Exercise 1.5 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitz continuous boundary Γ and let Γ be decomposed into two non-empty curves Γ_1 and Γ_2: $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. For sufficiently smooth functions f and g give a variational formulation of the* mixed boundary value problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma_1, \\ \partial_n u &= g \qquad \text{on } \Gamma_2. \end{aligned}$$

2 Unique solvability of variational formulations

In this section we deal with existence and uniqueness of a solution to the problem

$$u \in V: \; a(u,v) = L(v) \quad \forall v \in V. \tag{2.1}$$

Here, V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $a(\cdot,\cdot)$ is a bilinear form and $L: V \to \mathbf{R}$ is a linear form. We will need some properties of a and L.

Definition 2.1 1. *The linear form $L: V \to \mathbf{R}$ is called* continuous *or* bounded *if*

$$\exists C > 0: \; |L(v)| \le C\|v\| \quad \forall v \in V.$$

2. *Any continuous linear form $V \to \mathbf{R}$ is called a* linear functional *on V. The space consisting of all linear functionals $V \to \mathbf{R}$ is called* dual space of V *and is denoted by V' or $\mathcal{L}(V, \mathbf{R})$. Its norm is defined by*

$$\|L\|_{V'} := \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|} \quad \textit{for } L \in V'.$$

3. *The bilinear form $a: V \times V \to \mathbf{R}$ is called* continuous *or* bounded *if*

$$\exists C_a > 0: \; |a(v,w)| \le C_a \|v\| \, \|w\| \quad \forall v, w \in V.$$

4. *The bilinear form $a: V \times V \to \mathbf{R}$ is called V-*elliptic *(or just* elliptic *if the space is clear) if*

$$\exists \alpha > 0: \; a(v,v) \ge \alpha \|v\|^2 \quad \forall v \in V.$$

We now have all the properties of bilinear and linear forms to formulate the main result of this section (Theorem 2.1 below). However, for its proof we need two more classical results from functional analysis.

Proposition 2.1 (Banach fixed point theorem)
Let V be a Banach space *(a complete vector space not necessarily having an inner product) and let $\phi: V \to V$ be a* contraction, *i.e.*

$$\exists c, \; 0 \le c < 1: \quad \|\phi(v) - \phi(w)\| \le c\|v - w\| \quad \forall v, w \in V.$$

Then there exists a unique $u \in V$ such that

$$\phi(u) = u.$$

Proposition 2.2 (Riesz representation theorem)
Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Any element $w \in V$ defines a continuous linear form $L_w \in V'$ by $L_w(v) := \langle w, v \rangle$. On the other hand, for any continuous linear form $L \in V'$ there exists a unique element $\mathcal{R}L \in V$ such that

$$L(v) = \langle \mathcal{R}L, v \rangle \quad \forall v \in V.$$

Moreover, there holds $\|\mathcal{R}L\| = \|L\|_{V'}$, i.e.

$$\|\mathcal{R}\|_{V' \to V} := \sup_{G \in V' \setminus \{0\}} \frac{\|\mathcal{R}G\|}{\|G\|_{V'}} = 1.$$

Theorem 2.1 (Lax-Milgram) *Let V be a Hilbert space, $a(\cdot, \cdot)$ a continuous, V-elliptic bilinear form and L a continuous linear form on V. Then the variational problem* (2.1) *has a unique solution $u \in V$.*

Proof. By the continuity of a we obtain that for any fixed $u \in V$ the mapping

$$Au : \; v \mapsto Au(v) := a(u, v)$$

is linear and bounded:

$$Au(v) = a(u, v) \leq C_a \|u\| \, \|v\| \leq C\|v\| \quad \forall v \in V \quad \text{with } C := C_a\|u\|.$$

This means that

$$\|Au\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|Au(v)|}{\|v\|} \leq C_a\|u\| \quad \forall u \in V,$$

i.e. $A : \; V \to V'$ is linear and continuous:

$$\|A\|_{V \to V'} := \sup_{u \in V \setminus \{0\}} \frac{\|Au\|_{V'}}{\|u\|} \leq C_a.$$

