Algebraic number theory **LTCC 2008**

Problem Sheet 1

(1) Let K be a field extension of $\mathbb{Q}$ and let $\alpha \in K \setminus \{0\}$ be algebraic over $\mathbb{Q}$. Show that α^{-1} is algebraic over $\mathbb{Q}$.

(2) Prove Lemma 3.4, i.e. show that if K is an algebraic number field and $\alpha \in K$ then there exists $n \in \mathbb{N}$ such that $n\alpha \in R_K$.

(3) Prove the case $m \equiv 1 \pmod 4$ of Lemma 3.6, i.e. show that if $m \neq 1$ is a square-free integer such that $m \equiv 1 \pmod 4$ then $R_{\mathbb{Q}(\sqrt{m})} = \mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{m}}{2}$.

(4) Let R be a commutative ring and A an ideal of R. Show that A is a prime ideal if and only if the following condition holds: whenever $IJ \subseteq A$ for some ideals I, J then $I \subseteq A$ or $J \subseteq A$.

(5) Let $K = \mathbb{Q}(\sqrt{-14})$ and consider the ideal $A = (3, 1 + \sqrt{-14})$ of R_K. Show that $A^2 = (9, 7 + \sqrt{-14})$ and $A^4 = (5 + 2\sqrt{-14})$. Show that A^2 is not a principal ideal and deduce that the class number h_K is divisible by 4.

(6) Show that $\mu_{\mathbb{Q}(\sqrt{-1})} = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$.

(7) Let $m > 1$ be a square-free integer and $K = \mathbb{Q}(\sqrt{m})$. Show that K has a unique fundamental unit $\varepsilon = a + b\sqrt{m}$ with $a > 0$ and $b > 0$. [You can use without proof the fact that $R_K^\times \cong \{\pm 1\} \times \mathbb{Z}$, so in particular that K has at least one fundamental unit.] Find this fundamental unit for $\mathbb{Q}(\sqrt{7})$.

(8) Show that the regulator Reg_K of an algebraic number field K is well-defined, i.e. show that Definition 6.12 does not depend on any of the choices (choice and order of the fundamental units α_i, order of the embeddings σ_i, choice of the $(r + s - 1) \times (r + s - 1)$-minor of the $(r + s) \times (r + s - 1)$-matrix in the definition).