Cohomology of groups
LTCC Lecture Notes 2018

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Contents

CHAPTER 1

Projective resolutions

1. R-Modules

In this section we will quickly review the basic definitions of modules over a ring, projective resolutions and the definition of $\mathrm{Ext}^n(M,N)$. In general we denote a ring by R and assume that R has a unit.

Let R be a ring. A **left R-module** is an abelian group $(M,+)$ together with a multiplication

$$\begin{array}{rcl} R \times M & \to & M \\ (r,m) & \mapsto & rm \end{array}$$

satisfying the following axioms:

(M1) $r(m+n) = rm + rn$ for all $r \in R$ and $m, n \in M$
(M2) $(r+s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
(M3) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
(M4) $1_R m = m$ for all $m \in M$.

We usually write M_R - or M if it is clear which ring is meant. Right R-modules are defined analogously. If R is commutative a left R-module can be made into a right R-module by defining the multiplication by $(m,r) \mapsto rm$.

Let M and N be R-modules. A map $\alpha : M \to N$ is called R-linear or an R-module homomorphism if

- $\alpha(m+m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$
- $\alpha(rm) = r\alpha(m)$ for all $m \in M, r \in R$.

Let M and N be R-modules. We denote by $\mathrm{Hom}_R(M,N)$ the set of all R-linear maps $\alpha : M \to N$.

Remark. $\mathrm{Hom}_R(M,N)$ is an abelian group with addition defined pointwise. Furthermore $End_R(M) = \mathrm{Hom}_R(M,M)$ is a ring where multiplication is defined by composition of maps.

Naturality means that for every R-module homomorphism $\alpha : M \to N$ the following diagram commutes,

$$\begin{array}{ccc} \mathrm{Hom}_R(R,M) & \xrightarrow{\phi_M} & M \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_R(R,N) & \xrightarrow{\phi_N} & N \end{array}$$

where $\alpha_*(f) = \alpha \circ f$ and $\alpha \circ \phi_M = \phi_N \circ \alpha_*$.

A sequence

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

($i \in \mathbb{Z}$) of linear maps is called **exact at** M_i if $im(\alpha_{i+1}) = ker\alpha_i$.
The sequence is called exact if it is exact at every $M_i (i \in \mathbb{Z})$.

EXERCISE 1. Show that:

(1) $0 \longrightarrow L \xrightarrow{\alpha} M$ is exact if and only if α is a monomorphism.

(2) $M \xrightarrow{\beta} N \longrightarrow 0$ is exact if and only if β is an epimorphism.

(3) $0 \longrightarrow L \xrightarrow{\alpha} M \longrightarrow 0$ is exact iff α is an isomomorphism.

Remark. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

In particular, α is a monomorphism, β is an epimorphism and $im(\alpha) = ker(\beta)$. Hence $N \cong M/\alpha(L)$. Conversely, if $N \cong M/L$, then there is a short exact sequence

$$L \hookrightarrow M \twoheadrightarrow N.$$

Let us get back to the groups $\mathrm{Hom}_R(M, N)$: Let $\alpha \in \mathrm{Hom}_R(M, N)$ and let $\xi : N \to X$ be an R-module homomorphism. We then define

$$\xi_* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, X)$$

by $\xi_*(\alpha) = \xi \circ \alpha$. In other words, $\mathrm{Hom}_R(M, -)$ is a covariant functor. Now let $\psi : Y \to M$ be an R-module homomorphism. We define

$$\psi^* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(Y, N)$$

by $\psi^*(\alpha) = \alpha \circ \psi$. We say $\mathrm{Hom}_R(-, N)$ is a contravariant functor.

THEOREM 1.1. *Let X and Y be R-modules and let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following sequences are exact:

(1) $0 \longrightarrow \mathrm{Hom}_R(Y, L) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Y, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Y, N)$

(2) $0 \longrightarrow \mathrm{Hom}_R(N, X) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, X) \xrightarrow{\alpha^*} \mathrm{Hom}_R(L, X)$.

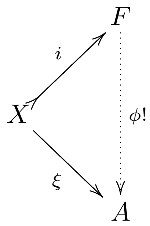
Proof: We leave (2) as exercise and do (1) in class. □

We say $\mathrm{Hom}_R(-, X)$ and $\mathrm{Hom}_R(Y, -)$ are left exact functors. Neither β_* nor α^* have to be surjective. We'll come back to conditions on X and Y for Hom to be an exact functor.

Projective modules are basically the bread and butter of homological algebra, so let's define them. But first, let's do free modules:

Let F be an R-module and X be a subset of F. We say F is **free on** X if for every R-module A and every map $\xi : X \to A$ there exists a unique R-module homomorphism $\phi : F \to A$ such that $\phi(x) = \xi(x)$ for all $x \in X$.

In other words F is free if there's a unique R-module homorphism ϕ making the following diagram commute:



A very hard look at this diagram now gives us the following lemma.

Proposition 1.2. *Let P be an R-module. Then the following statements are equivalent:*

(1) $\mathrm{Hom}_R(P, -)$ *is an exact functor*
(2) P *is a direct summand of a free module.*
(3) *Every epimorphism $M \twoheadrightarrow P$ splits.*
(4) *For every epimorphism $\pi : A \twoheadrightarrow B$ of R-modules and every R-module map $\alpha; P \to B$ there is an R-module homomorphism $\phi : P \to A$ such that $\pi \circ \phi = \alpha$.*

Every R-module satisfying the conditions of Proposition 1.2 is called a **projective R-module.**

Definition 1.3. Let M be an R-module. A projective resolution of M is an exact sequence

$$\cdots \longrightarrow P_{i+1} \xrightarrow{d_i} P_i \xrightarrow{d_{i+1}} \cdots \xrightarrow[1]{d} P_1 \xrightarrow[0]{d} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where every P_i, $i \geq 0, i \in \mathbb{Z}$, is a projective module.

We also use the short notation

$$\mathbf{P}_* \twoheadrightarrow M.$$

Given an R-module N, we apply $\mathrm{Hom}_R(-, N)$ to the projective resolution above to get a complex

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \cdots .$$

We define:

$$\mathrm{Ext}^n_R(M, N) = ker(\mathrm{Hom}_R(P_n, N) \to \mathrm{Hom}_R(P_{n+1}, N))/im(\mathrm{Hom}_R(P_{n-1}, N) \to \mathrm{Hom}_R(P_n, N)).$$

We use the convention that $P_i = 0$ for all $i < 0$.

Theorem 1.4. $\mathrm{Ext}^n_R(M, N)$ *is independent of the choice of projective resolution of M.*

Exercise 2. Prove that $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$.

Definition 1.5. Let M be an R-module. We say M has finite projective dimension over R, $\mathrm{pd}_R M < \infty$, if M admits a projective resolution $\mathbf{P}_* \twoheadrightarrow M$ of finite length. In particular, there exists an $n \geq 0$ such that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resulution of n. The smallest such n is called the projective dimension of M.