By the Riesz representation theorem there exists for any element $G \in V'$ (i.e. any continuous linear form $G : \; V \to \mathbf{R}$) a unique element $\mathcal{R}G \in V$ such that

$$G(v) = \langle \mathcal{R}G, v \rangle \quad \forall v \in V.$$

Therefore, our problem (2.1) is equivalent to

$$u \in V : \; \mathcal{R}Au = \mathcal{R}L.$$

We show that for sufficiently small $\rho > 0$ the mapping

$$\mathcal{C}_\rho : \begin{cases} V & \to & V \\ v & \mapsto & v - \rho(\mathcal{R}Av - \mathcal{R}L) \end{cases}$$

is a contraction. The Banach fixed point theorem then yields the unique existence of $u \in V$ such that

$$u - \rho(\mathcal{R}Au - \mathcal{R}L) = u, \quad \text{i.e.} \quad \mathcal{R}Au = \mathcal{R}L,$$

which proves the theorem.

To establish that, for small ρ, $\mathcal{C}_\rho$ is a contraction we use the ellipticity of a (with constant $\alpha > 0$), the norm property $\|\mathcal{R}\|_{V' \to V} = 1$ and the boundedness $\|A\|_{V \to V'} \le C_a$. This yields

$$\begin{aligned} \|v - \rho\mathcal{R}Av\|^2 &= \langle v - \rho\mathcal{R}Av, v - \rho\mathcal{R}Av\rangle = \|v\|^2 - 2\rho\langle \mathcal{R}Av, v\rangle + \rho^2\|\mathcal{R}Av\|^2 \\ &= \|v\|^2 - 2\rho\, a(v,v) + \rho^2\|\mathcal{R}Av\|^2 \\ &\le \|v\|^2 - 2\rho\alpha\|v\|^2 + \rho^2\|\mathcal{R}\|^2_{V' \to V}\|A\|^2_{V \to V'}\|v\|^2 \le (1 - 2\rho\alpha + \rho^2 C_a^2)\|v\|^2, \end{aligned}$$

i.e. $\mathcal{C}_\rho$ is a contraction for $\rho \in (0, 2\alpha/C_a^2)$. □

Remark 2.1 *The proof of the Lax-Milgram lemma implies that the mapping $A : V \to V'$ is an isomorphism (linear and bijective). Since*

$$\alpha\|v\|^2 \le a(v,v) = Av(v) \le \|Av\|_{V'}\|v\| \quad \forall v \in V$$

one finds that the inverse A^{-1} of A is continuous with norm

$$\|A^{-1}\|_{V' \to V} = \sup_{G \in V' \setminus \{0\}} \frac{\|A^{-1}G\|}{\|G\|_{V'}} \le \alpha^{-1}.$$

It follows that the variational formulation (2.1) *is* well-posed *in the sense that there exists a unique solution which depends continuously on the data (on L):*

$$\|u\| = \|A^{-1}L\| \le \alpha^{-1}\|L\|_{V'}.$$

Exercise 2.1 *Show, by using the Lax-Milgram lemma, that* (1.15) *has a unique solution provided that $f \in L_2(\Omega)$.*

Exercise 2.2 *Under appropriate conditions on f and g, prove existence and uniqueness of the solution to the variational formulation found in Exercise* 1.5.
Hint: *Use without proof that*

$$\exists C > 0 : \quad \|v\|_{L_2(\Gamma)} \le C\|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega).$$

3 Abstract error estimate for the finite element method

Let us recall the setting. The aim is to find an approximative solution to the continuous problem (2.1),

$$u \in V: \ a(u, v) = L(v) \quad \forall v \in V,$$

where we use the notation from before: V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $a(\cdot, \cdot)$ is a bilinear form and $L: \ V \to \mathbf{R}$ is a linear form.

The discrete version is as follows. *For a given finite-dimensional subspace $V_h \subset V$ find $u_h \in V_h$ such that*

$$a(u_h, v) = L(v) \quad \forall v \in V_h. \tag{3.1}$$

Theorem 3.1 *Let $a(\cdot, \cdot)$ be a continuous and V-elliptic bilinear form and let $L \in V'$ (i.e. L is a continuous linear form on V). Then the discrete variational problem* (3.1) *has a unique solution $u_h \in V_h$ and there holds the* stability estimate

$$\|u_h\| \leq \alpha^{-1} \|L\|_{V'}. \tag{3.2}$$

Proof. Since V_h is a subspace of V the continuity of a and L and the ellipticity of a remain true on V_h as forms $a: \ V_h \times V_h \to \mathbf{R}$ and $L: \ V_h \to \mathbf{R}$. Therefore, the unique existence of u_h follows from the Lax-Milgram lemma (Theorem 2.1) and the stability estimate is a discrete version of Remark 2.1. $\square$

The next theorem is the basis for error estimates of the finite element method.

