4 Sobolev spaces, trace theorem and normal derivative

Throughout, $\Omega \subset \mathbf{R}^n$ will be a sufficiently smooth, bounded domain.

We use the standard Sobolev spaces

$$H^0(\mathbf{R}^n) := L_2(\mathbf{R}^n),\ H^0(\Omega) := L_2(\Omega),\ H^k(\mathbf{R}^n),\ H^k(\Omega) \quad (k \text{ positive integer}).$$

Note that all these spaces are based on the use of weak derivatives up to order k. We will use the *Fourier transform* to redefine the norms in these spaces. Recall that the Fourier transform $\mathcal{F}$ is defined by (there are different normalisations possible)

$$\hat{v}(\xi) := \mathcal{F}v(\xi) := \int_{\mathbf{R}^n} e^{-i2\pi\xi\cdot x} v(x)\,dx \qquad (\xi \in \mathbf{R}^n).$$

Since

$$|\hat{v}(\xi)| = |\int_{\mathbf{R}^n} e^{-i2\pi\xi\cdot x} v(x)\,dx| \le \int_{\mathbf{R}^n} |e^{-i2\pi\xi\cdot x} v(x)|\,dx = \int_{\mathbf{R}^n} |v(x)|\,dx$$

it follows that $\hat{v}$ is well-defined whenever $v \in L_1(\mathbf{R}^n)$. The inversion formula for the Fourier transform is

$$\mathcal{F}^{-1}\hat{v}(x) := \int_{\mathbf{R}^n} e^{i2\pi\xi\cdot x} \hat{v}(\xi)\,d\xi.$$

One finds the following properties:

- If $v,\ \hat{v} \in L_1(\mathbf{R}^n)$ then $\mathcal{F}^{-1}\mathcal{F}v = v = \mathcal{F}\mathcal{F}^{-1}v$ wherever v is continuous.
- $\mathcal{F}$ generalises to a bounded linear mapping

  $$\mathcal{F}:\ L_2(\mathbf{R}^n) \to L_2(\mathbf{R}^n)$$

  and there holds

  $$(\mathcal{F}\varphi, \mathcal{F}v) = (\varphi, v) = (\mathcal{F}^{-1}\varphi, \mathcal{F}^{-1}v) \quad \forall \varphi, v \in L_2(\mathbf{R}^n),$$

  i.e., $\mathcal{F}$ is a unitary isomorphism. This property is known as *Plancherel's theorem.* The symbol $(\cdot,\cdot)$ denotes the L_2 inner product on $\mathbf{R}^n$ and will be used throughout, also for its extension by duality. When referring to the inner product on a subset of $\mathbf{R}^n$, e.g. on Ω, we add this subset as an index, e.g. $(\cdot,\cdot)_\Omega$.
- A conclusion from Plancherel's theorem is the relation

  $$\|v\|_{L_2(\mathbf{R}^n)} = \|\hat{v}\|_{L_2(\mathbf{R}^n)} \qquad \forall v \in L_2(\mathbf{R}^n).$$

Example 4.1 *Consider the one-dimensional case, i.e., $n = 1$.*
There holds $\|v'\|_{L_2(\mathbf{R})} = \|\mathcal{F}(v')\|_{L_2(\mathbf{R})}$, and for any $v \in H^1(\mathbf{R})$ with compact support we obtain

$$\mathcal{F}(v')(\xi) = \int_{\mathbf{R}} e^{-i2\pi x\xi} v'(x)\,dx = v(x)e^{-i2\pi x\xi}\Big|_{x=-\infty}^{\infty} - \int_{\mathbf{R}} -i2\pi\xi\, e^{-i2\pi x\xi} v(x)\,dx = i2\pi\xi\,\hat{v}(\xi).$$

Therefore,

$$\|v'\|_{L_2(\mathrm{R})} = \|i2\pi\xi\,\hat{v}(\xi)\|_{L_2(\mathrm{R})} = 2\pi\|\xi\,\hat{v}(\xi)\|_{L_2(\mathrm{R})}$$

and

$$\|v\|^2_{H^1(\mathrm{R})} = \|v\|^2_{L_2(\mathrm{R})} + \|v'\|^2_{L_2(\mathrm{R})} = \|\hat{v}\|^2_{L_2(\mathrm{R})} + 4\pi^2\|\xi\,\hat{v}\|^2_{L_2(\mathrm{R})} = \int_{\mathrm{R}} (1 + 4\pi^2\xi^2)|\hat{v}(\xi)|^2\,d\xi,$$

so that

$$\|v\|_{H^1(\mathrm{R})} \qquad \textit{and} \qquad \Big(\int_{\mathrm{R}} (1+\xi^2)|\hat{v}(\xi)|^2\,d\xi\Big)^{1/2} = \|(1+|\xi|^2)^{1/2}\hat{v}\|_{L_2(\mathrm{R})}$$

are equivalent norms.

