**Examples 4.24.** *1.) Let A be a C*-algebra, $x \in A$ with $x = x^*$. Then we can write x in a unique way as $x = x_+ - x_-$, where $x_+$ and $x_-$ are self-adjoint,* $\mathrm{Sp}(x_+), \mathrm{Sp}(x_-) \subseteq [0,\infty)$, *$x_+x_- = 0$: Define $f_+ : \mathbb{R} \to [0,\infty), t \mapsto \max\{0,t\}$ and $f_- : \mathbb{R} \to [0,\infty), t \mapsto -\min\{0,t\}$. Then set $x_+ := f_+(x)$, $x_- := f_-(x)$.*

*2.) Let A and x be as in 1.), and suppose that* $\mathrm{Sp}(x) \subseteq [0,\infty)$. *Then there is a unique self-adjoint element a in A with* $\mathrm{Sp}(a) \subseteq [0,\infty)$ *with $a^2 = x$: Define $a := \sqrt{x}$.*

**Remark 4.25.** We have $f(\mathrm{Sp}(x)) = \mathrm{Sp}(f(x))$ for $f \in C(\mathrm{Sp}(x))$ (or $f \in C_0(\mathrm{Sp}(x))$.

## 5. Positive elements in C*-algebras

**Definition 5.1.** *Let A be a C*-algebra. $a \in A$ is called positive ($a \geq 0$) if $a = a^*$ and* $\mathrm{Sp}(a) \subseteq [0,\infty)$.

**Remark 5.2.** We have seen that every positive element $a$ has a unique square root $\sqrt{a}$ in $A$.

**Lemma 5.3.** *Let A be a C*-algebra with unit 1, $a \in A$ self-adjoint, and $\lambda \in \mathbb{C}$ with $\lambda \geq \|a\|$. Then $a \geq 0$ if and only if $\|\lambda 1 - a\| \leq \lambda$.*

*Proof.* $a \geq 0 \Leftrightarrow \mathrm{Sp}(a) \subseteq [0,\infty) \Leftrightarrow \chi(a) = \mathrm{id}_{\mathrm{Sp}(a)} \geq 0 \Leftrightarrow \|\chi(\lambda 1 - a)\|_\infty = \|\lambda 1 - \chi(a)\|_\infty \leq \lambda \Leftrightarrow \|\lambda 1 - a\| \leq \lambda$. □

**Proposition 5.4.** *If a and b are positive, then $a+b$ is positive.*

*Proof.* Set $\lambda := \|a\| + \|b\|$. Then $\lambda \geq \|a+b\|$. We have $\|\lambda - (a+b)\| \leq \|\|a\| - a\| + \|\|b\| - b\| \leq \lambda$ by Lemma 5.3. □

**Proposition 5.5.** *Let A be a C*-algebra, $a \in A$. The following are equivalent:*

(1) *$a \geq 0$,*
(2) *There exists a self-adjoint $h \in A$ with $a = h^2$,*
(3) *There exists $x \in A$ with $a = x^*x$.*

*Proof.* (1) ⇒ (2): $h := \sqrt{a}$. (2) ⇒ (3): $x := h$. (3) ⇒ (1): We first show that $-x^*x \geq 0 \Rightarrow x = 0$. If $-x^*x \geq 0$, then $-xx^* \geq 0$ because $\mathrm{Sp}(x^*x) \cup \{0\} = \mathrm{Sp}(xx^*) \cup \{0\}$ by Proposition 2.6. Write $x = x_1 + \mathrm{i} \cdot x_2$ with $x_1 = \mathrm{Re}(x)$ and $x_2 = \mathrm{Im}(x)$. Then

$$x^*x + xx^* = (x_1^2 + \mathrm{i} \cdot x_1x_2 - \mathrm{i} \cdot x_2x_1 + x_2^2) + (x_1^2 + \mathrm{i} \cdot x_2x_1 - \mathrm{i} \cdot x_1x_2 + x_2^2) = 2x_1^2 + 2x_2^2.$$

So $x^*x = 2x_1^2 + 2x_2^2 - xx^* \geq 0$, so that $\mathrm{Sp}(x^*x) = \{0\} \Rightarrow x^*x = 0 \Rightarrow x = 0$. Now write $x^*x = u - v$, where $u = (x^*x)_+$, $v = (x^*x)_-$ and $uv = vu = 0$. Set $y := xv$. Then

$$-y^*y = -vx^*xv = -v(u-v)v = v^3 \geq 0,$$

so that by the above, $y = 0$. Hence $v^3 = 0 \Rightarrow v = 0$, and $x^*x = u \geq 0$. □