PROPOSITION 1.6. *Let M be an R-module. Then the following statments are equivalent:*

(1) $\mathrm{pd}_R M \leq n$.
(2) $\mathrm{Ext}^i_R(M, -) = 0$ *for all* $i > n$
(3) $\mathrm{Ext}^{n+1}_R(M, -) = 0$
(4) *Let* $0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *be an exact sequence with* P_i *projective for all* $0 \leq i \leq n-1$. *Then* K_{n-1} *is projective.*

EXERCISE 3. Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Prove the following:

(1) $\mathrm{pd}M' \leq sup\{\mathrm{pd}M, \mathrm{pd}M'' + 1\}$.
(2) $\mathrm{pd}M \leq sup\{\mathrm{pd}M'', \mathrm{pd}M'\}$.
(3) $\mathrm{pd}M'' \leq sup\{\mathrm{pd}M, \mathrm{pd}M' - 1\}$.

(This is an exercise in applying Theorem 1.7)

EXERCISE 4. Let M be an R-module such that $\mathrm{pd}M = n$. Then there exists a free R-module F such that

$$\mathrm{Ext}^n(M, F) \neq 0.$$

THEOREM 1.7. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. And let N be an arbitrary R-module. Then there are long exact sequences in cohomology*

(1)

$$\cdots \to \mathrm{Ext}^n(N, M'') \to \mathrm{Ext}^n(N, M) \to \mathrm{Ext}^n(N, M') \to \mathrm{Ext}^{n+1}(N, M'') \to \cdots$$

(2)

$$\cdots \to \mathrm{Ext}^n(M', N) \to \mathrm{Ext}^n(M, N) \to \mathrm{Ext}^n(M'', N) \to \mathrm{Ext}^{n+1}(M', N) \to \cdots$$

EXERCISE 5. **[Dimension shifting]** Let $K \hookrightarrow P \twoheadrightarrow M$ be the beginning of a projective resolution of M and let N be an R-module. Then for all $n \geq 1$,

$$\mathrm{Ext}^n(K, N) \cong \mathrm{Ext}^{n+1}(M, N).$$

Proof: Apply Theorem 1.7 and the fact that Ext vanishes on projectives. □

2. The Group Ring

Throughout we denote a group by G. Let $\mathbb{Z}G$ denote the free $\mathbb{Z}$-module with basis the elements of G. In particular, every $x \in \mathbb{Z}G$ can be written in a unique way as

$$x = \sum_{g \in G} n_g g$$

where $n_g \in \mathbb{Z}$and almost all $n_g = 0$. Define a multiplication on $\mathbb{Z}G$ as follows:

$$xy = (\sum_{g \in G} n_g g)(\sum_{h \in G} n_h h) = \sum_{g,h \in G} n_g n_h (gh).$$

this makes $\mathbb{Z}G$ into a ring, the **integral group ring.**

EXAMPLE 1.8. (1) Let $G = \langle x \rangle$ be infinite cyclic. Then $\mathbb{Z}G$ has $\mathbb{Z}$-basis $\{x^i \mid i \in \mathbb{Z}\}$ and can be identified with the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials $\sum_{i \in \mathbb{Z}} a_i x^i$, where almost all $a_i = 0$.

(2) Let G be cyclic order n and t be a generator for G. $\{1, t, t^2, ..., t^{n-1}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}G$ and $t^n - 1 = 0$ hence

$$\mathbb{Z}G \cong \mathbb{Z}[T]/T^n - 1.$$

DEFINITION 1.9. Let M be an abelian group and let G act on M

$$\begin{array}{ccc} G \times M & \to & M \\ (g, m) & \mapsto & gm \end{array}$$

such that for all $m, n \in M$ and $g, h \in G$:

- $1_G m = m$
- $(gh)m = g(hm)$
- $g(m + n) = gm + gn$

we say that M is a G-module.

A G-module can be made in a $\mathbb{Z}G$-module by "linearly extending" the action, i.e. $xm = (\sum_{g \in G} n_g g)m = \sum_{g \in G} n_g (gm)$. Furthermore, G is a subgroup of the multiplicative group $\mathbb{Z}G^*$ and hence there's the follwing universal property:
Let R be a ring and $f : G \to R^*$ be a group homomorphism. Then f can be extended uniquely to a ring homomorphism $\mathbb{Z}G \to R$. Hence

$$Hom_{rings}(\mathbb{Z}G, R) \cong Hom_{groups}(G, R^*)$$

and a G-module is nothing but a $\mathbb{Z}G$-module.

EXAMPLE 1.10. Every abelian group A is a trivial G-module with the action defined by $ag = a$ for all $a \in A, g \in G$. Hence for $x = \sum_{g \in G} n_g g$ it follows that $xa = \sum_{g \in G} n_g a$.

For every group G there is a ring homomorphism

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$$

defined by $\varepsilon(g) = 1$. for all $g \in G$. Hence for $x = \sum_{g \in G} n_g g$, $\varepsilon(x) = \sum_{g \in G} n_g$. The kernel of ε is called the **augmentation ideal** and is denoted by $\mathfrak{g}$ or IG.

LEMMA 1.11. *$\mathfrak{g}$ is a free $\mathbb{Z}$-module with basis*

$$X = \{g - 1 \mid 1 \neq g \in G\}.$$

ε is a G-module homomorphism and $\mathfrak{g}$ is a G-module.

LEMMA 1.12. (1) *Let S be generating set for G. Then $\mathfrak{g}$ is generated as a G-module by*

$$S - 1 = \{s - 1 \mid s \in S\}.$$

(2) *Let S be a set of elements of G such that $S - 1$ generates $\mathfrak{g}$ as a G-module. Then S generates the group G.*

Proof: We do (1) in class and leave (2) as an exercise. □

Now let Ω be a G-set and consider the free abelian group $\mathbb{Z}\Omega$ on Ω. The operation of G on Ω can be extended to a $\mathbb{Z}$-linear operation of G on $\mathbb{Z}\Omega$. Hence $\mathbb{Z}\Omega$ is a G-module, the so called **Permutation module**.

EXAMPLE 1.13. (1) Let $H \leq G$ be a subgroup and let G/H be the set of left cosets. Then $\mathbb{Z}[G/H]$ is a permutation module.
(2) Let $\Omega = \sqcup_{i\in I}\Omega_i$ (disjoint union). Then $\mathbb{Z}\Omega = \oplus_{i\in I}\mathbb{Z}\Omega_i$.

In particular, every permutation module can be expressed as

$$\mathbb{Z}\Omega = \bigoplus_{\omega\in\Omega^0} \mathbb{Z}[G/G_\omega],$$

where Ω^0 is a system of representatives of the orbits of the G-action and $G_\omega = \{g \in G \mid g\omega = \omega\}$ is the stabiliser (or isotropy group) of ω. We say G acts **freely** on Ω if all stabilisers are trivial.