This example easily generalises to higher dimensions ($n > 1$). Moreover, it leads us to the definition of Sobolev spaces on $\mathbf{R}^n$ for any positive real order.

Definition 4.1 *For $s > 0$ we define*

$$H^s(\mathbf{R}^n) := \Big\{ v \in L_2(\mathbf{R}^n);\ \|(1+|\xi|^2)^{s/2}\hat{v}\|_{L_2(\mathbf{R}^n)} < \infty \Big\}$$

with norm

$$\|v\|_{H^s(\mathbf{R}^n)} := \|(1+|\xi|^2)^{s/2}\hat{v}\|_{L_2(\mathbf{R}^n)}.$$

As in Example 4.1 one sees that, for integer s, this norm is equivalent to the usual one (based on derivatives). For non-integer s, $H^s(\mathbf{R}^n)$ is called a *fractional order Sobolev space.*

We are now in a position to analyse the trace operator in the half-space case. Consider the situation given in Figure 4.1. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we denote $x' := (x_1, \dots, x_{n-1})$. Then we define for $v \in C_0^\infty(\mathbf{R}^n)$ its trace onto the hyperplane $\mathbf{R}^{n-1} \times \{0\}$ by

$$\gamma_0 v(x') := v(x', x_n = 0), \qquad x' \in \mathbf{R}^{n-1}.$$

Theorem 4.1 (trace theorem, half-space case) *For $s > 1/2$ there exists a unique extension of γ_0 to a bounded linear operator*

$$\gamma_0 :\ H^s(\mathbf{R}^n) \to H^{s-1/2}(\mathbf{R}^{n-1}).$$

Proof. By density it suffices to consider $v \in C_0^\infty(\mathbf{R}^n)$. By the Fourier inversion formula we find that

$$\gamma_0 v(x') = \int_{\mathrm{R}^n} e^{i2\pi x\cdot\xi}\hat{v}(\xi)\,d\xi\Big|_{x_n=0} = \int_{\mathrm{R}^n} e^{i2\pi x'\cdot\xi'}\hat{v}(\xi)\,d\xi = \int_{\mathrm{R}^{n-1}} \Big(\int_{\mathrm{R}} \hat{v}(\xi',\xi_n)\,d\xi_n\Big) e^{i2\pi x'\cdot\xi'}\,d\xi'.$$

Therefore,

$$\mathcal{F}(\gamma_0 v)(\xi') = \int_{\mathrm{R}} \hat{v}(\xi',\xi_n)\,d\xi_n = \int_{\mathrm{R}} (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\hat{v}(\xi',\xi_n)\,d\xi_n$$

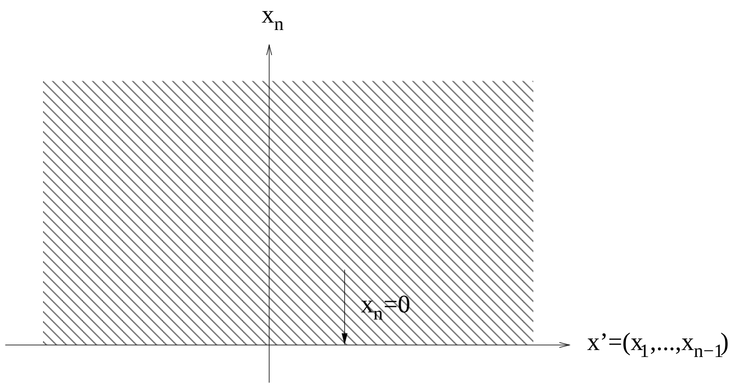

Figure 4.1: The trace in the half-space case.

and an application of the Cauchy-Schwarz inequality yields

$$|\mathcal{F}(\gamma_0 v)(\xi')|^2 \leq \int_{\mathrm{R}} (1+|\xi|^2)^{-s}\, d\xi_n \int_{\mathrm{R}} (1+|\xi|^2)^2 |\hat{v}(\xi',\xi_n)|^2\, d\xi_n.$$

