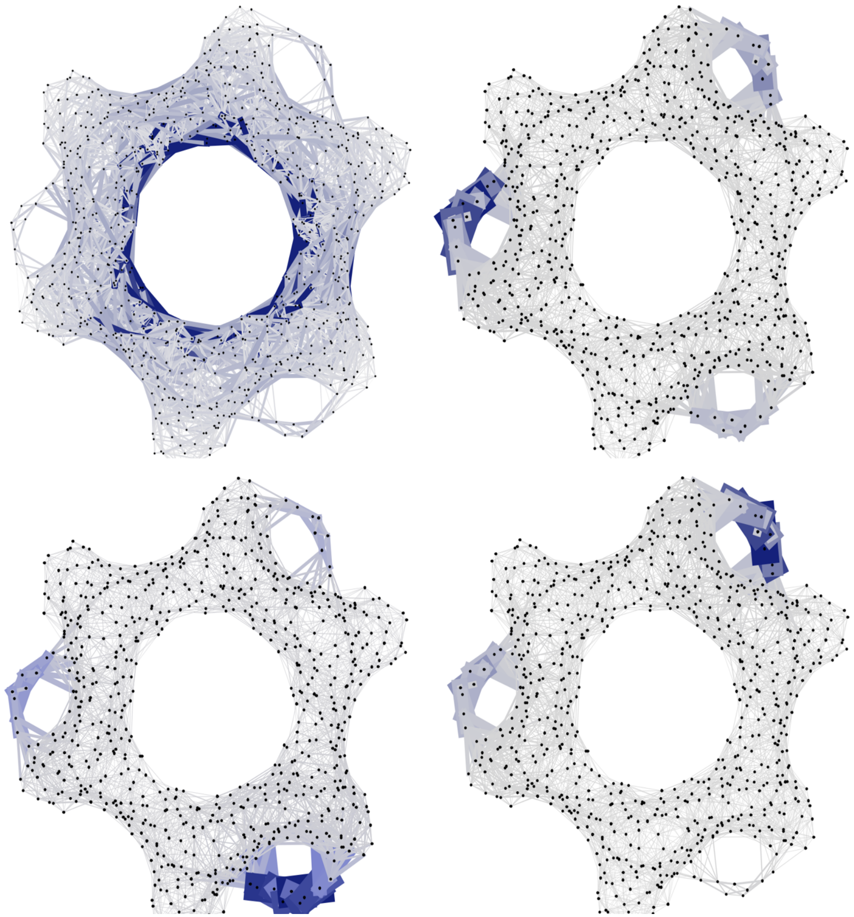
$$\textbf{dim ker}(\mathbf{L}_{[m]}) = \beta_m$$

The harmonic eigenvectors

are associated to the generators of the homology

They are in general non-uniform over the m-simplices of the simplicial complex

Harmonic eigenvectors



Wee et al. (2023)

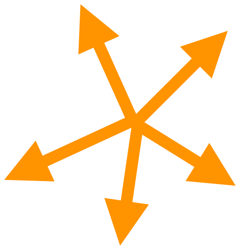
Hodge decomposition

The Hodge decomposition implies that topological signals can be decomposed

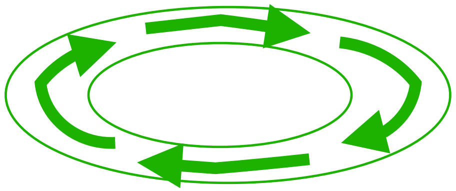
in a irrotational, harmonic and solenoidal components

$$\boxed{\mathbb{R}^{D_m} = \textbf{im}(\mathbf{B}_{[m]}^\top) \oplus \textbf{ker}(\mathbf{L}_{[m]}) \oplus \textbf{im}(\mathbf{B}_{[m+1]})}$$

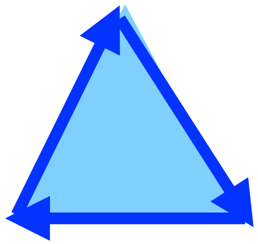
which in the case of topological signals of the links can be sketched as



Irrotational component
Gradient Flow



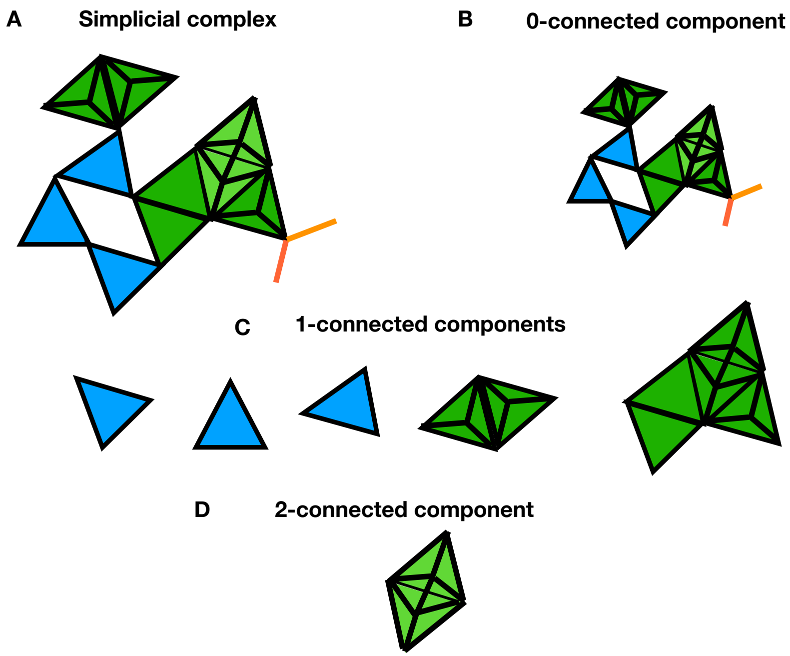
Harmonic component



Solenoidal component
Curl Flow

Geometrical properties of simplicial complexes

m-connected components

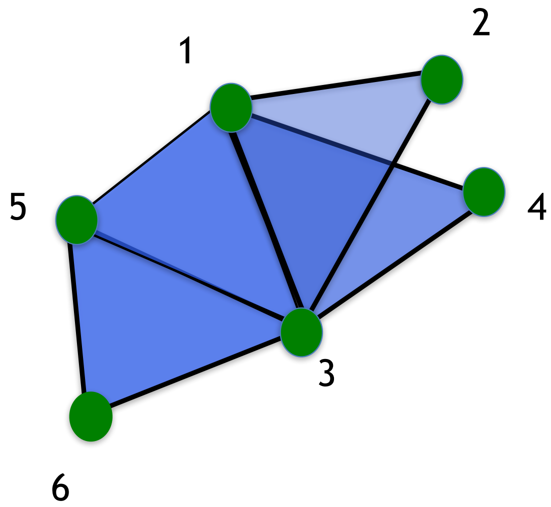


Generalized degree

The generalized degree $k_{m',m}(\alpha)$ of a m-face α

is given by the number

of m'-dimensional simplices or cells incident to the m-face α.



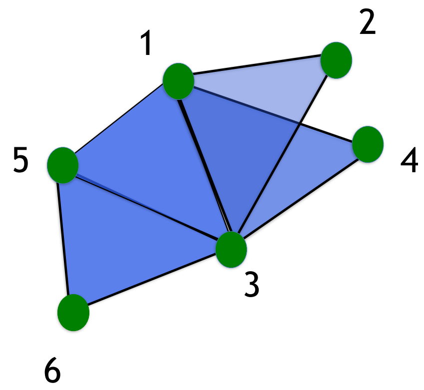
r	$k_{2,0}([r])$
[1]	3
[2]	1
[3]	4
[4]	1
[5]	2
[6]	1

$[r,s]$	$k_{2,1}([r,s])$
[1,2]	1
[1,3]	3
[1,4]	1
[1,5]	1
[2,3]	1
[3,4]	1
[3,5]	2
[3,6]	1
[5,6]	1

Incidence number

To each (d-1)-face α we associate the incidence number

$$n_\alpha = k_{d,d-1}(\alpha) - 1$$



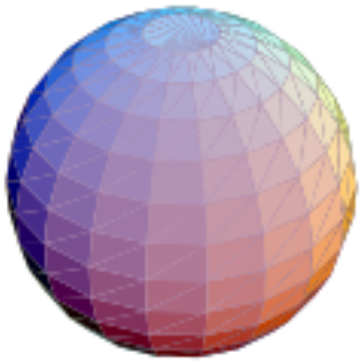
[Bianconi & Rahmede (2016)]

(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

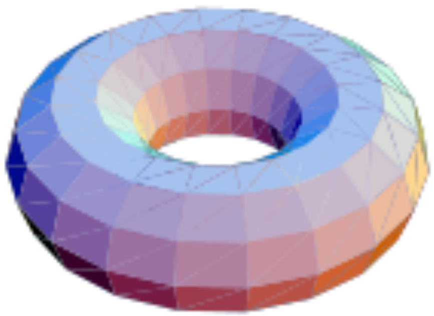
Discrete manifolds

Discrete manifolds of dimension d are d-dimensional simplicial or cell complexes, in which the neighbourhood of each node is isomorphic to a d-dimensional ball

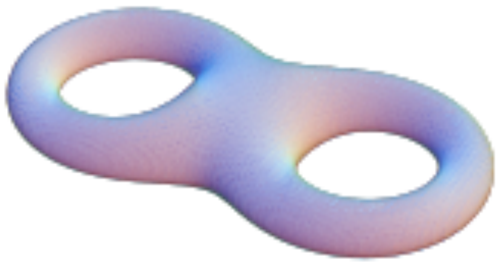
Sphere



Torus

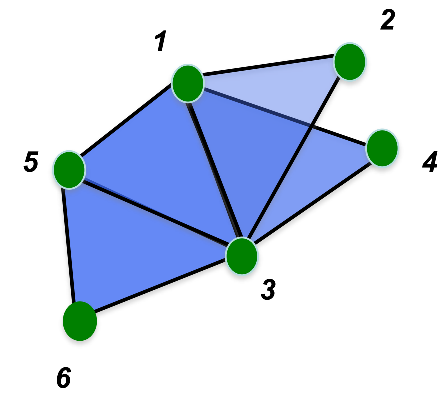


Double-Torus

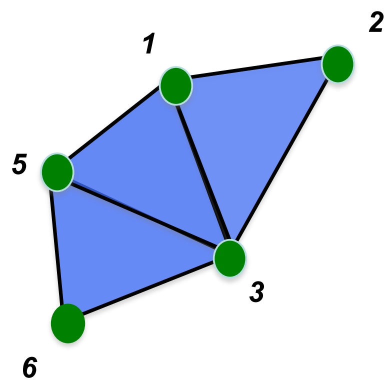


Discrete manifolds

A necessary condition for being a discrete manifold is that each (d-1)-face α is incident at most to two d-dimensional simplices or cells, i.e. $n_\alpha \in \{0,1\}$