LEMMA 1.14. *Let Ω be a free G-set and let Ω^0 be a system of representatives for the G-orbits. Then $\mathbb{Z}\Omega$ is a free G-module with basis Ω^0.*

LEMMA 1.15. *Let $H \leq G$ be a subgroup of G. Then $\mathbb{Z}G$ is free as a left H-module.*

Now let us define the cohomology groups:

DEFINITION 1.16. Let G be a group. Then the n-th cohomology group of G with coefficients in the G-module M is defined to be

$$\mathrm{H}^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

In chapter one we have determined the zeroth cohomology group Ext^0. Hence

$$\mathrm{H}^0(G, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G,$$

where M^G denote the G-fixed points of M. We have, so far, computed cohomology via projective resolutions and defined the projective resolution of a module M to be the shortest length of a projective resolution of M. One theme of this course will be cohomological finiteness conditions for groups, so let's make first definition.

DEFINITION 1.17. Let G be a group. The cohomological dimension of G, denoted $\mathrm{cd}G$ is defined to be

$$\mathrm{cd}G = \mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}.$$

The above Lemma 1.15 implies directly:

PROPOSITION 1.18. *Let $H \leq G$ be a subgroup of G. Then*

$$\mathrm{cd}H \leq \mathrm{cd}G.$$

REMARK 1.19. One can, of course always define the group ring RG for any ring R. $\mathrm{H}^*_R(G, -)$ and $\mathrm{cd}_R G$ are defined analogously. Something more here, adjoint functors?

We shall now spend some time on finding projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$. Let us begin with two easy examples:

EXAMPLE 1.20. (1) Let $G = \langle x \rangle$ be an infinite cyclic group. Then

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{*(x-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective (free) resolution of $\mathbb{Z}$.

(2) Let G be a cyclic group of order n generated by t. Then, as seen before, $\mathbb{Z}G \cong \mathbb{Z}[t]/(T^n - 1)$. Now $T^n - 1 = (T-1)(T^{n-1} + T^{n-2} + ... + T^o)$ and hence for each $x \in \mathbb{Z}G$ it follows that

$$(t-1)x = 0 \iff x = (t^{n-1} + ... + t + t^0)y = Ny \quad \text{some } y \in \mathbb{Z}G.$$

Hence there is a projective (free) resolution of $\mathbb{Z}$ of infinite length:

$$... \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} ... \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We will see later that G has no projective resolution of finite length.

COROLLARY 1.21. *Let G be a group with* $\mathrm{cd}G < \infty$, *then G is torsion-free.*

3. The Bar-resolution

The standard resolution
We define a free resolution

$$F_* \twoheadrightarrow \mathbb{Z}$$

as follows: Let

$$F_i = \mathbb{Z}(G^{n+1})$$

with the G-action defined diagonally:

$$(g_0, g_1, ..., g_n)g = (g_0 g, g_1 g, ..., g_n g).$$

The chain maps $\delta : F_n \to F_{n-1}$ are defined as by

$$\begin{array}{rccc} d_i : & F_n & \to & F_{n-1} \\ & (g_0, ..., g_i, ...g_n) & \mapsto & (g_0, ..., \hat{g}_i, ..., g_n) \end{array}$$

($\hat{g}_i$ denotes omitting this term) and then

$$\delta = \sum_{i=0}^{n} (-1)^i d_i.$$

Show that $\delta\delta = 0$. To see that $F_* \twoheadrightarrow \mathbb{Z}$ is exact , we note that there is a contracting homotopy (not a G-map)

$$\begin{array}{rccc} h : & F_n & \to & F_{n-1} \\ & (g_o, ..., g_n) & \mapsto & (1, g_0, ..., g_n). \end{array}$$

This resolution is called the standard resolution.

The Bar resolution Now define free G-modules

$$Q_n = \mathbb{Z}(G^n \times G)$$

where G acts on $G^n \times G$ as follows:

$$(g_0, ..., g_{n-1}, g_n)g = (g_o, ..., g_{n-1}, g_n g).$$

For all $n \geq 0$ there is G-module isomorphism

$$Q_n \cong F_n.$$

LEMMA 1.22. $\mathrm{Hom}_{\mathbb{Z}G}(Q_n, M) \cong C^n(G, M)$, *where $C^n(G, M)$ denotes the set of all functions $\varphi : G^n \to M$.*

REMARK 1.23. (1) $C^n(G, M)$ is an abelian group.
(2) As $\mathbb{Z}G$ module Q_n has basis $G^n \times \{1\}$.

(3) Induced by the cochain maps $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{n+1}, M)$ there is a cochain map
$$D : C^n(G, M) \to C^{n+1}(G, M)$$
defined by $(D\varphi)(g_0, ..., g_n) = \varphi(g_1, ..., g_n) - \varphi(g_0 g_1, g_2, ..., g_n) + \varphi(g_0, g_1 g_2, g_3, ..., g_n) - ... + (-1)^n(\varphi(g_0, ..., g_{n-1} g_n) + (-1)^{n+1}\varphi(g_0, ..., g_{n-1})g_n$.
Show $DD = 0$. As before, suppose $\varphi : G^n \to M$ such that $D\varphi = 0$ we call φ a n-cocycle.

DEFINITION 1.24. A 1-cocyle is called a **derivation**.

For $\varphi : G \to M$ and for all $g, h \in G$, $0 = (D\varphi)(g, h) = \varphi(h) - \varphi(gh) + \varphi(g)h$ implies
$$\varphi(gh) = \varphi(h) + \varphi(g)h.$$

REMARK 1.25. Let $\varphi : G \to M$ be a derivation. Then
(1) $\varphi(1) = 0$;
(2) $\varphi(g^{-1}) = -\varphi(g)g^{-1}$.

LEMMA 1.26. *Let M be a G-module. for every $m \in M$ and $g \in G$ the function $g \mapsto mg - m$ is a derivation. Such derivations are called* ***inner derivations****.*

A 2 cocyle is sometimes called a **factor set**.

EXERCISE 6. Let $\varphi : G \times G \to M$ be a factor set. Show that $\varphi(g, 1) = \varphi(1, 1)$ and $\varphi(1, k) = \varphi(1, 1)k$. for all $g, k \in G$.

EXAMPLE 1.27. Let G be an abelian group and let A be a trivial G-module. Then every multilinear map
$$\varphi : G \times ... \times G \to A$$
is a cocylce.
(1) Let $G = \mathbb{R}^n$ Then $det : \underbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}_{n} \to \mathbb{R}$ is a n-cocylce.
(2) Let $G = \mathbb{R}^n$ and $\langle v, w\rangle = \sum_{i=0}^{n} v_i w_i$ with $v, w \in \mathbb{R}^n$ is a 2-cocycle.

As shown above, the definition
$$\mathrm{H}^n(G, M) = \mathrm{H}^n(C^*(G, M))$$
is consistent with our previous definition of group cohomology via projective resolutions.

