Proposition 4.5. *(a) Let A be a C*-algebra with unit. Let $x \in A$ with $x^* = x$. Then $r(x) = \|x\|$.*

(b) The norm on a C-algebra A with unit is unique.*

Proof. (a) $x = x^*$ implies $x^2 = x^*x$, so that induction on n shows $\left\|x^{2^n}\right\| = \|x\|^{2^n}$. Hence $r(x) = \lim_n \sqrt[2^n]{\|x^{2^n}\|} = \lim_n \sqrt[2^n]{\|x\|^{2^n}} = \|x\|$.

(b) Assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two C*-norms on A. Then $\|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \|x^*x\|_2 = \|x\|_2^2$. □

Remark 4.6. Let A and B be C*-algebras. Then $A \oplus B$ is a C*-algebra with coordinate-wise operations and $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. More generally, given a family $(A_i)_{i \in I}$ of C*-algebras, then

$$\prod_{i \in I} A_i := \left\{ (x_i)_{i \in I} \colon x_i \in A_i \ \forall i, \sup_i \|x_i\| < \infty \right\}$$

is a C*-algebra with coordinate-wise operations and $\|(x_i)\| = \sup_i \|x_i\|$.

Let us now construct the unitalization of a C*-algebra. Given an algebra A, form the vectorspace $A \oplus \mathbb{C}$ and denote it by $\tilde{A}$, i.e., $\tilde{A} = \{(x, \lambda) \colon x \in A, \lambda \in \mathbb{C}\}$. $\tilde{A}$ becomes an algebra under component-wise addition and multiplication $(x,\lambda)(y,\mu) = (xy + \lambda y + \mu x, \lambda\mu)$. Then $1 := (0,1)$ is the unit of $\tilde{A}$. Moreover, $A \to \tilde{A}$, $x \mapsto (x,0)$ is an injective algebra homomorphism, which allows us to view A as a subset of $\tilde{A}$. If A is a Banach algebra, then $\tilde{A}$ becomes a Banach algebra with unit under the norm $\|(x,\lambda)\| = \|x\| + |\lambda|$. An involution on A extends to an involution on $\tilde{A}$ given by $(x,\lambda)^* = (x^*, \overline{\lambda})$.

Proposition 4.7. *Let A be a C*-algebra. There is a unique norm on $\tilde{A}$ making it into a C*-algebra.*

Proof. By Proposition 4.5 (b), it suffices to show existence.

Case 1: A has a unit e. Then $\tilde{A} \to A \oplus \mathbb{C}(1-e)$, $(x,\lambda) \mapsto (\lambda e + x, \lambda(1-e))$ is a *-algebra isomorphism, with inverse $(x - \lambda e, \lambda) \leftarrow\!\shortmid (x, \lambda(1-e))$. Note that $\mathbb{C}(1-e) \cong \mathbb{C}$ as *-algebras, so that we obtain $\tilde{A} \cong A \oplus \mathbb{C}$. By Remark 4.6, there is a C*-norm on $A \oplus \mathbb{C}$, hence on $\tilde{A}$.

Case 2: A has no unit. Let $\mathcal{L}(A)$ be the set of bounded linear operators on A. Define $\tilde{A} \to \mathcal{L}(A)$, $x \mapsto L_x$, where $L_x(b) := xb$. Then for $x = (a,\lambda) = \lambda 1 + a \in \tilde{A}$ (recall that $1 = (0,1)$), we define

$$\|x\| := \|L_x\| = \sup\{\|(\lambda 1 + a)b\| \colon b \in A, \|b\| \le 1\}.$$

$\|\cdot\|$ has the following properties:

For all $a \in A$, $\|a\|_A = \|L_a\| = \|a\|_{\tilde{A}}$. This is because $\|a\|\,\|a^*\| = \|a\|^2 = \|aa^*\| \le \|L_a\|\,\|a^*\|$, so that $\|a\| \le \|L_a\|$; and $\|L_a(z)\| = \|az\| \le \|a\|\,\|z\|$, so that $\|L_a\| \le \|a\|$.

It is submultiplicative: $\|L_{xy}\| = \|L_x L_y\| \le \|L_x\|\,\|L_y\|$.