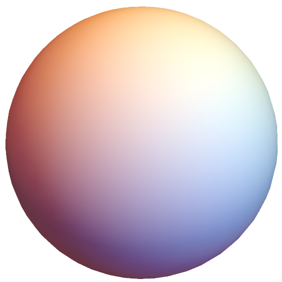


NOT A MANIFOLD



MANIFOLD

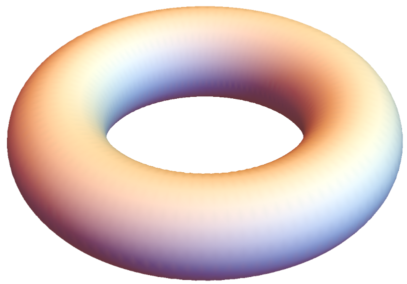
Example key manifolds and their Betti numbers



n-dimensional hypersphere

Betti numbers

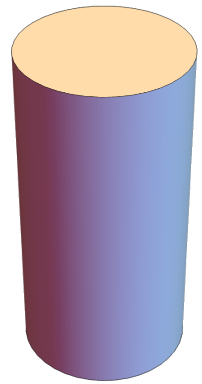
$$\beta_0 = \beta_{n-1} = 1$$
$$\beta_k = 0 \text{ for } 0 < k < n-1$$



n-dimensional torus (cell complex)

Betti numbers

$$\beta_k = \binom{n-1}{k}$$



n-dimensional cylider

Betti numbers

$$\beta_0 = \beta_{n-2} = 1$$
$$\beta_k = 0 \text{ for } k \neq 0, k \neq n-2$$

Lesson IV:
Global Topological Synchronisation and Network Geometry

- Global topological synchronisation and Master Stability Function

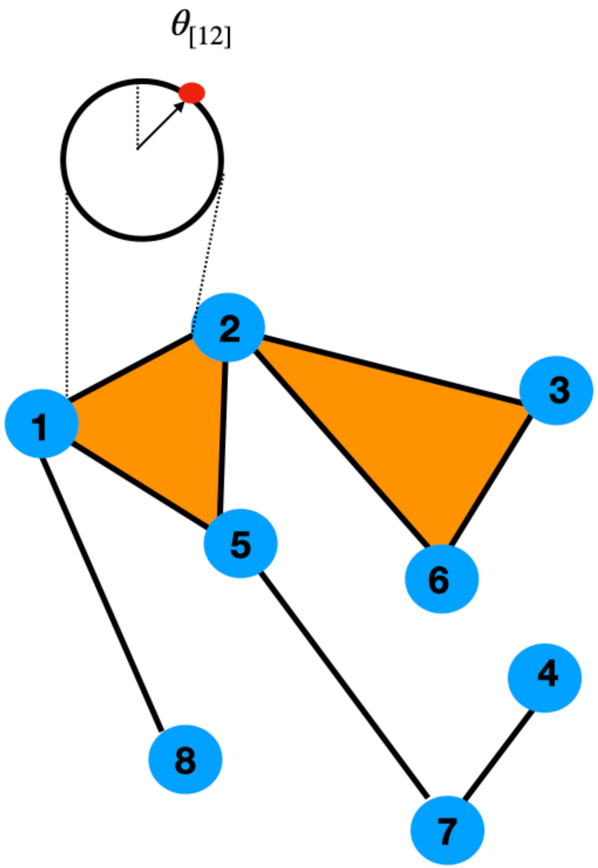
  - Global synchronisation on graphs
  - Global synchronisation on simplicial and cell complexes

- Weighted cohomology
  - Basics of weighted cohomology
  - The representation power of weighted simplicial complexes

- Global topological synchronisation on weighted simplicial complexes

Topological Kuramoto model

The higher-order simplicial Kuramoto model



How to define the higher-order Kuramoto model coupling higher dimensional topological signals?

A. P. Millán, J. J. Torres, and G.Bianconi,
***Physical Review Letters*, *124*, 218301 (2020)**

Topological synchronisation

We propose to study the higher-order Kuramoto model

defined as

$$\boxed{\dot{\boldsymbol{\phi}} = \hat{\boldsymbol{\omega}} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},}$$

where is the vector of phases associated to n-simplices

and the topological signals ad their internal frequencies are indicated by

$$\boldsymbol{\phi} = (\ldots, \theta_{\alpha} \ldots)^{\top}$$

$$\hat{\boldsymbol{\omega}} = (\ldots, \hat{\omega}_{\alpha} \ldots)^{\top}$$

with the internal frequencies

$$\hat{\omega}_{\alpha} \sim \mathcal{N}(\Omega, 1)$$

Dynamics learns topology

Topological Higher-order Kuramoto model

$$\dot{\boldsymbol{\phi}} = \hat{\omega} - \sigma \mathbf{B}_{[m+1]} \sin \mathbf{B}_{[m+1]}^{\top} \boldsymbol{\phi} - \sigma \mathbf{B}_{[m]}^{\top} \sin \mathbf{B}_{[m]} \boldsymbol{\phi},$$

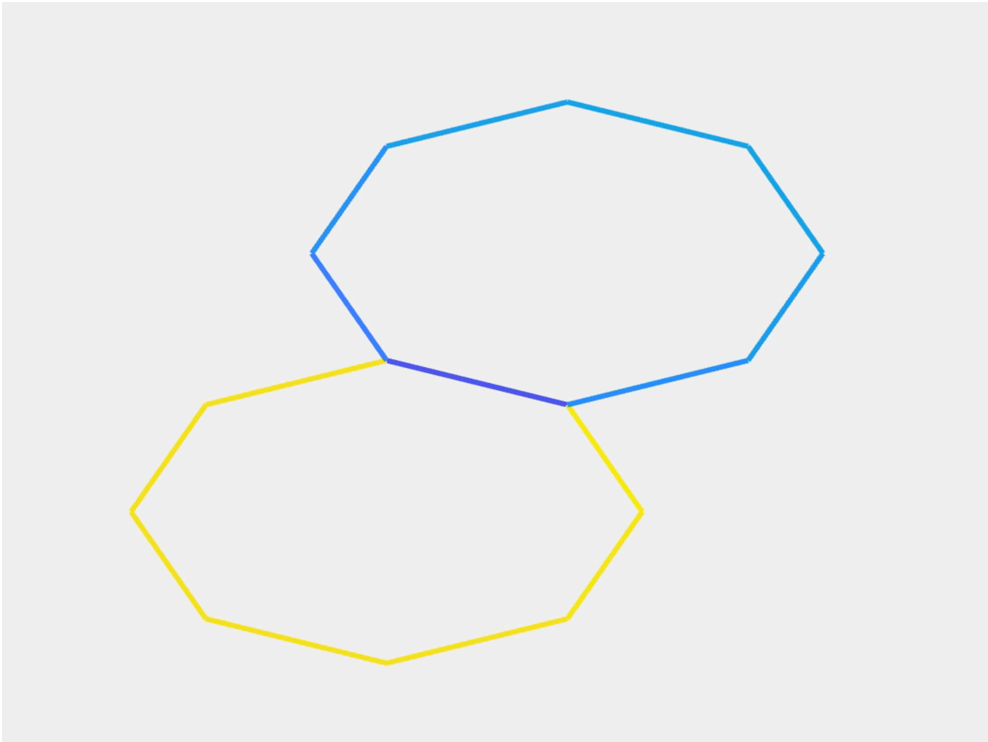
In the Topological Kuramoto model the free dynamics is localised on the m-dimensional holes

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

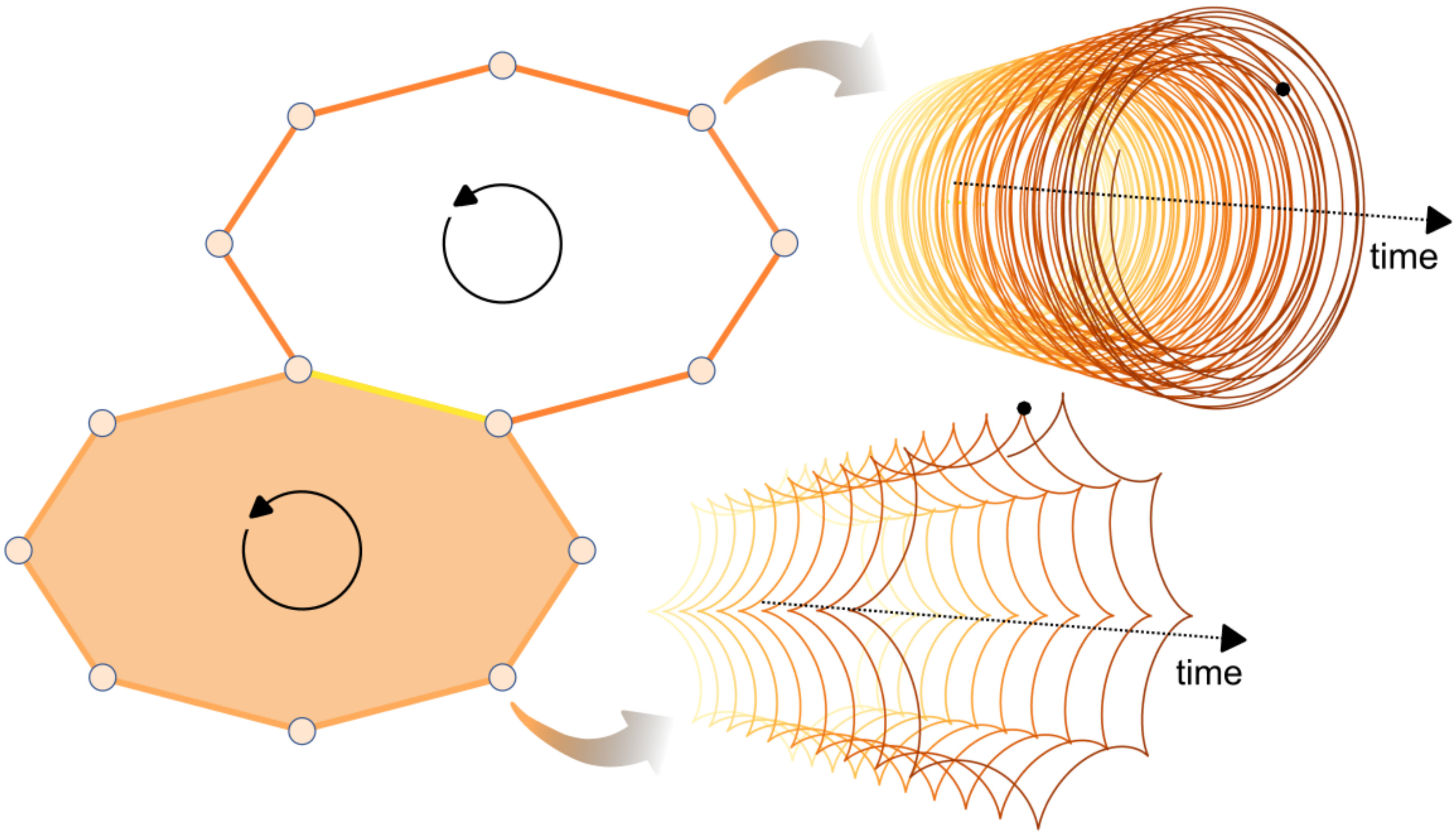
The free dynamics is localised on harmonic components