**Definition 5.6.** *For a C*-algebra A, define $A_+ := \{h \in A_{\mathrm{sa}}\colon h \geq 0\}$, where $A_{\mathrm{sa}} := \{x \in A\colon x = x^*\}$.*

*Given $x, y \in A_{\mathrm{sa}}$, we write $x \leq y$ if $y - x \geq 0$.*

Then $A_+$ is a convex cone (i.e., $h \in A_+, \lambda \geq 0 \Rightarrow \lambda h \in A_+$; $h_1, h_2 \in A_+ \Rightarrow h_1 + h_2 \in A_+$). We have $A_+ \cap (-A_+) = \{0\}$, $A_{\mathrm{sa}} = A_+ - A_+$ and $A_+$ is closed by Lemma 5.3. Moreover, "$\leq$" defines a partial order on $A_{\mathrm{sa}}$ (reflexive, antisymmetric, transitive).

**Theorem 5.7.** *Let A be a C*-algebra.*

(a) *$A_+ = \{x^*x\colon x \in A\}$.*
(b) *Given $a, b \in A_{\mathrm{sa}}$ and $c \in A$, we have $a \leq b \Rightarrow c^*ac \leq c^*bc$.*

(c) $0 \le a \le b \Rightarrow \|a\| \le \|b\|$.
(d) *Assume that A has a unit. If* $0 \le a \le b$ *and* $a$, $b$ *are invertible, then* $0 \le b^{-1} \le a^{-1}$.

*Proof.* (a) follows from Proposition 5.5.

(b): $c^*bc - c^*ac = c^*(b-a)c = c^*(\sqrt{b-a})(\sqrt{b-a})c \ge 0$ by (a).

(c): We may assume that $A$ has a unit 1, otherwise work in $\tilde{A}$. Then $0 \le a$ implies

$$\|a\| = \inf\{\lambda \ge 0\colon \lambda 1 \ge a\} \le \inf\{\lambda \ge 0\colon \lambda 1 \ge b\} = \|b\|.$$

(d): $a \le b \Rightarrow 1 = \sqrt{a^{-1}}a\sqrt{a^{-1}} \le \sqrt{a^{-1}}b\sqrt{a^{-1}} \Rightarrow d := \sqrt{a}b^{-1}\sqrt{a} = (\sqrt{a^{-1}}b\sqrt{a^{-1}})^{-1} \le 1 \Rightarrow b^{-1} = \sqrt{a^{-1}}d\sqrt{a^{-1}} \le \sqrt{a^{-1}}\sqrt{a^{-1}} = a^{-1}$. □

**Remark 5.8.** Given $0 \le a \le b$ in $A$ with $\alpha > 0$, we cannot in general conclude that $a^\alpha \le b^\alpha$. (Actually, if $0 \le a \le b$ always implies $a^2 \le b^2$, then $A$ must be commutative.)

**Example 5.9.** *Let* $A = M_2(\mathbb{C})$. *A becomes a C*-algebra under the usual matrix operations and involution given by*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}^* = \begin{pmatrix} \overline{\lambda_{11}} & \overline{\lambda_{21}} \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} \end{pmatrix}.$$

*Now consider*

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

*Then* $p = p^*$, $q = q^*$ *and* $p^2 = p$, $q^2 = q$, *so* $p, q \ge 0$, *and we have* $p \le p+q$, *but* $p^2 \not\le (p+q)^2$.

**Proposition 5.10.** *Let* $0 \le \beta \le 1$ *and* $0 \le a \le b$. *Then* $0 \le a^\beta \le b^\beta$.

*Proof.* Let $\alpha > 0$. $f_\alpha(t) := \frac{t}{1+\alpha t} = \alpha^{-1}(1-(1+\alpha t)^{-1})$. Then $f_\alpha(a) \le f_\alpha(b)$ by Theorem 5.7 (d). Let $t \ge 0$. Then

$$\int_0^\infty f_\alpha(t)\alpha^{-\beta}d\alpha = \int_0^\infty (1+\alpha t)^{-1}t\alpha^{-\beta}d\alpha = \int_0^\infty (1+\alpha)^{-1}t\alpha^{-\beta}t^\beta t^{-1}d\alpha,$$

where we applied the transformation $t\alpha \to \alpha$ in the last step. Let $\gamma := \int_0^\infty (1+\alpha)^{-1}\alpha^{-\beta}d\alpha$. Note that this integral converges only for $0 \le \beta \le 1$. Then

$$t^\beta = \gamma^{-1}\int_0^\infty f_\alpha(t)\alpha^{-\beta}d\alpha.$$

Thus

$$b^\beta - a^\beta = \gamma^{-1}\int_0^\infty (f_\alpha(b) - f_\alpha(a))\alpha^{-\beta}d\alpha \ge 0.$$

□

## 6. Approximate units, ideals, and quotients

**Definition 6.1.** *Let A be a C*-algebra. An approximate unit in A is an increasing net* $(u_\lambda)_{\lambda\in\Lambda}$ *with* $0 \le u_\lambda \le 1$ *such that* $\lim_\lambda u_\lambda x = x = \lim_\lambda x u_\lambda \; \forall x \in A$.