It is a norm: Let $x = \lambda 1 + a$ with $\lambda \neq 0$. Assume $\|x\| = \|L_x\| = 0$. Then $L_x = 0$, so that $xz = 0\ \forall z \in A$. So $\lambda z + az = 0$ $\forall z \in A \Rightarrow z = -\frac{a}{\lambda} z\ \forall z \in A$. Hence $-\frac{a}{\lambda}$ must be a unit in A. ↯

So $\tilde{A}$ is a Banach algebra with respect to $\|\cdot\|$. It remains to prove the C*-identity, or equivalently, $\|L_x\|^2 \le \|L_{x^*x}\|$. Given $x \in \tilde{A}$ and $\varepsilon > 0$, there is $a \in A$ with $\|a\| \le 1$ and $\|xa\| \ge \|L_x\| - \varepsilon$. Then

$$\|L_{x^*x}\| \ge \|a^*\|\,\|x^*xa\| \ge \|a^*x^*xa\| = \|xa\|^2 \ge (\|L_x\| - \varepsilon)^2.$$

□

Remark 4.8. Let A and B be algebras, $\varphi : A \to B$ an algebra homomorphism. Then φ extends to $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ given by $\tilde{\varphi}(\lambda 1 + x) = \lambda 1 + \varphi(x)$.

Theorem 4.9. *Let A be a Banach algebra with involution *, such that $\|x^*\| = \|x\|\ \forall x \in A$, and let B be a C*-algebra. Let $\varphi : A \to B$ be a *-homomorphism. Then $\|\varphi(x)\| \le \|x\|\ \forall x \in A$.*

Proof. As $\tilde{\varphi}$ is an algebra homomorphism, we have $\mathrm{Sp}_{\tilde{B}}(\tilde{\varphi}(a)) \subseteq \mathrm{Sp}_{\tilde{A}}(a)\ \forall a \in \tilde{A}$. Hence

$$\|x\|^2 \ge \|x^*x\| \ge r(x^*x) \ge r(\varphi(x)^*\varphi(x)) = \|\varphi(x)^*\varphi(x)\| = \|\varphi(x)\|^2.$$

□

Remark 4.10. Let A be an algebra with unit 1, and $u \in A$ invertible. Then $\mathrm{Sp}(u^{-1}) = \{\lambda^{-1} \colon \lambda \in \mathrm{Sp}(u)\}$.

Reason: We have $0 \notin \mathrm{Sp}(u), \mathrm{Sp}(u^{-1})$. So let $\lambda \neq 0$. We have to show $\lambda 1 - u \in GL(A) \Rightarrow \lambda^{-1}1 - u^{-1} \in GL(A)$. But this follows from $(\lambda 1 - u)z = 1 \Rightarrow (u^{-1} - \lambda^{-1}1)z = \lambda^{-1}u^{-1}$ (which shows that $\lambda^{-1}1 - u^{-1} \in GL(A)$).

Definition 4.11. *Let A be a C*-algebra. For $x \in A$, define $\mathrm{Sp}(x) := \mathrm{Sp}_A(x)$ if A has a unit, and define $\mathrm{Sp}(x) := \mathrm{Sp}_{\tilde{A}}(x)$ if A has no unit.*

Remark 4.12. 1) We had the algebra homomorphism $\pi : \tilde{A} \to \mathcal{L}(A)$ given by $\pi(x)(a) = xa$. Then $\mathrm{Sp}(x) = \mathrm{Sp}_{\pi(\tilde{A})}\pi(x)$ whether or not A has a unit.

2) If A has no unit, then we have $0 \in \mathrm{Sp}(x)\ \forall x \in A$.

3) If A has a unit, then $\mathrm{Sp}_{\tilde{A}}(x) = \mathrm{Sp}_A(x) \cup \{0\}$.

Theorem 4.13. *Let A be a C*-algebra.*

1) If A has a unit 1, and $u \in A$ is unitary, i.e., $uu^* = 1 = u^*u$, then $\mathrm{Sp}(u) \subseteq S^1 \subseteq \mathbb{C}$.

2) If $x \in A$ is self-adjoint, i.e., x satisfies $x = x^*$, then $\mathrm{Sp}(x) \subseteq \mathbb{R}$.

3) Let $B \subseteq A$ be a sub-C-algebra. Then for every $x \in B$, $\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x)$.*

4) Let $\varphi : A \to \mathbb{C}$ be an algebra homomorphism. Then $\varphi(x^*) = \overline{\varphi(x)}\ \forall x \in A$.