THEOREM 1.28. *Let G be a finite group and let M be a G-module. Then every element of $\mathrm{H}^n(G, M)$ for $n > 0$ has finite order dividing the order of G. In particular, for all $n > 0$.*
$$|G|\,\mathrm{H}^n(G, M) = 0.$$

CHAPTER 2

Groups acting on trees

1. Trees

Definition 2.1. Let X be a set. A free group on X is a group F together with a function $\iota : X \to F$ such that for any group G and any function $\phi : X \to G$ there is a unique homomorphism $\theta : F \to G$ such that $\theta\iota = \phi$.

Remark 2.2. Any two free groups on X are isomorphic and for every set X there exists a free group on X. Let F be the set of all reduced words in $X \cup X^{-1}$.

Definition 2.3. Let G be a group. A G-graph $\Gamma = (\Gamma, V, E, \iota, \tau)$ consists of two G-sets V (vertices) and E (edges) and G-maps $\iota, \tau : E \to V$.

We call ι the initial vertex (function) and τ the terminal vertex (function). In case $\iota(e) = \tau(e)$ for a $e \in E$, then e is a loop.

Definition 2.4. Let G be a group and let X be a subset of G. The Cayley-graph $\Gamma = \Gamma(G, X)$ with respect to X is the G-graph Γ defines as follows:

$$V = G \quad \text{and} \quad E = X \times G.$$

and $\iota(x, g) = g$ and $\tau(x, g) = xg$ for all $x \in X, g \in G$.

Let Γ be a G-graph. We define new initial and terminal vertex functions. Or, in other works we define edges e^1 and e^{-1} with an orientation. Let $e \in E$. now $\iota(e^1) = \iota(e)$ and $\tau(e^1) = \tau(e)$ whereas $\iota(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \iota(e)$. A **path** in Γ is a finite sequence

$$v_0 e_1^{\epsilon_1} v_1 e_2^{\epsilon_2} ... v_{n-1} e_n^{\epsilon_n} v_n$$

where $\epsilon_i \in \{1, -1\}, v_i \in V, e_i \in E$ and $\iota(e^{\epsilon_i}) = v_{i-1}$ and $\tau(e^{\epsilon_i}) = v_i$. We shorten this to

$$p = e_1^{\epsilon_1} e_2^{\epsilon_2} ... e_n^{\epsilon_n}$$

with $\iota(p) = v_0$ and $\tau(p) = v_n$. The inverse path is

$$p^{-1} = e_n^{-\epsilon_n} ... e_1^{-\epsilon_1}$$

Let q be a path such that $\iota(q) = \tau(p)$. We form a new path (p, q) by gluing.
A path is called **reduced** if for all $i = 1, ..., n-1$

$$e_i^{\epsilon_i} \neq e_{i+1}^{-\epsilon_{i+1}}.$$

A path is called a **tree** if for all vertices v, w there is a unique reduced path p such that $\iota(p) = v$ and $\tau(p) = w$. Such a path is called a geodesic.
A path p is called closed at the vertex v if $\iota(p) = \tau(p) = v$. p is simple closed if there is no repetition of vertices. A graph Γ is called a **forest** if there are no simple closed paths.
A graph Γ is connected if for all $v, v' \in V$ there is a path connecting v and v'.

PROPOSITION 2.5. *A graph Γ is a tree if and only if Γ is a connected forest.*

The augmented cellular chain complex $C_*(\Gamma) \twoheadrightarrow \mathbb{Z}$ of a G-graph $\Gamma = (\Gamma, V, E, \iota.\tau)$ is the following sequence of G-modules

$$0 \longrightarrow \mathbb{Z}E \xrightarrow{d} \mathbb{Z}V \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

where $d(e) = \tau(e) - \iota(e)$ and $\epsilon(v) = 1$. This is always a complex as $\epsilon d = 0$. Also ϵ is onto if and only if V is non-empty.

LEMMA 2.6.

(1) *A non-empty graph is connected if and only if*

$$\mathbb{Z}E \to \mathbb{Z}V \twoheadrightarrow \mathbb{Z}$$

is exact.

(2) *A graph is a forest if and only if*

$$0 \to \mathbb{Z}E \to \mathbb{Z}V$$

is exact.

(3) *A non-empty graph Γ is a tree if and only if*

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$$

is exact.

Let us now consider a little aside:

DEFINITION 2.7. Let M be a G-module. The split extension $M \rtimes G$ of M over G is the set $M \times G$ together with multiplication

$$(m, g)(n, h) = (mh + n, gh).$$

REMARK 2.8.

(1) $M \rtimes G$ is a group with identity $(0, 1)$.

(2) The map $\pi : M \rtimes G \to G$ defined by $\pi(m, g) = g$ is a homomorphism.

LEMMA 2.9. *There is a bijection between the set of group homomorphisms $\theta : G \to M \rtimes G$ satisfying $\pi\theta = id_G$ and the set of derivations $\varphi : G \to M$.*

Now back to trees:

THEOREM 2.10. *Let F be a free group on X. Then the Cayley graph $\Gamma(F, X)$ is a tree.*

EXERCISE 7. Let G be a group and suppose the the Cayley graph $\Gamma(G, X)$ is connected. Show that G is generated by X.

DEFINITION 2.11. **Free product with amalgamation** Let G_1, G_2 and A be groups and let $\alpha_1 : A \to G_1$ and $\alpha_2 : A \to G_2$ be group homomorphisms. The free amalgamated product of G_1 and G_2 over A is the group G satisfying:

There is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & G_1 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} \\ G_2 & \xrightarrow{\beta_2} & B \end{array}$$

satisfying the following universal property

Let H be a group with homomorphisms $\gamma_i : G_i \to H$ $(i = 1, 2)$ such that $\gamma_1\alpha_1 = \gamma_2\alpha_2$ then there exists a unique homomorphism $\phi : G \to H$ such that $\phi\beta_i = \gamma_i$ $(i = 1, 2)$.

We write

$$G = G_1 *_A G_2.$$

Exercise 8. Let $G = G_1 *_A G_2$ with $A \to G_1$ and $A \to G_2$ not necessarily monomorphisms. Denote by $\bar{G_1}$ and $\bar{G_2}$ the images of G_1 and G_2 in G. Prove that $G = \bar{G_1} *_A \bar{G_2}$.

We therefore assume from now on that $A \hookrightarrow G_i$ and $G_i \hookrightarrow G$ for $i = 1, 2$.

Lemma 2.12. *Let $G = K *_H L$. for every G-module M and derivations $\delta' : K \to M$ and $\delta'' : L \to M$ such that $\delta'|_H = \delta''|_H$. Then there is a unique derivation $\delta : G \to M$ such that $\delta|_K = \delta'$ and $\delta|_L = \delta''$.*

Remark 2.13. Let $G = K *_H L$. Then

(1) $H = K \cap L$;
(2) G is generated by K and L.

Theorem 2.14. *Let $G = K *_H L$ and define a G-graph $\Gamma = \Gamma(E, V, \iota, \tau)$ as follows:*

$$E = G/H \quad \textit{and} \quad V = G/K \sqcup G/L,$$

with $\iota(gH) = gK$ and $\tau(gH) = gL$. Then Γ is a tree.