Now, by the substitution $\xi_n = (1+|\xi'|^2)^{1/2} t$,

$$\begin{aligned} M_s(\xi') &:= \int_{\mathrm{R}} (1+|\xi|^2)^{-s}\, d\xi_n = \int_{\mathrm{R}} \frac{d\xi_n}{(1+|\xi'|^2+|\xi_n|^2)^s} \\ &= \frac{1}{(1+|\xi'|^2)^{s-1/2}} \int_{\mathrm{R}} \frac{dt}{(1+t^2)^s} < \infty \quad \text{iff} \quad s > 1/2. \end{aligned}$$

Therefore, we can bound

$$(1+|\xi'|^2)^{s-1/2} |\mathcal{F}(\gamma_0 v)(\xi')|^2 \leq C_s \int_{\mathrm{R}} (1+|\xi|^2)^2 |\hat{v}(\xi)|^2\, d\xi_n$$

for a constant C_s depending on s, and integration with respect to ξ' yields

$$\|\gamma_0 v\|_{H^{s-1/2}(\mathrm{R}^{n-1})} \leq C_s^{1/2} \|v\|_{H^s(\mathrm{R}^n)}.$$

□

So far we have dealt with Sobolev spaces on $\mathbf{R}^n$. For boundary value problems on Lipschitz domains this is obviously not enough.

Definition 4.2 *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain. For $s \geq 0$ we introduce the following spaces:*

$$H^s(\Omega) := H^s(\mathbf{R}^n)\Big|_{\Omega} \quad \textit{with norm} \quad \|v\|_{H^s(\Omega)} := \inf_{V|_\Omega = v} \|V\|_{H^s(\mathrm{R}^n)},$$

$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}} \quad \textit{with norm} \quad \|\cdot\|_{H^s(\Omega)},$$

and

$$\tilde{H}^s(\Omega) := \{v \in H^s(\Omega);\; v^0 \in H^s(\mathbf{R}^n)\} \quad \textit{with norm} \quad \|v\|_{\tilde{H}^s(\Omega)} := \|v^0\|_{H^s(\mathbf{R}^n)}$$

where v^0 *denotes the extension of* v *by* 0 *onto* $\mathbf{R}^n \setminus \Omega$.

For $s < 0$ *we define*

$$H^s(\Omega) := \Big(\tilde{H}^{-s}(\Omega)\Big)' \quad \textit{(dual space)} \quad \textit{with operator norm}$$

and

$$\tilde{H}^s(\Omega) := \Big(H^{-s}(\Omega)\Big)' \quad \textit{(dual space)} \quad \textit{with operator norm.}$$

Remark 4.1 *One can show that, for* $s > 0$, $\tilde{H}^s(\Omega) = H_0^s(\Omega)$ *if* $s \neq$ *integer* $+ 1/2$. *In the cases* $s =$ *integer* $+ 1/2$ *the spaces are different,* $\tilde{H}^s(\Omega) \subset H_0^s(\Omega)$ *in general.*

Without going into the details, we mention that on a Lipschitz surface or boundary Γ all the above spaces can be defined analogously when $|s| \leq 1$. To this end one uses a partition of unity and local transformations onto subsets of $\mathbf{R}^{n-1}$. Higher order spaces require more regularity of Γ.

The trace theorem can be generalised to Lipschitz domains.

Theorem 4.2 (trace theorem, general form)
Let $\Omega \subset \mathbf{R}^n$ *be a bounded Lipschitz domain with boundary* Γ.
(i) *For* $1/2 < s < 3/2$, γ_0 *has a unique extension to a bounded linear operator*

$$\gamma_0 :\; H^s(\Omega) \to H^{s-1/2}(\Gamma).$$

(ii) *For any* $s \in (1/2, 3/2)$ *and any* $v \in H^{s-1/2}(\Gamma)$ *there exists* $V := \mathcal{E}v \in H^s(\Omega)$ *such that* $\gamma_0(V) = v$ *and*

$$\|\mathcal{E}v\|_{H^s(\Omega)} \leq C_s(\Omega)\, \|v\|_{H^{s-1/2}(\Gamma)} \qquad \forall v \in H^{s-1/2}(\Gamma).$$

Remark 4.2 *Part* (ii) *of Theorem* 4.2 *means that* γ_0 *has a right-inverse:*

$$v = \gamma_0 V = \gamma_0 \mathcal{E} v$$

which is continuous, and that

$$\gamma_0 :\; H^s(\Omega) \to H^{s-1/2}(\Gamma)$$

is surjective, i.e., $\gamma_0\Big(H^s(\Omega)\Big) = H^{s-1/2}(\Gamma)$. *Of course, this right-inverse* $\mathcal{E}$ *is an extension operator.*