Topological Synchronisation

The dynamical ordered state has many minima
Each corresponding to a single homology class of the simplicial complex (hole)

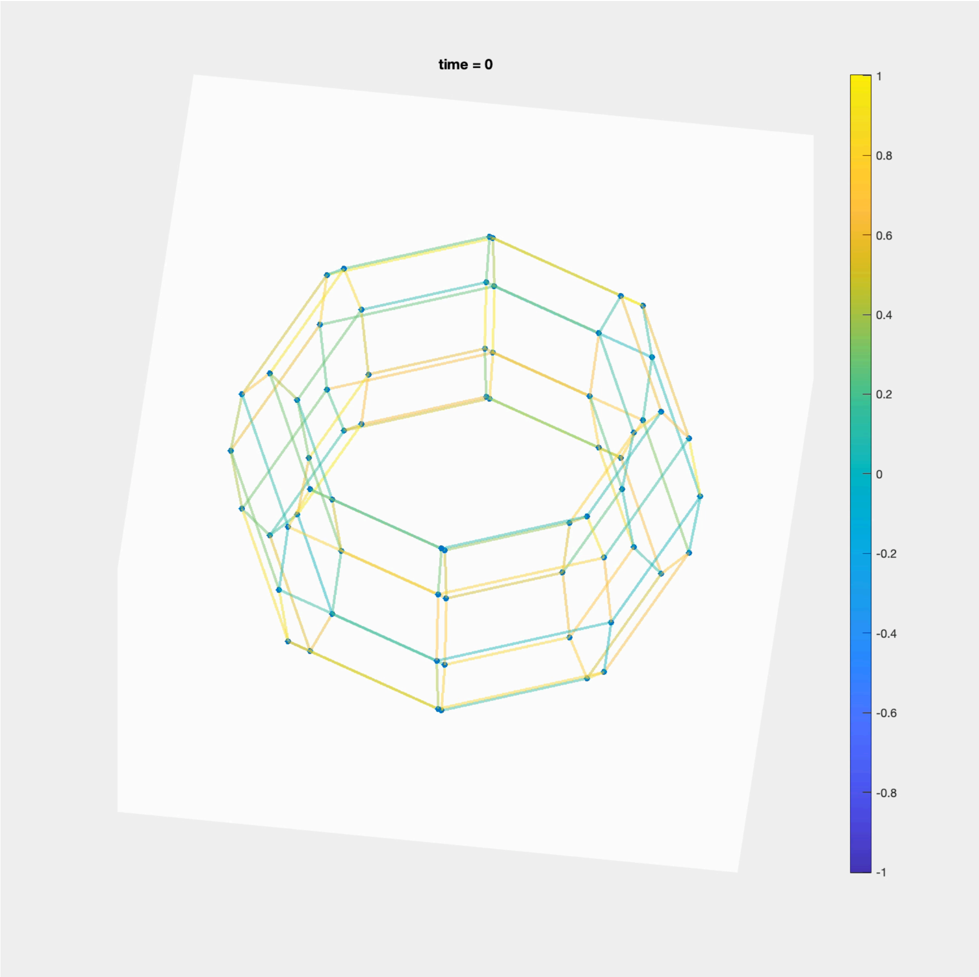


Topological Synchronisation



Global synchronisation of topological signals on simplicial and cell complexes

Global Topological Synchronization



Which are the topological and dynamical conditions under which we can observe Global Topological Synchronization?

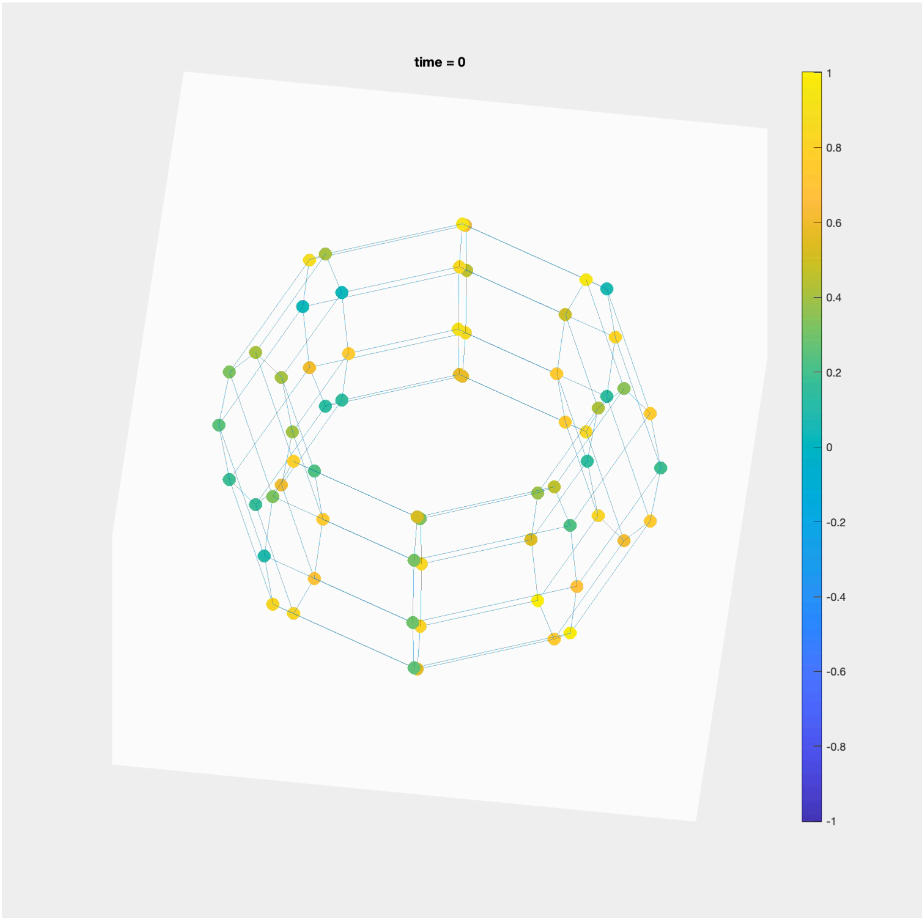
For example all the edges displaying the same dynamics?

Carletti, Giambagli Bianconi
PRL 2023

Wang, Muolo, Carletti, Bianconi
PRE 2024

Global synchronisation on graphs (When all the nodes display the same dynamics)

Global Synchronization of identical node oscillators



Uncoupled dynamics of identical node oscillators

Consider coupled identical oscillators defined on the nodes, captured by the 0-cochain $\mathbf{x} \in C^0$ with value $\mathbf{x}_r \in \mathbb{R}^d$ on each node r .

In absence of interactions these nodes obey the same dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

with arbitrary non-linear function $\mathbf{f}(\mathbf{x})$ eventually describing chaotic oscillators

The oscillators are identical but starting from different initial conditions can have very different dynamical evolution.

Coupled identical oscillators

On a graph oscillators on nearby nodes are coupled via the

graph Laplacian $\mathbf{L}_{[0]}$

Thus we consider the following coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_s [L_{[0]}]_{rs}\, \mathbf{h}(\mathbf{x}_s)$$

with arbitrary non-linear functions $\mathbf{f}(\mathbf{x}), \mathbf{h}(\mathbf{x})$.

Synchronised state

The globally synchronised state is a state in which each oscillator display the same dynamics

$$\mathbf{x}_r = \mathbf{x}^\star(t)\ \forall r \in Q_0(\mathscr{K}).$$

Where $\mathbf{x}^\star$ satisfies the dynamics

$$\frac{d\mathbf{x}^\star}{dt} = \mathbf{f}(\mathbf{x}^\star)$$

Does this solution ever exist?

Global synchronisation state of topological signals

The global synchronisation is a state in which

$$\mathbf{x}_r = \mathbf{x}_s \ \forall r, s \in Q_0(\mathscr{K})$$

The coupled dynamics for nodes signals

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[0]}]_{rs} \ \mathbf{h}(\mathbf{x}_s)$$

admits always a global synchronisation state in which all the nodes have the same dynamics.

In fact the harmonic eigenvector of the graph Laplacian is constant

$$\mathbf{u}^{harm} \propto \mathbf{1}$$

Synchronised state

The globally synchronised state is a state in which each oscillator display the same dynamics

$$\mathbf{x}_r = \mathbf{x}^\star(t) \ \forall r \in Q_0(\mathscr{K}).$$

Where $\mathbf{x}^\star$ satisfies the dynamics

$$\frac{d\mathbf{x}^\star}{dt} = \mathbf{f}(\mathbf{x}^\star)$$

Under which conditions is this solution stable for the coupled oscillators?

Master Stability Function for graphs

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the graph Laplacian.
- It is based on an expansion around a stable solution of the uncoupled dynamics.

Master Stability Function for graphs

Expanding for $\delta \mathbf{x}_r = \mathbf{x}_r - \mathbf{x}^\star$ we obtain

$$\frac{d\delta \mathbf{x}_r}{dt} = \mathbf{J}_\mathbf{f}(\mathbf{x}^\star)\delta \mathbf{x}_r - \sigma \sum_{s=1}^{N} L_{[0]}(r,s)\mathbf{J}_\mathbf{h}(\mathbf{x}^\star)\delta \mathbf{x}_s$$

This equation can be projected on the eigenmodes $\boldsymbol{\eta}_i$ of the graph Laplacian obtaining

$$\frac{d\boldsymbol{\eta}_i}{dt} = \left[\mathbf{J}_\mathbf{f}(\mathbf{x}^\star) - \sigma\lambda_i \mathbf{J}_\mathbf{h}(\mathbf{x}^\star)\right]\boldsymbol{\eta}_i$$

Therefore the synchronised state is stable if the maximum Lyapunov exponent of the above equation obeys $\Lambda_{max}(\lambda_i) < 0 \forall i$

Global synchronisation
of higher-order topological signals

Uncoupled dynamics of topological signals

Consider coupled identical oscillators defined on the n-simplices, captured by the m-cochain $\mathbf{x} \in C^m$ with $m > 0$ and values $\mathbf{x}_r \in \mathbb{R}^d$ on each m-simplex r .