Definition 2.15. **HNN-extensions** Let $H \leq K \leq G$ be groups ad let $t \in G$ such that $H^t \subseteq K$. G is an HNN-extension with respect to (K, H, t) if is satisfying the following universal property: Let G_1 be a group, $t_1 \in G$ and $\theta : K \to G$ be a homomorphism such that for all $h \in H$. $\theta(h^t) = \theta(h)^{t_1}$. Then there is a unique homomorphism $\hat{\theta}G \to G_1$ such that $\hat{\theta}|_K = \theta$ and $\hat{\theta}(t) = t_1$. We write

$$G = K *_{H,t} .$$

Theorem 2.16. *Let $G = K *_{H,t}$ be an HNN-extension. Then the G-graph $\Gamma = \Gamma(E, V, \iota, \tau)$ defined by*

$$V = G/K \quad \textit{and} \quad E = G/H$$

with $\iota(gH) = gK$ and $\tau(gH) = gtK$ is a tree.

Let T be a G-tree with one orbit of edges. Then there are one or two orbits of vertices. Let e be an edge. Then every vertex is in the orbit of $\iota(k)$ or in the orbit of $\tau(k)$.

One can also show that if $H \leq G$ and T a G-tree such that

(1) There is only one orbit of edges (with $H = G_e$ where $e \in E$.
(2) For every vertex v there is a $g \in G$ such that $gv \neq v$.

Then either $G = G_1 *_H G_2$ with $G_1 \neq G_2 \neq G$ or G is an HNN-extension with $G = K *_{H,t}$.

CHAPTER 3

Resolutions via Topology

In this section we shall see that we can construct resolutions once we have constructed models for classifying spaces. We shall introduce very quickly the basic topological notions used later. We shall, however introduce classifying spaces in a more general way than initially used. We will see how to construct classifying spaces for families of subgroups.

1. CW-complexes

In this section we only briefly introduce the concept of a CW-complex. The interested reader can find all detail in most Algebraic Topology textbooks, such as for example Hatcher's book [**8**], appendix.

A CW-complex can be thought of as built by the following proceedure:

(1) Start with a discrete set X^0, whose points are regarded as 0-cells. (This is the 0-skeleton).
(2) Inductively, from the $(n-1)$-skeleton X^{n-1} build the n-skeleton X^n by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. (This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identification $x \sim \varphi_\alpha x$ for $x \in D^n_\alpha$. Thus, as a set $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.)
(3) Put $X = \bigcup_n X^n$ where X is given the weak topology: A set $A \subset X$ is open if and only if $A \cap X^n$ is open for all n.

Example 3.1. A 1-dimensional CW-comples is just a graph with vertices the 0-cells and edges the 1-cells.

Example 3.2. $X = \mathbb{R}^2$ is a 2-dimensional CW-complex with $\mathbb{Z} \times \mathbb{Z}$ as the 0-cells, the open intervalls as the 1-cells and the interior of the unit squares as 2-cells.

Example 3.3. The sphere S^n has the structure of a CW-complex with one 0-cell and one n-cell.

Example 3.4. The real projective plane, $\mathbb{R}P^2$ can be seen as D^2 with antipodal points of $S^1 = \delta D^2$ identified. Hence $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^n$.

Exercise 9. How can we see that $\mathbb{R}P^n$ has a CW-structure, $e^0 \cup e^1 \cup ... \cup e^n$?

Exercise 10. How can we see that a closed orientable surface M_g of genus g ($M_1 = T$, the torus) has a CW-structure given by: $e^0 \cup e^1_1 \cup e^1_2 \cup ... \cup e^1_{2g} \cup e^2$, i.e. has one 0-cell, $2g$ 1-cells and one 2-cell? (Identify edges on a regular $4g$-gon.)

2. G-spaces

In this course, all our groups are discrete groups. One can, however, define classifying spaces for families for arbitrary topological groups. For detail see tomDieck's book on transformation groups [**5**].

DEFINITION 3.5. A G-space is a topological space X with a (continuous) left G-action

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

satisfying

(1) $ex = x$ for all $x \in X$ and $e = e_g$ the identity of G.
(2) (gh)x = g(hx) for all $x \in X$ and all $g, h \in G$.

EXAMPLE 3.6. (a) Let G be the infinite cyclic group with generator g, i.e. $G = \langle g \rangle$ and $X = \mathbb{R}$. X is a G-space with G acting by translation $g^i x = x + i$.
(b) Let $G = \mathbb{Z} \times \mathbb{Z}$. $X = \mathbb{R}^2$ is a G-space with G acting by translation.
(c) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y > o\}.$$

$Sl_2(\mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a, b, c, d \in \mathbb{Z}, det(A) = 1\}$ acts on $\mathbb{H}$ by Möbius-transformations, i.e $Az = \frac{az+b}{cz+d}$.
(Check this really makes $\mathbb{H}$ into a $Sl_2(\mathbb{Z})$-space)
The kernel of this action consists of scalar multiples in $Sl_2(\mathbb{Z})$ of the identity matrix I. Hence $\mathbb{H}$ is a G space for $G = PSl_2(\mathbb{Z}) = Sl_2(\mathbb{Z})/\{\pm I\}$.

DEFINITION 3.7. The stabilizer $G_x \leq G$ of a point $x \in X$ is the subgroup $\{g \in G \,|\, gx = x\}$.

Let us note that the Cartesian product $X \times Y$ of two G-spaces X and Y is again a G-space via the diagonal action $g(x, y) = (gx, gy)$ for all $x \in X, y \in Y$ and $g \in G$.

DEFINITION 3.8. Let $H \subseteq G$ be a subgroup of G. Write X^H for the subspace of H-fixed points

$$X^H = \{x \in X \,|\, hx = x, \forall h \in H\}$$

and X/H for the space of H-orbits,

$$X/H = \{Hx \,|\, x \in X\}.$$

Let $N_G(H)$ denote the normalizer of H in G:

$$N_G(H) = \{g \in G \,|\, gH = Hg\}.$$

Then the G-action on X restricts to an $N_G(H)$-action on X^H with H acting trivally. Hence X^H is a $N_G(H)/H$-space.

EXAMPLE 3.9. The space of left cosets G/H is a G-space via $(g, kH) \mapsto gkH$ for all $g, k \in G$.
(Fact: Every discrete G-space is a disjoint union of such G-spaces.)
Let $K \leq G$ a subgroup. Then $(G/H)^K$ consists of all cosets gH such that $KgH = gH \iff g^{-1}Kg \leq H$.