Proof. 1) $\lambda \in \mathrm{Sp}(u) \Rightarrow \lambda^{-1} \in \mathrm{Sp}(u^{-1}) = \mathrm{Sp}(u^*)$ by Remark 4.10. $u^*u = 1 \Rightarrow \|u\|^2 = \|u^*u\| = \|1\| \Rightarrow \|u\| = 1$. So we have $|\lambda| \le \|u\| = 1$ and $|\lambda^{-1}| \le \|u^*\| = 1$. This implies $\lambda \in S^1$.

2) We may assume that A has a unit, otherwise work in $\tilde{A}$. Define $u := \mathrm{e}^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$. Then $u^* = \mathrm{e}^{-ix} = u^{-1}$, i.e., u is unitary. Let $\lambda \in \mathrm{Sp}(x)$. Define $z := \sum_{n=1}^{\infty} \frac{i^n(x-\lambda)^{n-1}}{n!}$. [“$\frac{\mathrm{e}^{i(x-\lambda)}-1}{x-\lambda}$”] Then $\mathrm{e}^{ix} - \mathrm{e}^{i\lambda} = (x-\lambda)z\mathrm{e}^{i\lambda}$ is not invertible, so that $\mathrm{e}^{i\lambda} \in \mathrm{Sp}(u) \subseteq S^1$ by 1). So $\lambda \in \mathbb{R}$.

3) If $x = x^*$, the claim follows from 2) and Corollary 2.8. Now let x be arbitrary. We may assume that $1 \in B \subseteq A$, otherwise work with $\tilde{B} \subseteq \tilde{A}$. Let $b := \lambda - x \in B$. Suppose $a = b^{-1}$ exists in A. We have to show that $b \in GL(B)$. $ab = ba = 1 \Rightarrow a^*b^* = b^*a^* = 1 \Rightarrow bb^*a^*a = ba = 1 \Rightarrow bb^*$ is invertible in A, and self-adjoint, so bb^* is invertible in B (see above). So b is invertible in B.

4) We always have $\varphi(y) \in \mathrm{Sp}(y)$. So if $y \in A$ satisfies $y = y^*$, then $\varphi(y) \in \mathbb{R}$. For arbitrary x, write $x = \mathrm{Re}(x) + \mathrm{i} \cdot \mathrm{Im}(x)$, where $\mathrm{Re}(x) = \frac{x+x^*}{2}$, $\mathrm{Im}(x) = \frac{x-x^*}{2\mathrm{i}}$. Then $\mathrm{Re}(x)$, $\mathrm{Im}(x)$ are self-adjoint. So

$$\varphi(x^*) = \varphi(\mathrm{Re}(x) - \mathrm{i} \cdot \mathrm{Im}(x)) = \varphi(\mathrm{Re}(x)) - \mathrm{i} \cdot \varphi(\mathrm{Im}(x)) = \overline{\varphi(x)}.$$

□

Definition 4.14. *Let A be a C*-algebra, $M \subseteq A$. We define $C^*(M)$ as the smallest sub-C*-algebra of A which contains M, called the sub-C*-algebra of A generated by M, i.e.,* $C^*(M) = \bigcap\{B\colon B \subseteq A \text{ sub-C}^*\text{-algebra}, M \subseteq B\}$.

Remark 4.15. Let P_M be the linear span of products of elements in $M \cup M^*$. In other words, P_M is the set of non-commutative polynomials in M and M^*. Then P_M is a sub-*-algebra of A, and $P_M \subseteq C^*(M)$, so that $\overline{P_M} \subseteq C^*(M)$, and by minimality, we must have $\overline{P_M} = C^*(M)$.

Remark 4.16. Let D be a C*-algebra, φ, $\psi : C^*(M) \to D$ *-homomorphisms with $\varphi|_M = \psi|_M$. Then $\varphi = \psi$.

Definition 4.17. *Let A be a C*-algebra. An element $x \in A$ is called normal if $xx^* = x^*x$.*

Remark 4.18. x is normal if and only if $C^*(x)$ is commutative.