In absence of interactions these simplices obey the same dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

To insure invariance of the uncoupled equations upon change of orientation of each simplex we must impose that $\mathbf{f}(\mathbf{x})$ is an **odd** function, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Proof

Consider the uncoupled dynamics $$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r)$$

Upon change of orientation of the simplex r we have $\mathbf{x}_r \to -\mathbf{x}_r$.

Therefore the dynamics becomes $$\frac{d\mathbf{x}_r}{dt} = -\mathbf{f}(-\mathbf{x}_r)$$

Imposing invariance of the dynamics under this change of orientation implies that the function $\mathbf{f}(\mathbf{x})$ must be odd, i.e. $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$.

Coupled identical topological signals

- The coupled dynamics obeys

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [L_{[m]}]_{rq}\, \mathbf{h}(\mathbf{x}_q)$$

- where in order to ensure invariance under change of orientation of the simplifies $\mathbf{h}(\mathbf{x})$ should be an **odd** function.

Global synchronisation state of topological signals

Recall that for higher order topological signals, the signs of the signal is determined by the orientation of the simplex, i.e. $\mathbf{x}(\alpha_r) = -\mathbf{x}(-\alpha_r)$

For instance a positive sign of an edge flux is relative to the orientation chosen for that edge.

It follows that the state of global synchronisation is a state in which

$$\mathbf{x}_r = u_r \mathbf{x}^\star \text{ with } u_r \in \{1, -1\} \ \forall r \in Q_n(\mathcal{K})$$

$$\text{with } \frac{d\mathbf{x}^\star}{dt} = \mathbf{f}(\mathbf{x}^\star)$$

Global topological synchronisation

- It follows that the coupled dynamics

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_q [L_{[n]}]_{rq} \, \mathbf{h}(\mathbf{x}_q)$$

- can lead to global synchronisation only if the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$ admits an eigenvector $\mathbf{u}$ with elements of constant absolute value.

- Therefore for identical higher-order oscillators there are not only dynamical but also topological constraints to global synchronisation

Topological conditions for global synchronisation

- Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

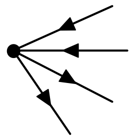
- Global synchronisation can only happen if there is one such vector $\mathbf{u}$ in the kernel of the Hodge Laplacian $\mathbf{L}_{[n]}$.

- Therefore we must have $\mathbf{L}_{[n]}\mathbf{u} = \mathbf{0}$ or, equivalently, $\mathbf{B}_{[n]}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

- For edge signals this condition implies:
  div $\mathbf{u} = \mathbf{0}$, curl $\mathbf{u} = \mathbf{0}$

Topological constraints for global synchronisation

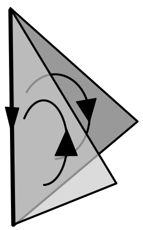
$\sigma^{(0)}$ *0-simplex*



$$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

$$\mathbf{B}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \end{pmatrix}$$

$\sigma^{(1)}$ *1-simplex*



$$\mathbf{B}_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ \times & \times & \times & \dots & \times \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\mathbf{B}_2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \times \\ \vdots \end{pmatrix}$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1.$

The condition $\mathbf{B}_{[n]}\mathbf{u} = \mathbf{0}$

This implies that:

The simplicial or cell complex must

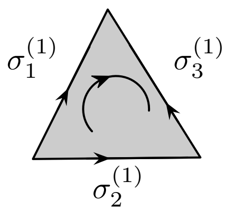
be balanced

Same number of positively and negatively

oriented simplices on the n-1 faces

Topological constraints for global synchronisation

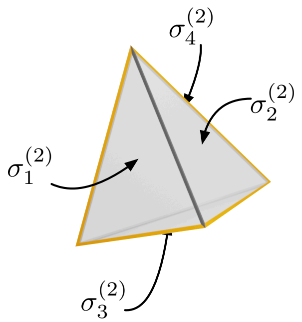
$\sigma^{(2)}$ *2-simplex*



$$\mathbf{B}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(1,1,1)\mathbf{B}_2 = -1 \neq 0$$

$\sigma^{(3)}$ *3-simplex*



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_3 = 0$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

The condition $\mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

On unweighted simplicial complexes

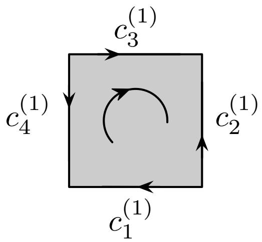
topological signals

of odd dimension can never achieve

global synchronisation

Topological constraints for global synchronisation

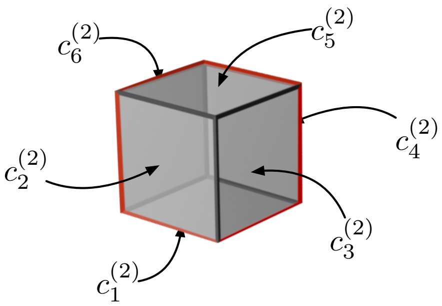
$c^{(2)}$ *2-cell*



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_2 = 0$$

$c^{(3)}$ *3-cell*



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

The condition $\mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

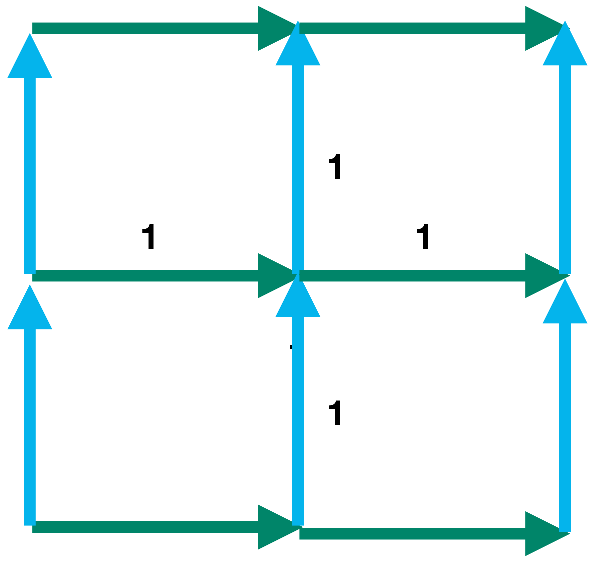
This implies that:

Cell complexes of any dimension can

achieve global synchronisation

overcoming topological obstruction

Square lattice with periodic boundary conditions (torus)



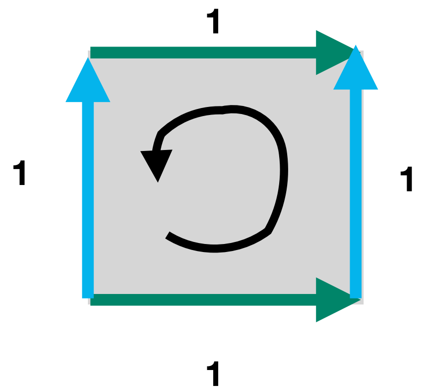
- Consider a square lattice with periodic boundary conditions (a torus).
- The eigenvector $\mathbf{u} = \mathbf{1}$ defined on each link of the network is in the kernel of $\mathbf{L}_{[1]}$, i.e. $\mathbf{1} \in \ker(\mathbf{L}_{[1]})$

Indeed $\mathbf{B}_{[1]}\mathbf{u} = \mathbf{0}, \mathbf{u}^{\top}\mathbf{B}_{[2]} = \mathbf{0}$

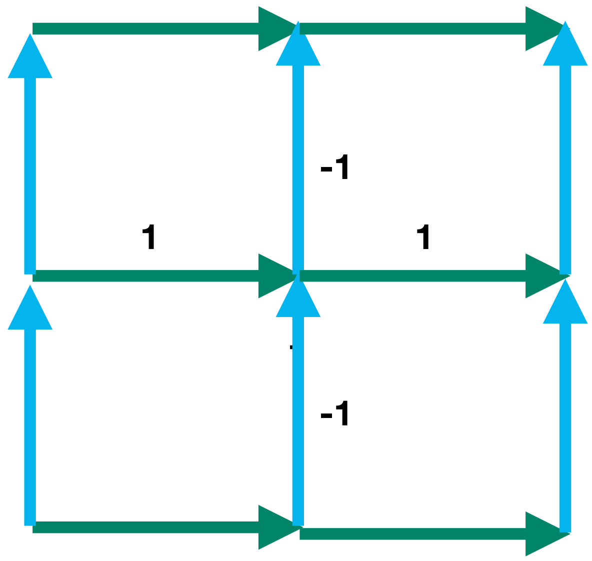
or

div $\mathbf{u} = \mathbf{0}$, curl $\mathbf{u} = \mathbf{0}$

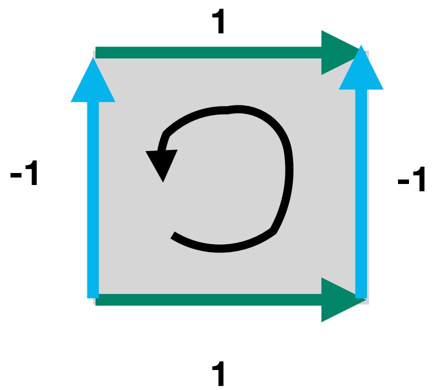
(see figures)



Square lattice with periodic boundary conditions (torus)



- Consider a square lattice with periodic boundary conditions (a torus).
- The eigenvector $\mathbf{u}$ defined on each link of the network and elements $u_r = 1$ on each x-type link and $u_r = -1$ on each y-type link is in the kernel of $\mathbf{L}_{[1]}$, i.e.$\mathbf{u} \in$ ker$(\mathbf{L}_{[1]})$



Indeed $\mathbf{B}_{[1]}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{B}_{[2]} = \mathbf{0}$

or

div $\mathbf{u} = \mathbf{0}$, curl $\mathbf{u} = \mathbf{0}$

(see figures)

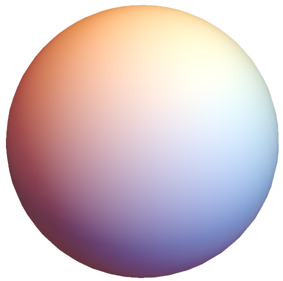
Properties of global synchronisation of topological signals

- The globally synchronised state is aligned with an harmonic eigenvector of the Hodge Laplacian, i.e. **requires topologies with holes that span the entire simplicial or cell complex**.

