5 Equations of viscous-inviscid interaction (triple-deck flow).

It is convenient to introduce a small parameter, $\varepsilon = Re^{-1/8}$, and write the non-dimensional Navier-Stokes equations as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \varepsilon^8 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{5.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \varepsilon^8 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{5.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.3}$$

For the flow field near the wall obstacle we take $x = 1 + \varepsilon^3 X$, with $X = O(1)$ and, to keep things a little more general, we also introduce a scaled time variable, $t = \varepsilon^2 T$.

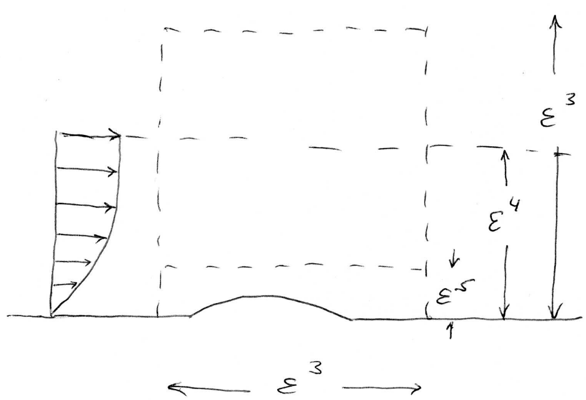

Figure 1: Local sublayers in flow past a hump.

5.1 Outer region or upper deck.

The normal coordinate is scaled according to $y = \varepsilon^3 y_1$, $y_1 = O(1)$, and the flow functions are perturbations to a uniform stream,

$$u = 1 + \varepsilon^2 u_1(X, y_1, T) + ..., \tag{5.4}$$

$$v = \varepsilon^2 v_1(X, y_1, T) + ..., \tag{5.5}$$

$$p = \varepsilon^2 p_1(X, y_1, T) + ..., \tag{5.6}$$

From the Navier-Stokes equations we have,

$$\frac{\partial u_1}{\partial X} = -\frac{\partial p_1}{\partial X}, \tag{5.7}$$

$$\frac{\partial v_1}{\partial X} = -\frac{\partial p_1}{\partial y_1}, \tag{5.8}$$

$$\frac{\partial u_1}{\partial X} + \frac{\partial v_1}{\partial y_1} = 0, \tag{5.9}$$

As we can see, the time is a parameter in effect hence the flow in the upper deck is quasi-stationary.

The system (5.7)-(5.9) can be handled in several ways. One can easily derive a Laplace's equation for p_1 or v_1, for instance, showing that the flow in the upper deck is potential. Alternatively, integrating (5.7) as $u_1 = -p_1$ and substituting for u_1 in the continuity equation (5.9), we recognize the resulting system,

$$\frac{\partial v_1}{\partial X} = -\frac{\partial p_1}{\partial y_1}, \; \frac{\partial p_1}{\partial X} = \frac{\partial v_1}{\partial y_1}, \tag{5.10}$$

as the Cauchy-Riemann conditions for a function, $F(z) = p_1 + iv_1$, analytic in the upper half plane of the complex variable $z = xX + iy_1$. Then, from the Cauchy integral formula, we have,

$$F(z) = \frac{1}{2\pi i}\oint_C \frac{F(\zeta)}{\zeta - z} d\zeta, \tag{5.11}$$

where z is a point in the upper half plane and the closed contour C consists of the X-axis and a semi-circle of an 'infinitely' large radius in the upper half plane.

For the purposes of matching with the subsequent expansion in the middle deck (main part of the boundary layer), we need a relation between the flow quantities in the limit as $y_1 \to 0$. Such a relation, derived from (5.11), is given by the Sokhotski-Plemelji formula,

$$F(z) = \frac{1}{2}F(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(\zeta)}{\zeta - z} d\zeta, \tag{5.12}$$

where z is now a point on the boundary, i.e. $z = X$, the integration proceeds along the X-axis, and the principal value of the integral is taken in (5.12). Taking the real part of (5.12) we obtain the relation between the pressure function and the normal velocity at $y_1 = 0$,

$$p_1(X, 0, T) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v_1(s, 0, T)}{s - X} ds. \tag{5.13}$$

Note: a student of aerodynamics will probably recognize the last equation as a thin airfoil formula for a symmetric airfoil without angle of attack.

5.2 Main part of the boundary layer or middle deck.

Here, as in the undisturbed boundary layer arriving at the location of the wall roughness, we have$y = \varepsilon^4 Y$, $Y = O(1)$. The flow functions expand as

$$u = U_0(Y) + \varepsilon u_2(X, Y, T) + ..., \tag{5.14}$$

$$v = \varepsilon^2 v_2(X, Y, T) + ..., \tag{5.15}$$

$$p = \varepsilon^2 p_2(X, Y, T) + ..., \tag{5.16}$$

where $U_0(Y)$ is the streamwise velocity in the boundary layer just ahead of the hump. The governing equations for the first disturbance terms are

$$U_0 \frac{\partial u_2}{\partial X} + v_2 \frac{dU_0}{dY} = 0, \tag{5.17}$$

$$\frac{\partial p_2}{\partial Y} = 0, \tag{5.18}$$

$$\frac{\partial u_2}{\partial X} + \frac{\partial v_2}{\partial Y} = 0. \tag{5.19}$$