Theorem 3.2 (Céa's lemma, quasi-optimal error estimate) *Let a be continuous and V-elliptic bilinear form and let $L \in V'$. Then, the solutions $u \in V$ and $u_h \in V_h$ of* (2.1) *and* (3.1), *respectively, satisfy*

$$\|u - u_h\| \leq \frac{C_a}{\alpha} \|u - v\| \quad \forall v \in V_h.$$

Here, α and C_a are the ellipticity and continuity constants of a, respectively (see §2*).*

Proof. If $\|u - u_h\| = 0$ then there is nothing to prove. Subtracting the equations of the continuous and discrete variational formulations, (2.1) and (3.1), yields the Galerkin orthogonality

$$a(u - u_h, w) = 0 \quad \forall w \in V_h.$$

We select an arbitrary $v \in V_h$ and define $w := u_h - v \in V_h$ so that $v = u_h - w$. Then, using the ellipticity of a, the Galerkin orthogonality and the continuity of a, we find that there holds

$$\begin{aligned} \alpha \|u - u_h\|^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u - u_h) + a(u - u_h, w) \\ &= a(u - u_h, u - u_h + w) = a(u - u_h, u - v) \leq C_a \|u - u_h\| \, \|u - v\|. \end{aligned}$$

Dividing by $\|u - u_h\| > 0$ and $\alpha > 0$ gives the result. $\square$

Céa's lemma provides an abstract error estimate by a term that includes the unknown solution u. However, it states two important facts. First, selecting any function $v \in V_h$, the norm $\|u - v\|$ is, up to a constant factor (independent of V_h), an upper bound for the error $\|u - u_h\|$. Therefore, based on Céa's lemma more specific error estimates can be derived if certain properties of u (regularity) are known. Second, Céa's lemma states that the finite element solution u_h is almost the best approximation of u among elements of V_h. ("Almost" refers to the factor C_a/α.) Céa's lemma is therefore also called a quasi-optimal error estimate. Note that the estimate can be formulated equivalently by

$$\|u - u_h\| \le \frac{C_a}{\alpha} \min_{v \in V_h} \|u - v\|$$

and that $\min_{v \in V_h} \|u - v\|$ is the distance of u to V_h (in the norm of V). Thus, the finite element method has the remarkable property of delivering the (almost) best approximation of an unknown function. Of course, this fact originates from the particular type of (elliptic) problems we are studying.

3.1 The energy norm

Let us assume that the bilinear form $a : V \times V \to \mathbf{R}$ is symmetric and positive definite. This means in fact that a is an inner product on V inducing a norm

$$\|v\|_a := \sqrt{a(v, v)}, \quad v \in V.$$

This norm is called *energy norm*. Its name has a physical motivation where $F(v) := \frac{1}{2}a(v, v) - L(v)$ relates to the energy of a physical system. By the ellipticity and continuity of a there holds

$$\alpha\|v\|^2 \le a(v, v) = \|v\|_a^2 = a(v, v) \le C_a\|v\|^2 \quad \forall v \in V,$$

that is

$$\alpha^{1/2}\|v\| \le \|v\|_a \le C_a^{1/2}\|v\| \quad \forall v \in V. \tag{3.3}$$

Therefore, $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent norms in V. The Galerkin orthogonality

$$a(u - u_h, v) = 0 \quad \forall v \in V_h$$

then is in fact an orthogonality: the finite element error $u - u_h$ is orthogonal to V_h with respect to the inner product $a(\cdot, \cdot)$. As we have seen in §1.3, the Galerkin orthogonality is equivalent to the best approximation property with respect to the norm induced by the inner product, in this case

$$\|u - u_h\|_a \le \|u - v\|_a \quad \forall v \in V_h. \tag{3.4}$$

Therefore, in the case of a symmetric, elliptic bilinear form, Céa's lemma (Theorem 3.2) can be improved to a best approximation property by switching from the norm $\|\cdot\|$ to the energy norm $\|\cdot\|_a$.