- Since the Hodge Laplacian has an harmonic space with dimension given by the Betti number, the same simplicial or cell complex can sustain different globalised states (see tori)

Example of manifolds sustaining global synchronisation

Synchronisation of $(n-1)$-dimensional topological signal

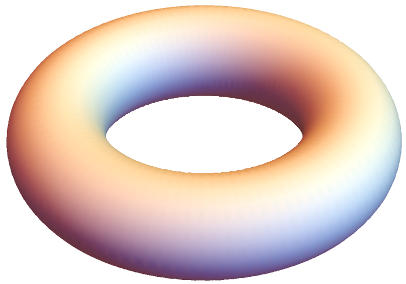


n-dimensional hypersphere

Betti numbers

$$\beta_0 = \beta_{n-1} = 1$$
$$\beta_k = 0 \text{ for } 0 < k < n-1$$

Synchronisation of any k-dimensional topological signal



n-dimensional torus (cell complex)

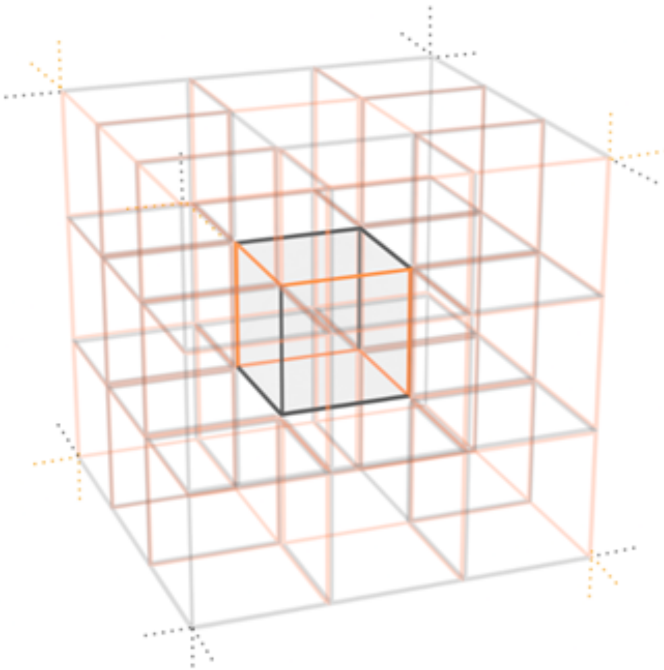
Betti numbers

$$\beta_k = \binom{n-1}{k}$$

Master Stability Function for simplicial and cell complexes

- The Master Stability Function establishes the dynamical conditions ensuring the stability of global synchronisation.
- It depends on the non-zero spectrum of the Hodge Laplacian.
- It should account for the possible degeneracy of the zero eigenvalue (a dimension of the kernel greater than one)
- It is based on an expansion around a stable solution of the uncoupled dynamics.

Global topological synchronization of unweighted d-dimensional Tori



d-dimensional Tori admit,

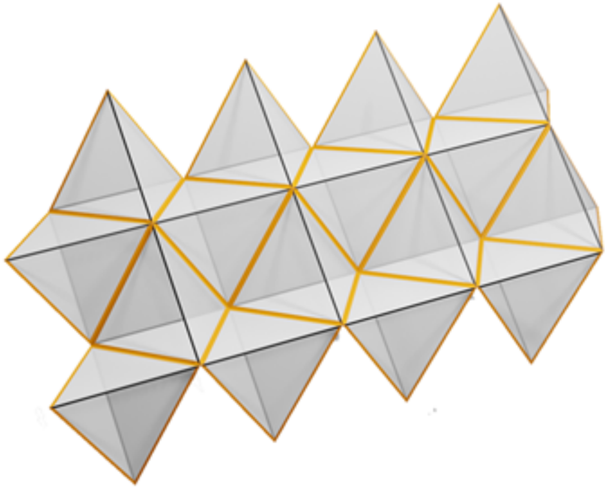
under suitable dynamical conditions,

global synchronization of any

m-dimensional topological signal with

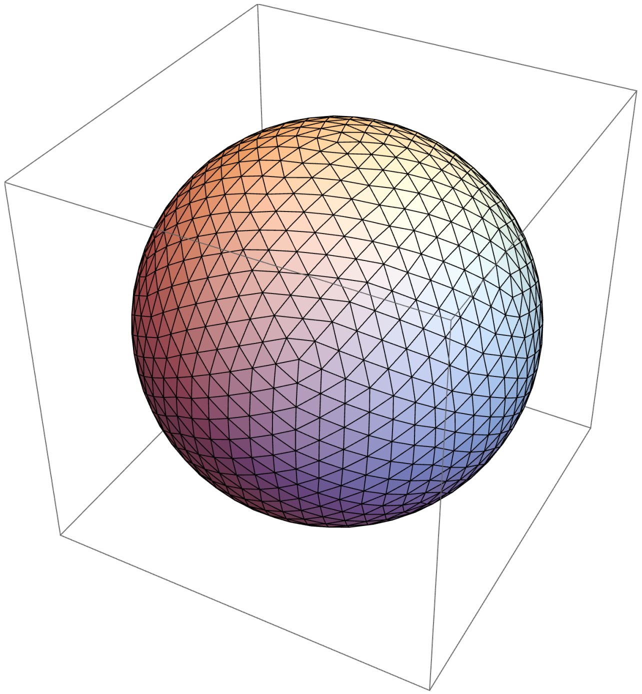
$$0 \leq m \leq d$$

Global synchronization on unweighted simplicial complexes



d-dimensional unweighted simplicial complexes never admit global synchronization of $m < d$ topological signals of odd dimension, for example edge signal on a 2 dimensional simplicial complexes

Global synchronization on Manifolds



On d-dimensional unweighted

discrete manifolds with no boundary, i.e. $n_\alpha = 1\,\forall\alpha$

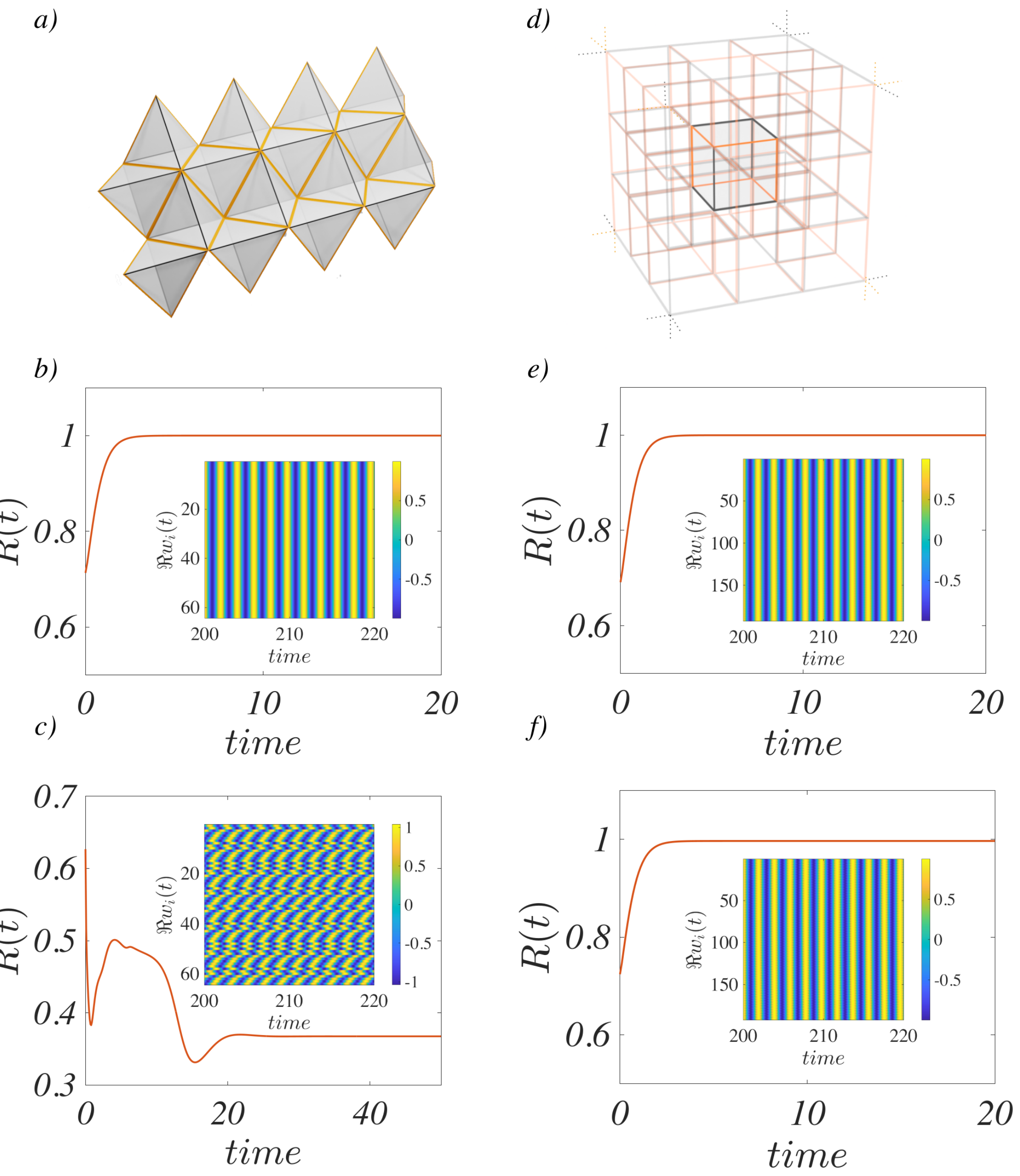
being represented as a

simplicial complexes or a cell

complex, d-dimensional topological signals

can always globally synchronize

even if d is odd.