Having the trace operator at hand we can now make an interpretation of the Dirichlet boundary condition. Studying the Poisson equation with Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega, \qquad u|_\Gamma = g$$

we conclude two things. First, the equation $u|_\Gamma = g$ means that $\gamma_0 u = g$ in $H^{1/2}(\Gamma)$ since the variational formulation of the Poisson equation is posed in $H^1(\Omega)$ (subject to the Dirichlet condition). Second, the Dirichlet condition makes sense only for $g \in H^{1/2}(\Gamma)$. If $g \notin H^{1/2}(\Gamma)$ then there does not exist a solution $u \in H^1(\Omega)$ of the given boundary value problem. This is a conclusion of the surjectivity of the trace operator.

Besides the trace operator γ_0, in §1 we were concerned about the definition of the normal derivative $\partial_n v$ of a function $v \in H^1(\Omega)$. We now deal with this operator.

The origin for the definition of the normal derivative is the first Green's formula, in the form

$$\int_\Omega -\Delta v\, w = \int_\Omega \nabla v \cdot \nabla w - \int_\Gamma \partial_n v\, w.$$

This leads us to the definition of $\partial_n v$ for $v \in H^1(\Omega)$ by

$$\langle \partial_n v, w\rangle_\Gamma := \int_\Omega \nabla v \cdot \nabla W + \int_\Omega \Delta v\, W$$

where $W \in H^1(\Omega)$ is any extension of $w \in H^{1/2}(\Gamma)$. The notation $\langle \Phi, \varphi\rangle_\Gamma$ means the application of the functional Φ to φ defined on Γ, in this case it is the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. For Φ, $\varphi \in L_2(\Gamma)$ it is simply the $L_2(\Gamma)$-inner product between Φ and φ.

Lemma 4.1

$$\partial_n :\ \{v \in H^1(\Omega);\ \Delta v \in \tilde{H}^{-1}(\Omega)\} \to H^{-1/2}(\Gamma)$$

is well-defined and continuous when defining

$$\int_\Omega \Delta v\, W := (\Delta v, W)_\Omega$$

as duality between $\tilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$.

Proof. (i) First we show that the definition of $\langle \partial_n v, w\rangle_\Gamma$ is independent of the extension W of w. Let W_1, $W_2 \in H^1(\Omega)$ be two extensions of w, i.e., $\gamma_0 W_1 = \gamma_0 W_2 = w$. Then

$$\int_\Omega \nabla v \cdot \nabla (W_1 - W_2) + \int_\Omega \Delta v\, (W_1 - W_2) = 0$$

by the second Green identity since $W_1 - W_2 \in H_0^1(\Omega)$. This proves that $\langle \partial_n v, \gamma_0(W_1 - W_2)\rangle_\Gamma = 0$ as wanted.

(ii) Now we show the boundedness of ∂_n. Let $\mathcal{E}:\ H^{1/2}(\Gamma) \to H^1(\Omega)$ denote the extension operator from Theorem 4.2(ii). We estimate

$$\begin{aligned}
\|\partial_n v\|_{H^{-1/2}(\Gamma)} = \sup_{w\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\langle \partial_n v, w\rangle_\Gamma}{\|w\|_{H^{1/2}(\Gamma)}} \leq C \sup_{w\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\langle \partial_n v, w\rangle_\Gamma}{\|\mathcal{E}w\|_{H^1(\Omega)}} \\
= \quad C \sup_{w\in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\int_\Omega \nabla v\cdot\nabla\mathcal{E}w + \int_\Omega \Delta v\,\mathcal{E}w}{\|\mathcal{E}w\|_{H^1(\Omega)}} \leq C \sup_{W\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \nabla v\cdot\nabla W + \int_\Omega \Delta v\, W}{\|W\|_{H^1(\Omega)}} \\
\leq \quad C \sup_{W\in H^1(\Omega)\setminus\{0\}} \frac{\|v\|_{H^1(\Omega)}\,\|W\|_{H^1(\Omega)} + \|\Delta v\|_{\tilde{H}^{-1}(\Omega)}\,\|W\|_{H^1(\Omega)}}{\|W\|_{H^1(\Omega)}} \\
= \quad C\left(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{\tilde{H}^{-1}(\Omega)}\right).
\end{aligned}$$

□

Remark 4.3 $\Delta v \in L_2(\Omega)$ *implies* $\Delta v \in \tilde{H}^{-1}(\Omega)$.