For the velocity components we can write the solution in the form,

$$u_2 = A(X, T) \frac{dU_0}{dY}, \; v_2 = -\frac{\partial A(X, T)}{\partial X} U_0(Y), \tag{5.20}$$

with the function $A(X, T)$ undetermined at this stage, whereas the pressure does not change across the middle deck,

$$p_2 = p_2(X, T). \tag{5.21}$$

We can now perform matching between the upper and middle decks. For the pressure function we have

$$p_2(X, T) = p_1(X, 0, T). \tag{5.22}$$

For the vertical velocity, recall that $U_0(Y) \to 1$ as $Y \to \infty$,therefore

$$v_1(X, 0, T) = -\frac{\partial A(X, T)}{\partial X}. \tag{5.23}$$

This allows us to re-write (5.13) in the form

$$p_1(X, 0, T) = p_2(X, T) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\partial A(s, T)}{\partial s} \frac{ds}{s - X}. \tag{5.24}$$

5.3 The viscous sublayer or lower deck.

Here we write $y = \varepsilon^5 y_{3,}\ y_3 = O(1)$. The flow functions expand as

$$u = \varepsilon u_3(X, y_3, T) + ...,\ v = \varepsilon^3 v_3(X, y_3, T) + ...,\ p = \varepsilon^2 p_3(X, y_3, T) + ... \,. \tag{5.25}$$

Substitution into the Navier-Stokes equations (5.1)-(5.3) gives the system of boundary-layer equations,

$$\frac{\partial u_3}{\partial T} + u_3 \frac{\partial u_3}{\partial X} + v_3 \frac{\partial u_3}{\partial y_3} = -\frac{\partial p_3}{\partial X} + \frac{\partial^2 u_3}{\partial y_3^2}, \tag{5.26}$$

$$\frac{\partial p_3}{\partial y_3} = 0, \tag{5.27}$$

$$\frac{\partial u_3}{\partial X} + \frac{\partial v_3}{\partial y_3} = 0. \tag{5.28}$$

Matching the pressure function with the solution in the main deck yields

$$p_3(X, T) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\partial A(s, T)}{\partial s} \frac{ds}{s - X}. \tag{5.29}$$

Matching the streamwise velocity component we have

$$u_3(X, y_3, T) = \lambda_0 y_3 + \lambda_0 A(X, T) + ... \text{ as } y_3 \to \infty, \tag{5.30}$$

using the property $U_0(Y) = \lambda_0 Y + ...$ as $Y \to 0$.

Far upstream (in terms of the flow in the triple-deck region) the flow is expected to be unperturbed,

$$u_3(X, y_3, T) \to \lambda_0 y_3,\ v_3(X, y_3, T) \to 0,\ p_3(X, T) \to 0 \text{ as } X \to -\infty. \tag{5.31}$$

It remains to specify the shape of the wall obstacle. Suppose that, in the original non-dimensional variables,

$$y = \varepsilon^5 f\left(\frac{x-1}{\varepsilon^3}, \frac{t}{\varepsilon^2}\right), \tag{5.32}$$

for some shape function f. In the scaled triple-deck variables we then have the no-slip conditions of the form

$$u_3 = 0,\ v_3 = \frac{\partial f}{\partial T} \text{ at } y_3 = f(X, T). \tag{5.33}$$

An example of computations for the triple-deck equations in the case of flow past a corner is shown in Figure 2 (from V.V. Sychev, A.I. Ruban, Vic.V. Sychev, G.L. Korolev, Asymptotic theory of separated flows, CUP, 1998)

Discussion. The function $A(X, T)$ in (5.20) is commonly known as the displacement function. Suggest the reason for this terminology.

Exercise. Show that the undisturbed shear coefficient, λ_0, can be eliminated from the sublayer equations by an Affine transformation.

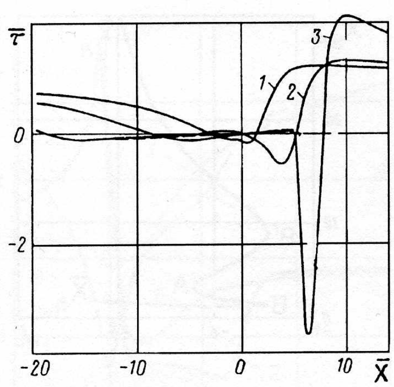

Fig. 2.21 Skin-friction distribution in the interaction region for a concave corner.

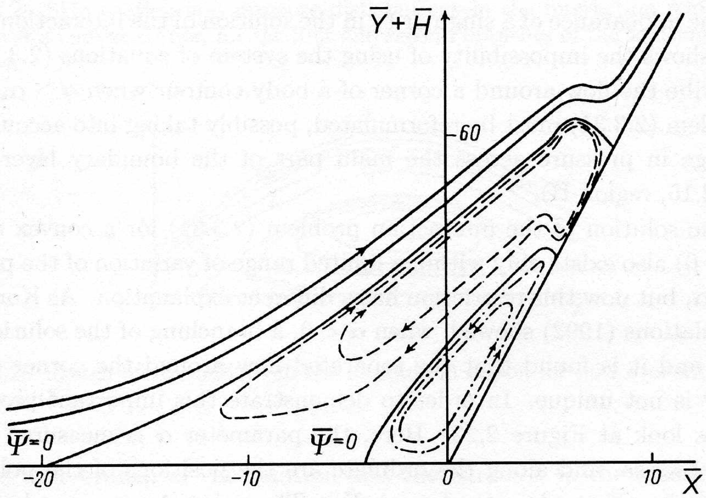

Fig. 2.22 Sketch of streamlines in the interaction region for flow over a concave corner with $\alpha = 7$.

Figure 2: Flow past a corner, the wall shape is given by $y = \alpha x$ (in scaled variables).