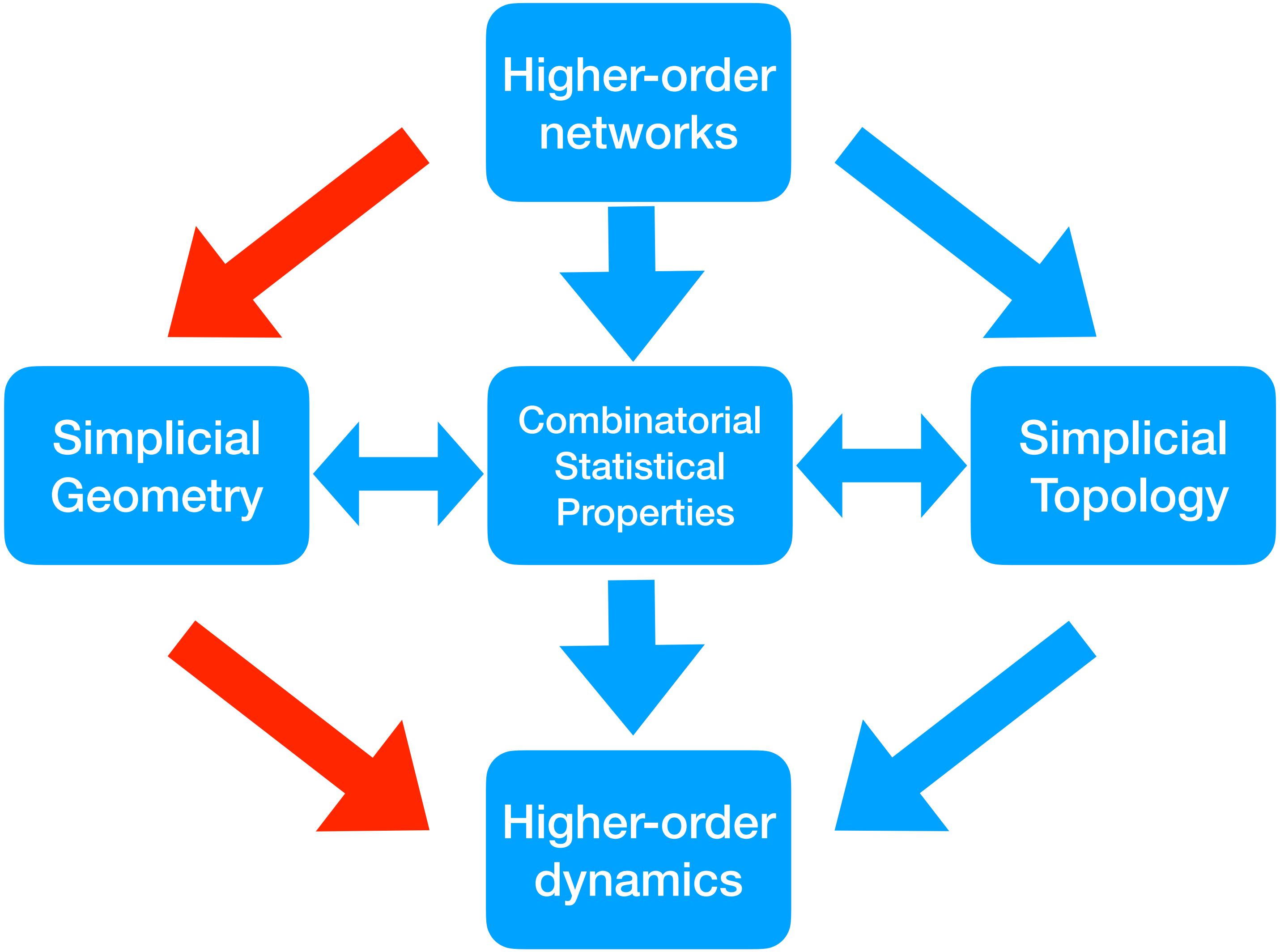
Global Topological synchronisation

Stuart-Landau coupled oscillators for 1 and 2 dimensional topological signals on a 2d Waffle and a 2d Tori

Edge signal globally synchronise on the Tori cell complex but they do not in the Waffle simplicial complex

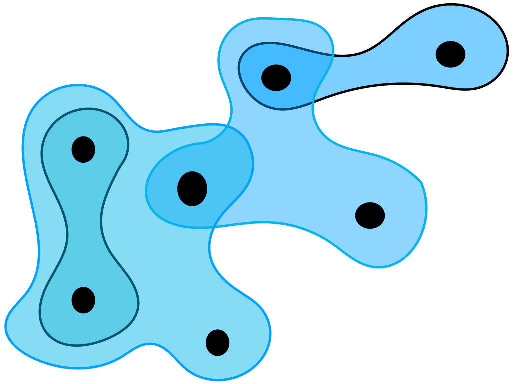
Carletti, Giambagli, Bianconi (2022)

Higher-order structure and dynamics

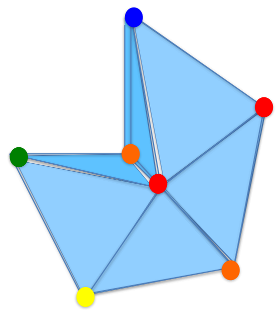


Higher-order networks

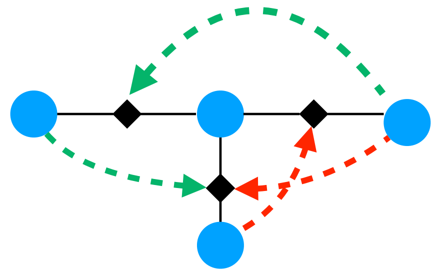
Higher-order networks are characterising the interactions between two or more nodes



Hypergraph



Simplicial complex

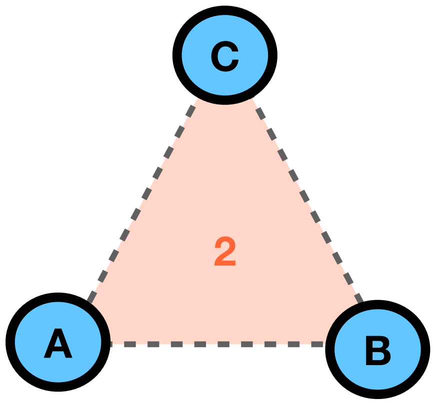


Network with triadic interactions

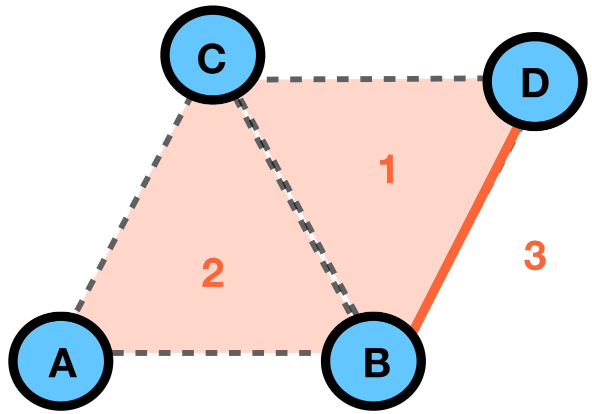
Hypergraphs or simplicial complexes?

Hypergraphs

1 Article: [A,B]



2 Articles: [A,B,C]



2 Articles: [A,B,C]
1 Article: [B,C,D]
3 Articles: [B,D]

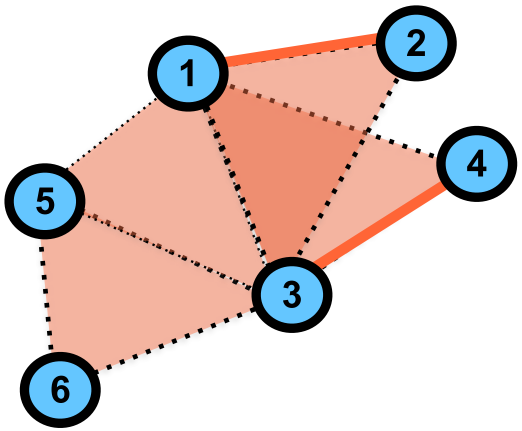
Hypergraphs

HYPERGRAPH

A hypergraph $\mathcal{G} = (V, E_H)$ is defined by a set V of N nodes and a set E_H of hyperedges, where a $(m+1)$-hyperedge indicates a set of $m+1$ nodes

$$e = [v_0, v_1, v_2, \ldots, v_m],$$

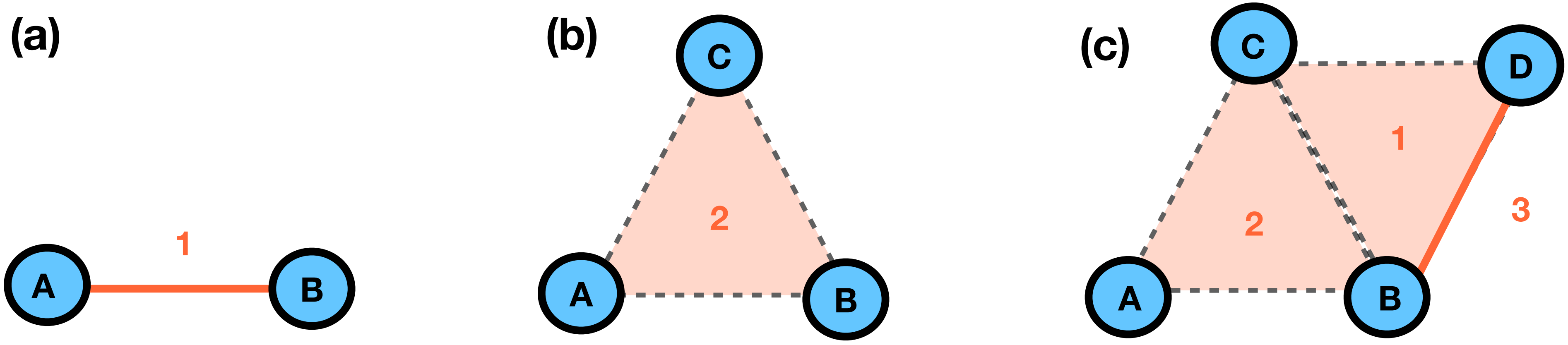
with generic value of $1 \leq m < N$.
An hyperdge describes the many-body interaction between the nodes.



There are no constraints imposing whether an arbitrary hyperedge α should belong to the hypergraph $\mathscr{H}$

$$\mathscr{H} = \{[1,2], [3,4], [1,2,3], [1,3,4], [1,3,5], [3,5,6]\}$$

Hypergraphs



Bare affinity weights (orange) $\omega_\alpha \geq 0$

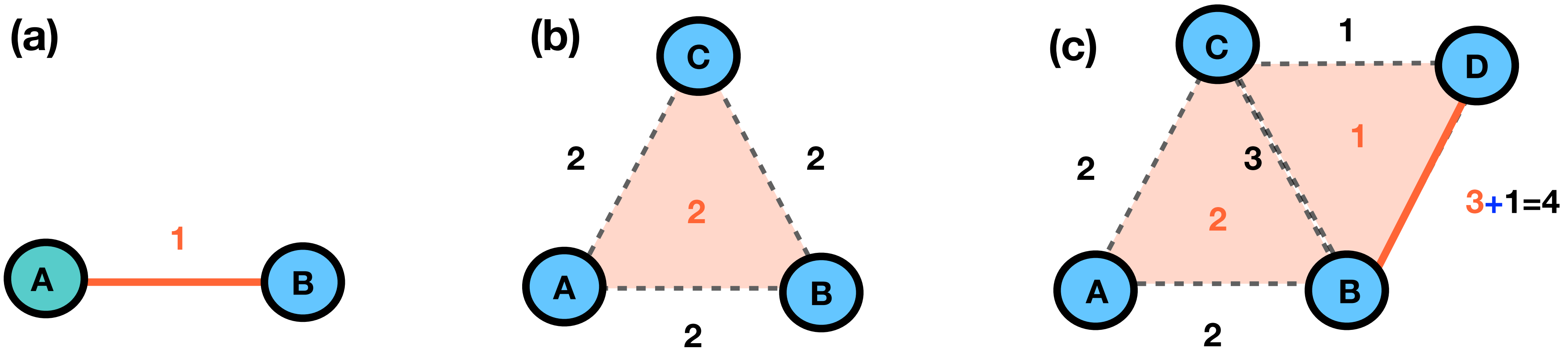
1 Article: [A,B]

2 Articles: [A,B,C]

2 Articles: [A,B,C]
1 Article: [B,C,D]
3 Articles: [B,D]

Baccini et al. (2022)

The Representation Power of Weighted Simplicial Complexes



Topological weights

$$\boxed{w_\alpha = \omega_\alpha + \sum_{\alpha' \supset \alpha} w_{\alpha'} > 0}$$

The transformation is invertible!

