C*-ALGEBRAS

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1. Introduction

This course is an introduction to C*-algebras, which belongs to the mathematical area of operator algebras. Basically, the goal is to study operators on Hilbert spaces, but not each single operator at a time in an isolated way. Instead, the idea is to look at collections of operators, so that we allow for interactions, and then study the structures obtained in this way. These structures are what we call operator algebras.

So what is the motivation to study operator algebras? Here is what the founding fathers, Murray and von Neumann, wrote in one of the first papers on operator algebras:

> **ON RINGS OF OPERATORS**
>
> **By F. J. Murray* and J. v. Neumann**
>
> **(Received April 3, 1935)**
>
> ***Introduction***
>
> **1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.**

Murray and Von Neumann, On rings of operators, Annals of Math. (2) 37 (1936), no. 1, 116–229.

All these predictions turned out to be true. Even today, there are strong interactions between operator algebras and operator theory, quantum physics, and the theory of unitary group representations. Actually, much more turned out to be true, as there are also very interesting connections between operator algebras and topology, geometry or dynamical systems.

In this course, we will develop the basics of a particular class of operator algebras, so-called C*-algebras. Our goal is to prove two main structural results. The first one tells us how commutative C*-algebras look like. This result provides the justification why we often view C*-algebra theory as non-commutative topology. The second result gives a description of general C*-algebras. It justifies why we can think of C*-algebras as operator algebras, and it also explains the name “C*-algebras”. Along the way, we will also get to know more about the structure of C*-algebras and learn techniques how to work with them.

We refer the reader to [1, 2, 3, 4] for more detailed treatments of the subject.

References

[1] K.R. Davidson, *C*-algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, 1996.

[2] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory*, Pure and Applied Mathematics, 100, Academic Press, 1983.

[3] G.J. Murphy, *C*-algebras and operator theory*, Academic Press, 1990.

[4] G.K. Pedersen, *C*-algebras and their automorphism groups*, London Mathematical Society Monographs, 14, Academic Press, 1979.

2. Banach algebras

Definition 2.1. *An algebra is a vector space with an associative bilinear multiplication.*

A normed algebra A is an algebra with norm $\|\cdot\|$, such that $\|xy\| \leq \|x\| \, \|y\|$ for all $x,y \in A$.

A Banach algebra is a complete, normed algebra.

From now on, all algebras will be over $\mathbb{C}$.

The completion of a normed algebra is a Banach algebra.

Example 2.2. *Let G be a discrete group. Then $\ell^1(G) := \{f : G \to \mathbb{C}\colon \sum_{x\in G} |f(x)| < \infty\}$ is a Banach space with respect to $\|f\|_1 := \sum_{x\in G} |f(x)|$. Define the convolution $(f * g)(s) = \sum_{t\in G} f(t)g(t^{-1}s)$. With the convolution as multiplication, $\ell^1(G)$ becomes a Banach algebra.*

Proposition 2.3. *Let A be a Banach algebra with unit 1. Then*

1.) *For $x \in A$, $\|x\| < 1 \Rightarrow 1 - x$ is invertible. More generally, $\|x\| < \lambda \Rightarrow \lambda - x$ is invertible.*
2.) *$GL(A)$ is open and $GL(A) \to GL(A)$, $x \mapsto x^{-1}$ is continuous.*

Here $\lambda - x$ stands for $\lambda 1 - x$, and $GL(A)$ is the set of invertible elements in A.

Proof. 1.) $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, and we have $\lambda - x = \lambda(1 - \lambda^{-1}x)$.

2.) Let $x \in GL(A)$, $y \in A$, $\|x - y\| < \frac{1}{\|x^{-1}\|}$. Then $y = x - (x-y) = x(1 - x^{-1}(x-y))$, and $1 - x^{-1}(x-y)$ is invertible by 1.). The formula for $(1 - x^{-1}(x-y))^{-1}$ from 1.) shows continuity. $\square$

Definition 2.4. *Let A be an algebra with unit 1. For $x \in A$, define the spectrum of x as*

$$\mathrm{Sp}\,(x) := \{\lambda \in \mathbb{C}\colon \lambda - x \notin GL(A)\}.$$

Proposition 2.5. *Let A be a Banach algebra with unit. For $x \in A$, $\mathrm{Sp}\,(x)$ is closed, and $\mathrm{Sp}\,(x) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \leq \|x\|\}$.*

