ltccsoln1.tex 8 October 2012

SOLUTIONS 1

Q1.

(i) *Area of a triangle*: $A = \frac{1}{2}bh$: area = half base times perpendicular height.
Proof: In the acute-angled case, drop a perpendicular from the vertex to the base. The rectangle on this base with this height has area bh. Its area is that of four triangles, similar in pairs. One of each pair gives the triangle, which thus has area a half this, as required.

In the obtuse-angled case, divide the triangle into elements parallel to the base. Push these over to make the triangle acute-angled, without change of area, and use the above result.

(ii) *Area of a polygon*: Triangulate, and use (i).

(iii) *Area of a circle.*

(a) Without calculus (as the Greeks did it): divide the circle into a large even number of equi- angular segments. Re-arrange into a pile, with the even-numbered segments pointing one way and the odd-numbered ones the other. The pile is approximately a rectangle, with base the radius r, and with height approximately πr (by symmetry, half the circumference $2\pi r$ is on each side). This gives $A \sim r.\pi r = \pi r^2$, and the approximation can be made as accurate as we like by taking the subdivision of the circle fine enough.

(b) With calculus: use plane polar co-ordinates, with element of area $dA = dr.rd\theta = rdrd\theta$. Then $A = \int\int rdrd\theta = \int_0^r udu. \int_0^{2\pi} d\theta = \frac{1}{2}r^2.2\pi = \pi r^2$.

(iv) *Area of an ellipse.* Use plane cartesians, element of area $dA = dx.dy$. If the ellipse is round (semi-axes $a = b$), it is a circle and $A = \pi a^2$ by (iii). If not, squash it to make it round, with radius a ($a < b$ say). Then $dA \to (a/b)dA$, giving 'squashed area' πa^2. 'Unsquashing' blows this up by a factor of b/a, giving area $A = (b/a).(\pi a^2) = \pi ab$.

Q2. In Q1, we have exhausted our available plane co-ordinate systems, and so there are no more easy examples to hand!

In general, we must sub-divide, by super-imposing a square grid ('graph paper'), and counting squares, (a) inside, (b) round the edge.

Q3. We should *not* expect the general region in the plane to have an area. The above square-counting method fails with regions that are 'all edge and no middle', and we can make the edge as badly behaved as we like.

Q4. (i) The integral does not exist as a Riemann integral. Between any two reals there are both (infinitely many) rationals and (infinitely many) irrationals. So all upper Riemann sums – on $[0,1]$ say – are 1 and all lower Riemann sums are 0, regardless of how fine we make the partition.

(ii) The integral exists as a Lebesgue integral, and is 0. For, *almost all* reals are irrational. So the indicator of the rationals is a.e. 0, so integrates to 0 (we can change an integrand on a set of measure 0 without changing the integral.

The contrast here indicates how vastly more general the Lebesgue integral is than the Riemann integral. Recall also that a function f is Riemann integrable iff it is continuous a.e. The indicator of the rationals is *discontinuous everywhere*, so as far from being Riemann integrable as it could be.

Q5 (Georges BOULIGAND, 1935). For the region S_1 with area A_1 with base the hypotenuse, side 1: use cartesian coordinates to approximate its area, arbitrarily closely, by decomposing it into small squares of area $dA_1 = dxdy$.

For each such small square on side 1, construct similar small squares on sides 2 and 3, of areas dA_2, dA_3.

By Pythagoras' theorem, $dA_1 = dA_2 + dA_3$.

Summing, we get $A_1 = A_2 + A_3$ arbitrarily closely, and so exactly.

NHB