Baccini et al. (2022)

Weighted simplicial complexes

can encode hypergraphs

without any loss of information

At the same time they allow a

topological and geometrical investigation

of higher-order networks

Baccini et al. (2022)
ICML Prize Challenges in Topological ML 2024

Cohomology and metric matrices
(Weights)

Recall from lecture 1 the definition of coboundary operator and its dual

Cochains

m-cochains

A m-dimensional cochain $f \in C^m$ is a linear function $f : C_m \longrightarrow \mathbb{R}$, that associates to every m-chain of the simplicial complex a value in $\mathbb{R}$.

L^2 norm between cochains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathcal{K})} f_r^2$$

This scalar product can be generalised by introducing metric matrices (see next)

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{[m+1]}^\top \mathbf{f} \equiv \bar{\mathbf{B}}_{[m+1]} \mathbf{f}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \longrightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

If follows that if $f' \in C^m$ is given by $f' = \delta_m^* g$.

Then $\mathbf{f}' = \bar{\mathbf{B}}_{[m+1]}^\top \mathbf{g} = \mathbf{B}_{[m+1]} \mathbf{g}$

How this definition change is we introduce a non-trivial metric?

Metric matrices

We introduce the $N_m \times N_m$ metric matrices $\mathbf{G}^{-1}_{[m]}$ typically taken to be diagonal with elements

$$\mathbf{G}^{-1}_{[m]}(\alpha_r, \alpha_r) = w(\alpha_r)$$

where $w(\alpha_r)$ indicates the affinity weight (inverse of a "distance") associated to the simplex α_r.

For a graph, typical choices of these matrices are

$$\mathbf{G}^{-1}_{[1]}([r,s],[r,s]) = w([r,s])$$ weight of the link

$$\mathbf{G}^{-1}_{[0]}([r],[r]) = \sum_{s \in Q_0(\mathcal{K})} w([r,s])$$ strength (weighted degree) of the node

Scalar product between co-chains

We define a scalar product between m-cochains as

$$\langle f, f \rangle = \mathbf{f}^\top \mathbf{G}_{[m]}^{-1} \mathbf{f}$$

Which has an element by element expression

$$\langle f, f \rangle = \sum_{r \in Q_m(\mathscr{K})} f_r [G_{[m]}^{-1}]_{rs} f_s$$

For $\mathbf{G}_{[m]} = \mathbf{I}$ we recover the standard L^2 norm.

Coboundary operator

Coboundary operator δ_m

The coboundary operator $\delta_m : C^m \to C^{m+1}$ associates to every m-cochain of the simplicial complex $(m+1)$-cochain

$$\delta_m f = f \circ \partial_{m+1}.$$

Therefore we obtain

$$(\delta_m f)[v_0, v_1, \ldots, v_{m+1}] = \sum_{p=0}^{m+1} (-1)^p f([v_0, v_1, \ldots, v_{p-1}, v_{p+1} \ldots v_{m+1}])$$

If follows that if $g \in C^{m+1}$ is given by $g = \delta_m f$.

Then $\mathbf{g} = \mathbf{B}_{[m+1]}^{\top} \mathbf{f} \equiv \bar{\mathbf{B}}_{[m+1]} \mathbf{f}$

Adjoint of the coboundary operator

Adjoint operator δ_m^*

The adjont of the coboundary operator $\delta_m^* : C^{m+1} \longrightarrow C^m$ satisfies

$$\langle g, \delta_m f \rangle = \left\langle \delta_m^* g, f \right\rangle$$

where $f \in C^m$ and $g \in C^{m+1}$.

Ajoint operator δ_m^*

We define the matrix $\bar{\mathbf{B}}^*_{[m+1]}$ as the matrix representing δ_m^*,

i.e. if $f' = \delta_m^* g$, then $\mathbf{f}' = \bar{\mathbf{B}}^*_{[m+1]}\mathbf{g}$

From the definition it follows that

$$\boxed{\bar{\mathbf{B}}^*_{[m+1]} = \mathbf{G}_{[m]}\bar{\mathbf{B}}^\top_{[m+1]}\mathbf{G}^{-1}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}}$$

Hence if $\mathbf{G}_{[m]} = \mathbf{I}, \mathbf{G}_{[m+1]} = \mathbf{I}$ then $\mathbf{B}^*_{[m+1]} = \mathbf{B}_{[m+1]}$

Proof

We define the matrix $\mathbf{B}^*_{[m+1]}$ as the matrix representing δ^*_m,

i.e. if $f' = \delta^*_m g$, then $\mathbf{f}' = \mathbf{B}^*_{[m+1]}\mathbf{g}$

We have the scalar product

$$\langle g, \delta_m f \rangle = \mathbf{g}\mathbf{G}^{-1}_{[m+1]}\bar{\mathbf{B}}_{[m+1]}\mathbf{f}$$

$$\langle \delta^*_m g, f \rangle = \mathbf{g}(\mathbf{B}^*)^\top_{[m+1]}\mathbf{G}^{-1}_{[m]}\mathbf{f}$$

If follows that for any **f** and **g**

$$\mathbf{g}\mathbf{G}^{-1}_{[m+1]}\bar{\mathbf{B}}_{[m+1]}\mathbf{f} = \mathbf{g}\left(\bar{\mathbf{B}}^*_{[m+1]}\right)^\top \mathbf{G}^{-1}_{[m]}\mathbf{f}$$

Hence

$$\mathbf{B}^*_{[m+1]} = \mathbf{G}_{[m]}\bar{\mathbf{B}}^\top_{[m+1]}\mathbf{G}^{-1}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}$$

Weighed Hodge Laplacian

The Hodge-Laplacians

The m-dimensional Hodge-Laplacian L_m is defined as

$$L_m = L_m^{up} + L_m^{down}$$

where up and down m-dimensional Hodge Laplacians are given by

$$\begin{aligned} L_m^{up} &= \delta_m^* \delta_m, \\ L_m^{down} &= \delta_{m-1} \delta_{m-1}^*. \end{aligned}$$

The weighted Hodge Laplacian obeys Hodge decomposition

Weighted Asymmetric Hodge Laplacians

The weighted asymmetric Hodge Laplacians

$$\tilde{\mathbf{L}}^{up}_{[m]} = \bar{\mathbf{B}}^{*}_{[m+1]}\bar{\mathbf{B}}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}$$

$$\tilde{\mathbf{L}}^{down}_{[m]} = \bar{\mathbf{B}}_{[m]}\bar{\mathbf{B}}^{*}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]}$$

On a graph with weights of the edges indicated as w_{ij} and weights on the nodes indicated as s_i, we have

$\tilde{\mathbf{L}}_0 = \mathbf{G}_{[0]}\mathbf{B}_{[1]}\mathbf{G}^{-1}_{[1]}\mathbf{B}^{\top}_{[1]}$ of elements $\tilde{\mathbf{L}}_{[0]}(i,j) = \delta_{ij} - \frac{1}{s_i}w_{ij}$

Hodge decomposition for weighted Hodge Laplacians

The weighted asymmetric Hodge Laplacians

$$\tilde{\mathbf{L}}^{up}_{[m]} = \bar{\mathbf{B}}^{*}_{[m+1]}\bar{\mathbf{B}}_{[m+1]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}$$

$$\tilde{\mathbf{L}}^{down}_{[m]} = \bar{\mathbf{B}}_{[m]}\bar{\mathbf{B}}^{*}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]}$$

obey Hodge decomposition, i.e. $\tilde{\mathbf{L}}^{up}_{[m]}\tilde{\mathbf{L}}^{down}_{[m]} = \mathbf{0}, \quad \tilde{\mathbf{L}}^{down}_{[m]}\tilde{\mathbf{L}}^{up}_{[m]} = \mathbf{0}$

Proof:

$$\tilde{\mathbf{L}}^{up}_{[m]}\tilde{\mathbf{L}}^{down}_{[m]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]} = \mathbf{0}$$

$$\tilde{\mathbf{L}}^{down}_{[m]}\tilde{\mathbf{L}}^{up}_{[m]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1}_{[m]}\mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]} = \mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]} = \mathbf{0}$$

Weighted Symmetric Hodge Laplacians

The weighted symmetric Hodge Laplacians are given by

$$\hat{\mathbf{L}}^{up}_{[m]} = \mathbf{G}^{1/2}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{G}^{1/2}_{[m]}$$

$$\hat{\mathbf{L}}^{down}_{[m]} = \mathbf{G}^{-1/2}_{[m]}\mathbf{B}^{\top}_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1/2}_{[m]}$$

On a graph with weights of the edges indicated as w_{ij} and weights on the nodes indicated as s_i, we have

$\hat{\mathbf{L}}_0 = \mathbf{G}^{1/2}_{[0]}\mathbf{B}_{[1]}\mathbf{G}^{-1}_{[1]}\mathbf{B}^{\top}_{[1]}\mathbf{G}^{1/2}_{[0]}$ of elements $\hat{\mathbf{L}}_{[0]}(i,j) = \delta_{ij} - \dfrac{1}{\sqrt{s_i}}w_{ij}\dfrac{1}{\sqrt{s_j}}$

Hodge decomposition for weighted symmetric Hodge Laplacians

The weighted symmetric Hodge Laplacians

$$\hat{\mathbf{L}}^{up}_{[m]} = \bar{\mathbf{B}}^*_{[m+1]}\bar{\mathbf{B}}_{[m+1]} = \mathbf{G}^{1/2}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^\top_{[m+1]}\mathbf{G}^{1/2}_{[m]}$$

$$\hat{\mathbf{L}}^{down}_{[m]} = \bar{\mathbf{B}}_{[m]}\bar{\mathbf{B}}^*_{[m]} = \mathbf{G}^{-1/2}_{[m]}\mathbf{B}^\top_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1/2}_{[m]}$$

obey Hodge decomposition, i.e. $\mathbf{L}^{up}_{[m]}\mathbf{L}^{down}_{[m]} = \mathbf{0}, \quad \mathbf{L}^{down}_{[m]}\mathbf{L}^{up}_{[m]} = \mathbf{0}$

Proof:

$$\hat{\mathbf{L}}^{up}_{[m]}\hat{\mathbf{L}}^{down}_{[m]} = \mathbf{G}^{1/2}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^\top_{[m+1]}\mathbf{B}^\top_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{G}^{-1/2}_{[m]} = \mathbf{0}$$

$$\hat{\mathbf{L}}^{down}_{[m]}\hat{\mathbf{L}}^{up}_{[m]} = \mathbf{G}^{-1/2}_{[m]}\mathbf{B}^\top_{[m]}\mathbf{G}_{[m-1]}\mathbf{B}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^\top_{[m+1]}\mathbf{G}^{1/2}_{[m]} = \mathbf{0}$$

Asymmetric and symmetric weighted Hodge Laplacians are isospectral

The weighted symmetric and asymmetric Hodge Laplacians obey Hodge decomposition,

Thus to prove that they are isospectral it is enough to show that the up-Laplacians are isospectral and the down-Laplacians are isospectral.