Definition 3.10. A G-CW-complex consists of a G-space X together with a filtration

$$X^0 \subset X^1 \subset X^2 \subset ... \subset X$$

by G-subcomplexes such that

(1) Each X^n is closed in X.
(2) $\bigcup_{n\in\mathbb{N}} X^n = X$.
(3) X^0 is a discrete subspace of X.
(4) For each $n \geq 1$ there exists a discrete G-space Δ_n together with G-maps $F : S^{n-1} \times \Delta_n \to X^{n-1}$ and $\hat{f} : D^n \times \Delta_n \to X^n$ such that the following diagramme is a push-out:

$$\begin{array}{ccc} S^{n-1} \times \Delta_n & \to & X^{n-1} \\ \downarrow & & \downarrow \\ D^n \times \Delta_n & \to & X^n \end{array}$$

(5) A subspace Y of X is open if and only if $Y \cap X^n$ is open for all $n \geq 0$.

A map $f : X \to Y$ of G-CW-complexes is a G-map if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. If $G = \{e\}$ the trivial group, then a G-CW-complex is just a CW-complex as in Chapter 1. All our examples in 3.6 are G-CW-complexes.

Example 3.11. Let $G = C_2$ be the cyclic group of order 2. Then the sphere, S^2 is a G-CW-complex with G acting by the antipodal map.

Definition 3.12. (1) A G-CW-complex is called finite dimensional if $X^n = X$ for some $n \geq 0$. The least such n is called the dimension of X. (In case $dim(X) < \infty$, Axiom 5 above is redundant.)
(2) A G-CW-complex is said to be of finite type, if there are finitely many G-orbits in each dimension. (Equivalently, as X/G is a CW-complex, X/G only has finitely many cells in each dimension.)
(3) A G-CW-complex is called cocompact if X is finite dimensional and of finite type. (Equivalently, X/G is a finite CW-complex.)

All the examples we've seen so far, are cocompact. Before we can move on to defining classifying spaces, we need to have a quick look at an important construction, the join construction:

Definition 3.13. Let $I = [0, 1]$ and let X, Y be G-CW-complexes. We define the join of X and XY to be:

$$X * Y = (I \times X \times Y)/ \sim,$$

where $\sim$ is the equivalence relation generated by $(0, x, y_1) = (0, x, y_2)$ and $(1, x_1, y) = (1, x_2, y)$.

Hence the dimension of$X * Y$ is equal to $1 + dim(X) + dim(Y)$. Furthermore, the join of two G-spaces is again a G-space with diagonal G-action. One can also show that the join of two G-CW-complexes is again a G-CW-comples.

Example 3.14. (1) $X * \{pt\} = CX$ the cone on X.
(2) $X * S^0 = \Sigma X$ the suspension on X.
(3) The n-fold join $\{pt\} * ... * \{pt\}$ is a $n-1$-simplex

Lemma 3.15. [**12**] *Let X be a non-empty and Y be a n-connected space. Then $X * Y$ is $n + 1$-connected. In particular, the infinite join of non-empty G-CW-complexes is contractible.*

A space X is called 0-connected if it is non-empty and path-connected; it is callen n-connected if X is 0-connected and for each $1 \leq i \leq n$, the homotopy group $\pi_i(X)$ is trivial. For detail on connectedness and higher homotopy groups see [**15**, Chapter 11].

3. Classifying spaces

Let $\mathfrak{F}$ denote a family of subgroups of a group G. This is a collection of subgroups closed under conjugation and finite intersection. The following are examples of such families:

- $\mathfrak{F} = \mathfrak{All}$, the family of all subgroups of G
- $\mathfrak{F} = \mathfrak{Fin}$, the family of all finite subgroups of G
- $\mathfrak{F} = \mathfrak{V}\mathcal{C}$, the family of all virtually cyclic subgroups of G. (A group is virtually cyclic if it has a cyclic subgroup of finite index)
- $\mathfrak{F} = \{e\}$, the family consisting only of the trivial subgroup.

Later on, we will mainly be concerned with $\mathfrak{F} = \{e\}$, but will also talk about $\mathfrak{F} = \mathfrak{F}in$.

Definition 3.16. A G-CW-complex X is called a classifying space for the family $\mathfrak{F}$, or a model for $E_{\mathfrak{F}}G$, if for each subgroup $H \leq G$, the following holds:

$$X^H \simeq \begin{cases} * & \text{if} \quad H \in \mathfrak{F} \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem 3.17. *For each group G there exists a model for $E_{\mathfrak{F}}G$.*

Proof To prove existence one could follows either Milnor's [**11**] or Segal's [**16**] construction of EG, the classifying space for free actions. We shall follow Milnor's model here: Let

$$\Delta = \bigsqcup_{H \in \mathfrak{F}} G/H$$

be the discrete G-CW-complex as in example 3.9. Now form the n-fold join

$$\Delta_n = \underbrace{\Delta * ... * \Delta}_{n}$$

and put

$$X = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

Example 3.9 now implies that $\Delta^H = \emptyset \iff H \notin \mathfrak{F}$. Furthermore, since

$$\Delta^H * ... * \Delta^H = (\Delta * ... * \Delta)^H,$$

Lemma 3.15 implies that $X^H \simeq *$ for $H \in \mathfrak{F}$ and $X^H = \emptyset$ otherwise and X is therefore a model for $E_{\mathfrak{F}}G$. □

This construction, however gives us an infinite dimensional model, which is not of finite type. In this course we will try to find "nice" models.

REMARK 3.18. when considering the family $\mathfrak{F} = \mathfrak{Fin}$, the we denote the classifyibg space $E_{\mathfrak{F}}G$ by $\underline{E}G$. This is the classifying space for proper action.

Let G be torsion-free and X be a model for $\underline{E}G$. Then X is contractible and G acts freely ($X^{\{e\}} \simeq *$ and $X^H = \emptyset$ for all $\{e\} \neq H \leq G$). Hence X is a model for EG, the classifying space for free actions, or equivalently the universal cover of a $K(G, 1)$, an Eilenberg-Mac Lane space.

EXAMPLE 3.19. (Examples for torsion-free groups)

(a) $G = \mathbb{Z}$. Then $\mathbb{R}$ is a model for EG by Example 3.6 (a)
(b) $G = \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R}^2$ is a model for EG by Example 3.6 (b).
(c) Let G be the free group on 2 generators, $G = \langle x, y \rangle$. Then the Cayley-graph is a tree, which is a model for EG .

EXAMPLE 3.20. (Examples for groups with torsion)

(a) If G is a finite group, then $\{*\}$ is a model for $\underline{E}G$.
(b) Let $G = D_\infty$ be the infinite dihedral group. Then $\mathbb{R}$ is a model for $\underline{E}G$, where the generator for the infinite cyclic group acts by translation and the generator of order two acts by reflection.
(c) Let G be a wallpaper group, i.e. an extension of $\mathbb{Z} \times \mathbb{Z}$ with a finite subgroup of O_2, the group of 2×2 orthogonal matrices. Then $\mathbb{R}^2$ is a model for $\underline{E}G$.
(d) Let $G = PSL_2(\mathbb{Z})$. We've seen in example 3.6 (c) that G acts by Möbius transformations on $\mathbb{H}$ the upper half plane. This is a 2-dimensional model for $\underline{E}G$.