Proof. $\mathbb{C} \setminus \mathrm{Sp}\,(x)$ is the pre-image of $GL(A)$ under $\lambda \mapsto \lambda - x$. The second statement follows from Proposition 2.3, 1.). $\square$

Proposition 2.6. *Let A be an algebra with unit. For $x,y \in A$, we have $\mathrm{Sp}\,(xy) \cup \{0\} = \mathrm{Sp}\,(yx) \cup \{0\}$.*

Proof. Let $0 \neq \lambda \in \mathbb{C}$. We want to show that $\lambda - xy \in GL(A) \Leftrightarrow \lambda - yx \in GL(A)$. We may assume $\lambda = 1$ (otherwise divide by λ), and it is enough to show $\Rightarrow$ by symmetry. If $1 - xy$ is invertible, then there exists $u \in A$ with $u(xy-1) = (xy-1)u = 1$ (where 1 is the unit in A). Set $w := yux - 1$. Then

$$(yx-1)w = yxyux - yux - yx + 1 = y(xy-1)ux - yx + 1 = 1,$$
$$w(yx-1) = yuxyx - yux - yx + 1 = yu(xy-1)x - yx + 1 = 1.$$

□

Proposition 2.7. *Let B be a Banach algebra with unit 1, $1 \in A \subseteq B$ a closed subalgebra. For $x \in A$, we have $\mathrm{Sp}_B(x) \subseteq \mathrm{Sp}_A(x)$. Moreover, if $\lambda \in \partial \mathrm{Sp}_A(x)$, then $\lambda \in \mathrm{Sp}_B(x)$.*

Proof. The first claim follows from $GL(A) \subseteq GL(B)$. For the second claim, suppose $\lambda \in \partial \mathrm{Sp}_A(x)$. Then there is a sequence $(\lambda_n)_n$ in $(\mathrm{Sp}_A(x))^c$ such that $\lim_n \lambda_n = \lambda$. Assume that $\lambda \notin \mathrm{Sp}_B(x)$. Then $(\lambda - x)^{-1}$ exists in B. But $(\lambda_n - x)^{-1}$ converges to $(\lambda - x)^{-1}$ in B by Proposition 2.3, 2.), so that $(\lambda - x)^{-1}$ lies in A because A is closed. $\lightning$ □

Corollary 2.8. *In the situation of the proposition above, $\mathrm{Sp}_A(x) \subseteq \mathbb{R}$ implies $\mathrm{Sp}_A(x) = \mathrm{Sp}_B(x)$.*

Theorem 2.9. *Let A be a Banach algebra with unit 1. For $x \in A$, we have $\mathrm{Sp}(x) \neq \emptyset$. If we define $r(x) := \max\{|\lambda| \colon \lambda \in \mathrm{Sp}(x)\}$, then $r(x) = \lim_n \sqrt[n]{\|x^n\|}$.*

Proof. We have that $\lambda - x$ is invertible if $|\lambda| > r(x)$. If $(\lambda - x)^{-1}$ exists for all $\lambda \in \mathbb{C}$, then $\lambda \mapsto (\lambda - x)^{-1}$ would be holomorphic. But we have $\lim_{\lambda \to \infty} (\lambda - x)^{-1} = 0$. Hence, by Liouville's Theorem, this would imply that $(\lambda - x)^{-1} \equiv 0$. $\lightning$

We have $\lambda - x = \lambda(1 - \frac{x}{\lambda})$, so that $(\lambda - x)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} (\frac{x}{\lambda})^n$ for all $|\lambda| > r(x)$. Now convergence of $\sum_{n=0}^{\infty} (\frac{x}{\lambda})^n$ implies that $\left\|(\frac{x}{\lambda})^n\right\| \leq 1$ for all but finitely many n, so that $\limsup_n \sqrt[n]{\|x^n\|} \leq |\lambda|$. So $r(x) \geq \limsup_n \sqrt[n]{\|x^n\|}$. But for $\lambda \in \mathrm{Sp}(x)$, we have $\lambda^n \in \mathrm{Sp}(x^n)$ because $\lambda^n - x^n = (\lambda - x) \sum_{k=0}^{n-1} \lambda^k x^{n-1-k}$, and thus $|\lambda^n| \leq \|x^n\|$, which implies $r(x) \leq \liminf_n \sqrt[n]{\|x^n\|}$. □