Theorem 4.19. *Let B be a commutative C*-algebra with unit. Then $\chi : B \to C(\operatorname{Spec} B)$ is an isometric *-isomorphism.*

Proof. It is clear that χ is an algebra homomorphism. By Theorem 4.13, we have $\chi(x^*)(\varphi) = \varphi(x^*) = \overline{\varphi(x)} = \overline{\chi(x)(\varphi)}$. χ is isometric since $\|\chi(b)\|_\infty^2 = \left\|\overline{\chi(b)}\chi(b)\right\|_\infty = \|\chi(b^*b)\|_\infty = r(b^*b) = \|b^*b\| = \|b\|^2$. Moreover, χ is surjective since $\chi(B) \subseteq C(\operatorname{Spec} B)$ is a closed sub-*-algebra which separates points. So the Stone-Weierstrass Theorem implies $\chi(B) = C(\operatorname{Spec} B)$. ☐

Definition 4.20. *Let B be a commutative C*-algebra. We define $\operatorname{Spec} B := \operatorname{Spec} \tilde{B} \setminus \{\tilde{0}\}$, where $\tilde{0} : \tilde{B} \to \mathbb{C}$ is the extension of the zero homomorphism $0 : B \to \mathbb{C}$.*

Remark 4.21. $\operatorname{Spec} B$ is locally compact. Restricting the isomorphism $\tilde{B} \cong C(\operatorname{Spec} \tilde{B})$ from Theorem 4.19 to B, we obtain an isomorphism $B \cong \{f \in C(\operatorname{Spec} \tilde{B})\colon f(\tilde{0}) = 0\} \cong C_0(\operatorname{Spec} B)$.

Remark 4.22. Let B be a commutative C*-algebra with unit e. Then $\tilde{B} \cong B \oplus \mathbb{C}(1-e)$ as C*-algebras. Given $\varphi \in \operatorname{Spec} B$, we obtain $\varphi' \in \operatorname{Spec} \tilde{B}$ by setting $\varphi'(b + \lambda(1-e)) = \varphi(b)$. The map $\varphi \mapsto \varphi'$ identifies our old definition of the spectrum, i.e., $\{\varphi : B \to \mathbb{C}\colon \varphi \text{ homomorphism}, \varphi(e) = 1\}$, with our new definition $\operatorname{Spec} \tilde{B} \setminus \{\tilde{0}\}$.

Theorem 4.23. *Let A be a C*-algebra and $x \in A$ normal.*

(a) Suppose A has a unit 1. Then $\operatorname{Spec} C^*(x,1) \to \operatorname{Sp}(x)$, $\varphi \mapsto \varphi(x)$ is a homeomorphism.

(b) Let A be arbitrary, i.e., not necessarily with unit. Then $\operatorname{Spec} C^*(x) \to \operatorname{Sp}(x) \setminus \{0\}$, $\varphi \mapsto \varphi(x)$ is a homeomorphism.

Proof. (a) $\varphi \mapsto \varphi(x)$ is injective, surjective since $\operatorname{Sp}(x) = \{\varphi(x)\colon \varphi \in \operatorname{Spec} C^*(x,1)\}$, and continuous by definition of the topology on $\operatorname{Spec} C^*(x,1)$. As our spaces are compact, this implies that our map is a homeomorphism.

(b) We know by (a) that $\operatorname{Spec}(C^*(x)\tilde{}) \to \operatorname{Sp}(x)$, $\varphi \mapsto \varphi(x)$ is a homeomorphism, and it sends $\tilde{0}$ to 0. Our claim follows. ☐

Functional calculus. 1.) Let A be a C*-algebra with unit 1, and $x \in A$ normal. Then we have an isomorphism $C^*(x,1) \cong C(\operatorname{Spec} C^*(x,1)) \cong C(\operatorname{Sp}(x))$ sending x to $\mathrm{id}_{\operatorname{Sp}(x)}$. We denote the inverse $C(\operatorname{Sp}(x)) \to C^*(x,1) \subseteq A$ by $f \mapsto f(x)$. This gives rise to functional calculus. We have $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $\overline{f}(x) = f(x)^*$, and this generalizes functional calculus with polynomials or absolutely convergent power series. If $f(z) = \sum_{m,n=0}^\infty \lambda_{m,n} z^m \overline{z}^n$, viewed as a continuous function on $\mathbb{C}$ or $\operatorname{Sp}(x)$, then $f(x) = \sum_{m,n=0}^\infty \lambda_{m,n} x^m x^{*n}$.

2.) Let A be a general C*-algebra, not necessarily with unit, and $x \in A$ normal. Then we have an isomorphism $C^*(x) \cong C_0(\operatorname{Sp}(x)) = \{f \in C(\operatorname{Sp}(x))\colon f(0) = 0\}$. Again, the inverse gives rise to functional calculus $C_0(\operatorname{Sp}(x)) \to C^*(x) \subseteq A$, $f \mapsto f(x)$.