    Considering the two generators, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ we can see that $G \cong C_2 * C_3$ the free product of a cyclic group of order 2 and a cyclic group of order 3. Hence, the dual tree T is a 1-dimensional model for $\underline{E}G$.

For the interested reader I will include a very brief overview of some of the homotopy theory behind the above construction:

DEFINITION 3.21. A G-space X is called proper of for each pair of points $x, y \in X$ there are open neighbourhoods V_x of x and V_y of y such that the closure of $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is a compact subset of G.

If G is discrete this means that the above set is finite. Hence a G-CW complex X is proper if and only if all stabilizers are finite.

THEOREM 3.22. *(**J.H.C. Whitehead**, see* [**?**]*, Chapter I)*
A G-map $f : X \to Y$ between two G-CW-complexes is a G-homotopy equivalence if for all $H < G$ and all $x_0 \in X^H$ the induced map

$$\pi_*(X^H, x_0) \to \pi_*(Y^H, f(x_0))$$

is bijective.

Now, the following theorem explains why we call $\underline{E}G$ the classifying space for proper actions:

THEOREM 3.23. *(See* [**?**, Theorem 2.4]*) Let X be a proper G-CW-complex. Then, up to G-homotopy, there is a unique G-map $X \to \underline{E}G$.*

EXERCISE 11. Show that any two models for $\underline{E}G$ are G-homotopy equivalent.

4. Projective resolutions

In this section we will construct projective resolutions by considering classifying spaces. We will look at EG, the classifying space for free actions. But let us begin with the augmented cellular chain complex.

Let X be a G-CW-complex. It's augmented cellular chain complex is a chain complex of G-modules

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_o(X) \twoheadrightarrow \mathbb{Z},$$

in which each $C_i(X)$ is the free abelian group on the orbits of i-cells. Hence

$$C_i(X) \cong \bigoplus_{orbit-reps\,\sigma} \mathbb{Z}[G/G_\sigma].$$

Recall that a G-CW-complex X is a model for EG if X is contractible and G acts freely on X.

Proposition 3.24. *Let X be a model for* EG*. Then the augmented cellular chain complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Remark 3.25. Let $\mathbf{C}$ be a chain complex. We say $\mathbf{C}$ is acyclic of and only if $H_*(\mathbf{C}) = \mathbf{0}$. A G-CW-complex X is called acyclic if it has the homology of a point. Hence an acyclic G-CW-complex with a free G-action would also give us a free resolution of $\mathbb{Z}$.

Definition 3.26. We say a group G has finite geometric dimension ($\mathrm{gd}G < \infty$) if it admits a finite dimensional model for EG. The smallest such dimension is called the geomtric dimension of G.

We can now state the following corollary to Proposition 3.24:

Corollary 3.27. *For each group G:*

$$\mathrm{cd}G \leq gdG.$$

The converse is almost true and involves rather more than we can cover here. It comes in two parts, which were proved using very different methods.

Theorem 3.28. [**17, 18**][**Stallings-Swan**] *Let G be a group. Then*

$$\mathrm{cd}G = 1 \iff \mathrm{gd}G = 1 \iff G \textit{is free.}$$

Theorem 3.29. [**7**][**Eilenberg-Ganea**] *Let G be a group such that* $\mathrm{cd}G \geq 3$. *Then*

$$\mathrm{cd}G = \mathrm{gd}3.$$

This leaves the case when G is a group with $\mathrm{cd}G = 2$. It is still unknown whether there is a group G, which does not admit a 2-dimensional model for EG, although there are some candidates for examples [**?**] (Bestvina).

Example 3.30. Let G be a free group on S. We now construct a model X for EG . We take a fixed vertex x_0 and we then have a unique orbit of vertices . Hence

$$C_0(X) = \mathbb{Z}V = \mathbb{Z}G.$$

As basis for $C_1(X) = \mathbb{Z}E$ we take for each $s \in S$ an oriented 1-cell e_s. We assume the initial vertex is x_0. Otherwise we translate by a suitable g. Hence the initial vertex of e_s is x_0 and the terminal vertex is sx_0. and we get a map

$$\begin{array}{rccc} \delta: & \mathbb{Z}E & \rightarrow & \mathbb{Z}V \\ & e_s & \mapsto & sx_0 - x_0 = (s-1)x_0 \end{array}$$

$\mathbb{Z}E = C_i(X) = \mathbb{Z}G^{(S)}$ and we have a free resolution of length 1:

$$0 \longrightarrow \mathbb{Z}G^{(S)} \xrightarrow{\delta} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

This is the resolution we have seen in chapter 2.

(a) Let $G = \langle t \rangle$ be infinite cyclic. Then $X = \mathbb{R}$ and we recover the resolution in section 1:

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(b) We can also view X as a G-CW-comples with one orbit of 0-cells and s orbits of one cells $\{g, gs\}$ with the action induced by left translation of G on itself. E.g for $G = \langle s, t \rangle$ we get the tree seen in chapter 2

EXAMPLE 3.31. Let G be a finite cyclic group, i.e. $G = \langle t \,|\, t^n - 1 \rangle$. The circle S^1 is a G-CW-complex with n vertices and n 1-cells. There is one orbit of vertices $\{v, tv, ..., t^{n-1}v\}$ and one orbit of 1-cells $\{e, te, ..., t^{n-1}e\}$ and $(t-1)(e + te + ... + t^{n-1}e) = 0$. Consider

$$\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\ .$$

Hence $H_1(S^1)$ is generated by 1 element $e + te + .. + t^{n-1}e = Ne$ and we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\eta(1) = N$. We now splice these sequences together to obtain a free resolution

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of infinite length, as seen in Section 3 in Chapter 1.

And we retrieve the following results from Section 3 in Chapter 1:

PROPOSITION 3.32. *Let G be a finite cyclic group. Then*

$$\mathrm{H}^{2k}(G, \mathbb{Z}) \neq 0)$$

for all $k > 0$. In particular

$$\mathrm{cd}G = \infty.$$

COROLLARY 3.33. *Let G be a group of finite cohomological dimension. Then G is torsion-free.*

CHAPTER 4

Cohomological finiteness conditions

1. Induction and Coinduction

PROPOSITION 4.1. *Let R and S be rings, A a right S-module, B a right R-module and a left S-module and C a right R-module. Then there is a natural isomorphism*

$$\mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)),$$

the so called adjoint isomorphism.

$\mathrm{Hom}_R(B, C)$ is a right S-module via $(\varphi s)(b) = \varphi(sb)$. Contravariance of $\mathrm{Hom}_R(-, C)$ leads to this 'switch from right to left'.

REMARK 4.2. Let $\alpha : S \to R$ be a ring homomorphism. Then every R-module M can be viewed as an S-module via $sm = \alpha(s)m$ for all $s \in S, m \in M$. This is called **Restriction of scalars**.