Asymmetric and symmetric weighted Hodge Laplacians are isospectral

Let us prove that

$$\hat{\mathbf{L}}^{up}_{[m]} = \mathbf{G}^{1/2}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{G}^{1/2}_{[m]}, \qquad \tilde{\mathbf{L}}^{up}_{[m]} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}$$

are isospectral

Proof: $\hat{\mathbf{L}}^{up}_{[m]}\hat{\mathbf{v}} = \mathbf{G}^{1/2}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{G}^{1/2}_{[m]}\hat{\mathbf{v}} = \lambda\hat{\mathbf{v}}$

Multiplying both sides by $\mathbf{G}^{1/2}_{[m]}$ It follows $\mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\mathbf{G}^{1/2}_{[m]}\hat{\mathbf{v}} = \lambda\mathbf{G}^{1/2}_{[m]}\hat{\mathbf{v}}$

If we define

$\tilde{\mathbf{v}} = \mathbf{G}^{1/2}_{[m]}\hat{\mathbf{v}}$ it follows $\tilde{\mathbf{L}}^{up}_{[m]}\tilde{\mathbf{v}} = \mathbf{G}_{[m]}\mathbf{B}_{[m+1]}\mathbf{G}^{-1}_{[m+1]}\mathbf{B}^{\top}_{[m+1]}\tilde{\mathbf{v}} = \lambda\tilde{\mathbf{v}}$

Asymmetric and symmetric weighted Hodge Laplacians are isospectral

- The proof that $\tilde{\mathbf{L}}_{[m]}^{down}$ and $\hat{\mathbf{L}}_{[m]}^{down}$ are isospectral is left as an exercise.

Global synchronisation
of higher-order topological signals
on weighted simplicial complexes

Metric matrices (weights) can
overcome topological obstruction

Coupled identical topological signals on weighted SC

- The coupled dynamics obeys

$$\frac{d\mathbf{x}_r}{dt} = \mathbf{f}(\mathbf{x}_r) - \sigma \sum_{\beta} [\hat{L}_{[n]}]_{rq}\, \mathbf{h}(\mathbf{x}_q)$$

- where $\hat{\mathbf{L}}_{[n]}$ is the symmetric weighted Laplacian.

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Topological conditions for global synchronisation

- Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.
- Global synchronisation can only happen if there is one such vector $\mathbf{u}$ in the kernel of the Hodge Laplacian $\hat{\mathbf{L}}_{[n]}$.
- Therefore we must have $\hat{\mathbf{L}}_{[n]}\mathbf{u} = \mathbf{0}$ or, equivalently,
- $\mathbf{B}_{[n]}\mathbf{G}_{[n]}^{-1/2}\mathbf{u} = \mathbf{0}, \mathbf{u}^\top \mathbf{G}_{[n]}^{1/2}\mathbf{B}_{[n+1]} = \mathbf{0}$

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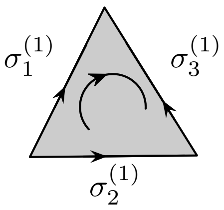
Topological constraints for global synchronisation on unweighted SC

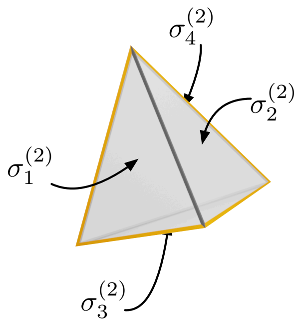
$\sigma^{(2)}$ *2-simplex*



$$\mathbf{B}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(1,1,1)\mathbf{B}_2 = -1 \neq 0$$

$\sigma^{(3)}$ *3-simplex*



$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1)\mathbf{B}_3 = 0$$

Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1.$

The condition $\mathbf{u}^\top \mathbf{B}_{[n+1]} = \mathbf{0}$

On unweighted simplicial complexes

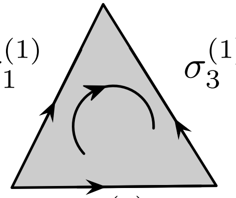
topological signals

of odd dimension can never achieve

global synchronisation

Avoiding the topological constraints for global synchronisation with geometry

$\sigma^{(2)}$ *2-simplex*



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Assume $\mathbf{u}$ is a vector of elements $|u_r| = 1$.

The condition $\mathbf{u}^\top \mathbf{G}_{[n]}^{1/2} \mathbf{B}_{[n+1]} = \mathbf{0}$ with $\mathbf{u} = \mathbf{1}$ implies

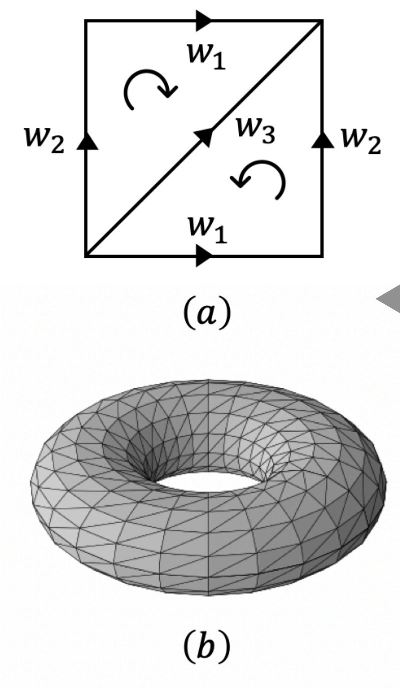
$$(1,1,1)\begin{pmatrix} w_1^{1/2} & 0 & 0 \\ 0 & w_2^{1/2} & 0 \\ 0 & 0 & w_3^{1/2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$$

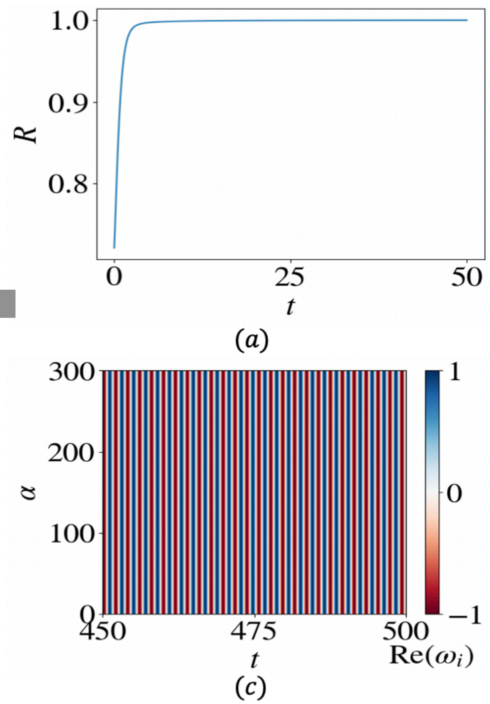
Thus if $w_1^{1/2} + w_2^{1/2} - w_3^{1/2} = 0$

the weights can overcome the topological obstruction

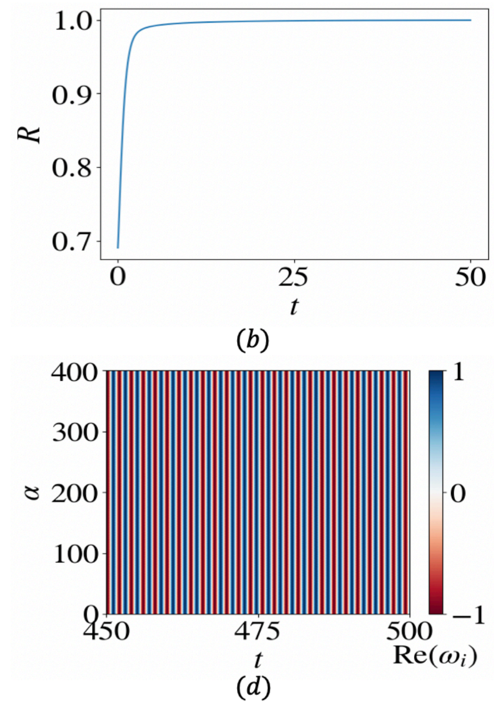
Edge signal globally synchronise on weighted SC

Triangulated Torus



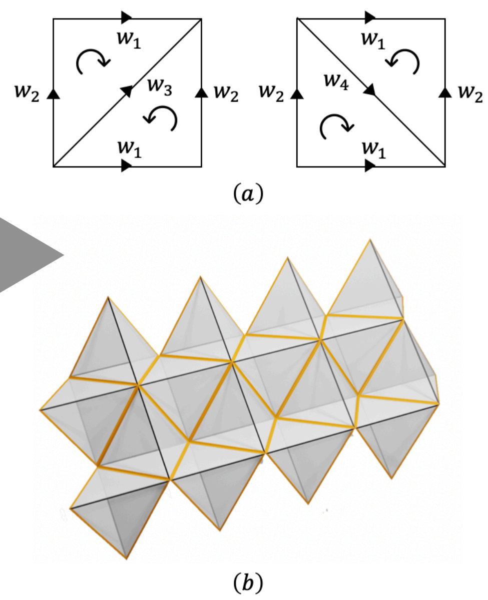


Global synchronisation on triangulated torus



Global synchronisation on triangulated waffle

Triangulate Waffle



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Lesson IV:
Global Topological Synchronisation and Network Geometry

- Global topological synchronisation and Master Stability Function
  - Global synchronisation on graphs
  - Global synchronisation on simplicial and cell complexes
- Weighted cohomology
  - Basics of weighted cohomology
  - The representation power of weighted simplicial complexes
- Global topological synchronisation on weighted simplicial complexes