**Examples 6.2.** *1) Let* $A = C_0(\mathbb{R})$. *Define* $u_N$ *by setting* $u_N \equiv 1$ *on* $[-N,N]$ *and* $u_N \equiv 0$ *on* $[-N-1,N+1]^c$, *and extend* $u_N$ *linearly on* $[-N-1,-N] \cup [N,N+1]$. *Then* $(u_N)$ *is an approximate unit in A.*

*2) Let* $A = \mathcal{K}(H)$, *the C*-algebra of compact operators on a Hilbert space H, i.e., the closure of the *-algebra of all finite rank operators. Let* $e_1, e_2, \ldots$ *be an orthonormal basis for H. Let* $p_n \in A$ *be the orthogonal projection onto the linear span of* $\{e_1, \ldots, e_n\}$. *Then* $p_1 \le p_2 \le p_3 \le \ldots$, *and* $(p_n)$ *is an approximate unit in A.*

**Theorem 6.3.** *Let A be a C*-algebra. Then A has an approximate unit.*

*Proof.* Let
$$\Lambda := \{h \in A\colon h \geq 0, \|h\| < 1\}.$$
First let us show that $\Lambda$ is directed. Given $a, b \in A_+$ with $0 \leq a \leq b$, we have
$$a(1+a)^{-1} = ((1+a)-1)(1+a)^{-1} = 1-(1+a)^{-1} \leq 1-(1+b)^{-1} = b(1+b)^{-1}.$$
Now let $a, b \in \Lambda$ be arbitrary. Let $a' := a(1-a)^{-1}$, $b' := b(1-b)^{-1}$, so that $a = a'(1+a')^{-1}$ and $b = b'(1+b')^{-1}$. Then
$$a = a'(1+a')^{-1} \leq (a'+b')(1+a'+b')^{-1},\ b = b'(1+b')^{-1} \leq (a'+b')(1+a'+b')^{-1}.$$
Moreover, $\left\|(a'+b')(1+a'+b')^{-1}\right\| = \max\left\{\frac{t}{1+t}\colon t \in \mathrm{Sp}\,(a'+b')\right\} < 1$. So $(a'+b')(1+a'+b')^{-1} \in \Lambda$ and is a common upper bound for $a$ and $b$. This shows that $\Lambda$ is (upward) directed.

Now, given $h \geq 0$ in $A$ and $n \in \mathbb{N}$, we have $h(\frac{1}{n}+h)^{-1} \in \Lambda$, and $h(1-h(\frac{1}{n}+h)^{-1}) \leq \frac{1}{n}$ (work in $\tilde{A}$ if needed). This is because
$$t\left(1-t(\frac{1}{n}+t)^{-1}\right) = t\frac{\frac{1}{n}}{\frac{1}{n}+t} \leq \frac{1}{n}\ \forall t \geq 0.$$
For $h \geq 0$ and $g \in \Lambda$ with $h(\frac{1}{n}+h)^{-1} \leq g$, we have
$$\|h-gh\|^2 = \left\|h(1-g)^2h\right\| \leq \|h(1-g)h\| \leq \left\|h\left(1-h(\frac{1}{n}+h)^{-1}\right)h\right\| \leq \frac{1}{n}\|h\|,$$
and similarly $\|h-hg\|^2 \leq \frac{1}{n}\|h\|$. Hence, for $\varepsilon > 0$ and $h \geq 0$ there exists $\lambda_0$ ($:= h(\frac{1}{n}+h)^{-1}$, where $\frac{1}{n}\|h\| < \varepsilon$) so that $\|h-gh\| < \varepsilon$ and $\|h-hg\| < \varepsilon\ \forall g \geq \lambda_0$. Now, given an arbitrary $x \in A$, apply the above to $h = x^*x$. Then we obtain that $\|x-gx\|^2 = \|(1-g)x^*x(1-g)\| = \|(1-g)h(1-g)\| \leq \|h-gh\|\,\|1-g\| \to 0$ as $g \to \infty$ in $\Lambda$. Similarly for $\|x-xg\|^2$. □

**Remark 6.4.** The same proof as for Theorem 6.3 shows that if $A$ is an ideal in a C*-algebra $B$, then we can find a net $(u_\lambda)$ in $A$ satisfying the same properties as in Definition 6.1. For us, ideal always means two-sided ideal.