3. Commutative Banach algebras

Theorem 3.1 (Gelfand-Mazur)**.** *Let A be a Banach algebra with unit, and suppose that A is a field. Then $A = \mathbb{C}$.*

Proof. For $x \in A$, let $\lambda \in \mathrm{Sp}(x) \neq \emptyset$. Then $\lambda - x$ is not invertible, which implies $\lambda - x = 0$, so that $x = \lambda \in \mathbb{C}$. □

Definition 3.2. *Let A be a commutative Banach algebra with unit. Define*

$$\mathrm{Spec}\, A := \{\varphi : A \to \mathbb{C} \text{ unital algebra homomorphism}\}.$$

Proposition 3.3. *Let A be a commutative Banach algebra with unit. Then there is a one-to-one correspondence between* {maximal ideals of A} *and* Spec$A$ *given by* $\ker \varphi \leftarrow\!\shortmid \varphi$ *and* $I \mapsto (\varphi : A \twoheadrightarrow A/I = \mathbb{C})$.

Remark 3.4. Spec$A$ is a compact space with respect to the topology of point-wise convergence ($\lim_\alpha \varphi_\alpha = \varphi$ if and only if $\lim_\alpha \varphi_\alpha(x) = \varphi(x)$ for all $x \in A$).

Example 3.5. *Let $A = C(X)$, where X is a topological space which is compact Hausdorff. Then A, together with $\|\cdot\|_\infty$, is a commutative Banach algebra with unit. In this case, $X \to \mathrm{Spec}\, A$, $x \mapsto \mathrm{ev}_x$ is a homeomorphism. Here $\mathrm{ev}_x(f) = f(x)$ for all $f \in C(X)$.*

Remark 3.6. Let A be a commutative Banach algebra with unit. For $x \in A$, we have $\mathrm{Sp}(x) = \{\varphi(x) \colon \varphi \in \mathrm{Spec}\, A\}$.

Definition 4.1. *Let A be an algebra. An involution * on A is a map $A \to A$, $x \mapsto x^*$ such that*

(i) ** is antilinear ($(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$),*
(ii) ** is antimultiplicative ($(xy)^* = y^* x^*$),*
(iii) ** is involutive ($(x^*)^* = x$).*

Definition 4.2. *A C*-algebra is a Banachalgebra A with an involution * such that $\|x^* x\| = \|x\|^2$ for all $x \in A$.*

Examples 4.3. *1) Let H be a Hilbert space, $\mathcal{L}(H)$ the set of all bounded linear operators $H \to H$. Then $\mathcal{L}(H)$ is an algebra under point-wise addition, and composition as multiplication. Together with the operator norm ($\|x\| = \sup_{\xi \in H, \|\xi\| \le 1} \|x\xi\|$), $\mathcal{L}(H)$ becomes a Banach algebra. An involution is given by sending x to its adjoint x^* ($\langle x\xi, \eta \rangle = \langle \xi, x^* \eta \rangle$). In this way, $\mathcal{L}(H)$ becomes a C*-algebra. We have*

$$\|x\|^2 = \sup_{\xi} \|x\xi\|^2 = \sup_{\xi} \langle x\xi, x\xi \rangle = \sup_{\xi} \langle x^* x\xi, \xi \rangle \le \|x^* x\|,$$

and we always have $\|x^* x\| \le \|x^*\| \, \|x\|$. So we get $\|x\| = \|x^*\|$ and thus $\|x^* x\| = \|x\|^2$.

2) Every closed subalgebra of a C-algebra, which is invariant under the involution, is itself a C*-algebra.*

3) Let X be a locally compact Hausdorff space. Define

$$C_0(X) := \{ f \to \mathbb{C}\colon f \text{ continuous}, f \text{ vanishes at } \infty \},$$

where f vanishes at ∞ means $\forall \varepsilon > 0 \ \exists K \subseteq X$ compact such that $|f(t)| < \varepsilon \ \forall t \notin K$. Then $C_0(X)$, together with point-wise addition, multiplication, involution (complex conjugation), and $\|\cdot\|_\infty$ is a C-algebra.*

Remark 4.4. 1) Let A be a C*-algebra. Then $\|x^*\| = \|x\|$ for all $x \in A$. If A has a unit 1, then $1^* = 1$ and $\|1\| = 1$.

2) The condition $\|x^* x\| = \|x\|^2$ (C*-identity) can be replaced by $\|x\|^2 \le \|x^* x\|$.