REMARK 4.3. **Extension of Scalars** Let $\alpha : S \to R$ be a ring homomorphism. As above, R can be viewed as a left S-module vial $sr = \alpha(s)r$ for all $s \in S, r \in R$. Now let M be a right S-module and form a $\mathbb{Z}$-module

$$M \otimes_S R.$$

The right action of R on itself commuted with the left action of S. Hence $M \otimes_S R$ can be viewed as a right R-module via

$$(m \otimes r)r' = m \otimes rr'.$$

We now apply the adjoint isomorphism 4.1 to obtain a natural isomorphism

$$\mathrm{Hom}_R(M \otimes_S R, N) \cong \mathrm{Hom}_S(M, N).$$

We say extension of scalars is left adjoint to restriction of scalars.

REMARK 4.4. **Coextension of scalars** This construction is dual to that in 4.3. Let M be a right S-module. Then

$$\mathrm{Hom}_S(R, M)$$

is a right R-module via $f^{r'}(r) = f(rr')$. Now it follows from 4.1 that for all R-modules N and S-modules M there is a natural isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M)) \cong \mathrm{Hom}_S(M, N).$$

We say Coextension of scalars is right adjoint to restriction of scalars.

EXAMPLE 4.5. Let $S = \mathbb{Z}G$ for a group G and $R = \mathbb{Z}$. Consider the augmentation map $\epsilon : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$, which is a ring homomorphism. extension of scalars sends a G-module M to

$$M \otimes_{\mathbb{Z}G} \mathbb{Z} \cong M_G,$$

where $M_G = M/L$, where L is the submodule generated by all $mg - m$. Also note, that

$$M_G \cong M/\mathfrak{g}.$$

On the other hand, coextension of scalars gives $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) = M^G = \mathrm{H}^o(G, M)$.

We will be insterested under which circumstances these constructions preserve exactness, send projectives to projectives or injectives to injectives. Note, that so far, it is only clear that restriction preserves exactness.

Lemma 4.6. (1) *Extension of scalars sends projective S-modules to projective R-modules.*
(2) *Coextension of scalars sends injective S-modules to injective R-modules.*
(3) *Let R be flat as an S-module. Then under restriction, injective R-modules become injective S-modules.*
(4) *Let R be projective as an S-module. Then under restriction projective mR-modules become projective S-modules.*

Lemma 4.7. *Let G be a group. Then every right G-module can be viewed as a left G-module and vice versa. The operation is given by $gm = mg^{-1}$ for all $g \in G, m \in M$.*

From now on let's consider group rings again. Let $H \leq G$ be a subgroup. Then the inclusion induces a ring-homomorphism

$$\mathbb{Z}H \hookrightarrow \mathbb{Z}G.$$

Extension of scalars becomes **Induction from H to G.** Let M be an H-module. Then.

$$Ind_H^G M = M \otimes_{\mathbb{Z}H} \mathbb{Z}G = M\!\uparrow_H^G$$

Coextension of scalars becomes **Coinduction from H to G.** Let M be an H-module. Then:

$$Coind_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

Let N be a G-module. Then restriction of scalars is usually denoted by

$$Res_H^G N = N\!\downarrow_H^G .$$

Proposition 4.8. *The G-module $M\!\uparrow_H^G$ contains M as a H-submodule. Furthermore,*

$$M\!\uparrow_H^G \cong \bigoplus_{g \in E} Mg$$

where E is a system of representatives for the right cosets Hg.

Note that $\mathbb{Z}\!\uparrow_H^G \cong \mathbb{Z}[H\backslash G]$ is a permutation module.

Proposition 4.9. **Frobenius reciprocity** *Let $H \leq G$ be a subgroup of the group G. Let M be an H-module and N be a G-module. Then there is an isomorphism of G-modules*

$$N \otimes M\!\uparrow_H^G \cong (N\!\downarrow_H^G \otimes M)\!\uparrow_H^G .$$

This implies that for every H-module N

$$N \otimes \mathbb{Z}[H\backslash G] \cong N \otimes_{\mathbb{Z}H} \mathbb{Z}G,$$

where on the left we have a diagonal G-action, wheras on the right hand side the G-action only comes from the action on $\mathbb{Z}G$. In particular, if M is a G-module with underlying abelian group M_0 then

$$M \otimes \mathbb{Z}G \cong M_0 \otimes \mathbb{Z}G.$$

In particular, if M_0 is a free abelian group, $M \otimes \mathbb{Z}G$ is a free G-module.

PROPOSITION 4.10. **Mackey's formula** *Let $H \leq G$ and $K \leq G$ and let E denote a system of representatives for the double cosets KgH. For each K-module M there is a K-module isomorphism:*

$$(M\uparrow_H^G)\downarrow_K^G \cong \bigoplus_{g\in E} (Mg\downarrow_{K\cap H^g}^{H^g})\uparrow_{K\cap H^g}^K .$$

In particular, if N is a normal subgroup of G then

$$(M\uparrow_H^G)\downarrow_H^G \cong \bigoplus_{g\in H\backslash G} Mg.$$

We can identify $Mg\downarrow_{K\cap H^g}^{H^g}$ with $M\downarrow_{K\cap H^g}^{H}$ whereby the second restriction is with respect to the map: $K \cap g^{-1}Hg \to H$ mapping $k \mapsto gkg^{-1}$.

PROPOSITION 4.11. *Let $|G : H| < \infty$. Then*

$$Ind_H^G M \cong Coind_H^G M$$

for every H-module M.

EXERCISE 12.
(1) Show that induction is invariant under conjugation,i.e. show that for every H-module M and $g \in G$
$$M\uparrow_H^G \cong Mg\uparrow_{H^g}^G .$$
(2) Let $|G : H| = \infty$. Show that for any H-module M:
$$(M\uparrow_H^G)^G = 0.$$

THEOREM 4.12. **Eckmann-Shapiro Lemma** *Let $H \leq G$ and let M be an H-module. Then*

$$\mathrm{H}^*(H, M) \cong \mathrm{H}^*(G, Coind_H^G M).$$

REMARK 4.13. Let $|G : H| < \infty$. Then
(1) $\mathrm{H}^*(H, \mathbb{Z}) \cong \mathrm{H}^*(G, \mathbb{Z}[H\backslash G])$ and
(2) $\mathrm{H}^*(H, \mathbb{Z}H) \cong \mathrm{H}^*(G, \mathbb{Z}G)$.

Finally we will make a remark on the exactness of induction:

PROPOSITION 4.14. *Let $A \hookrightarrow B \twoheadrightarrow C$ be a short exact sequence of $\mathbb{Z}H$-modules. Then*

$$A\uparrow_H^G \hookrightarrow B\uparrow_H^G \twoheadrightarrow C\uparrow_H^G$$

is an exact sequence of $\mathbb{Z}G$-modules.

EXERCISE 13. Let k be a field and let G be a finite group. Prove that a kG-module is projective if and only if it is injective. (Hint: every k-module is free).

THEOREM 4.15. **Mayer Vietoris sequence:** *Let $G = K *_H L$ be a fee product with amalgamation and let M be a G-module. Then the following sequence in cohomology is exact:*

$$\cdots \to \mathrm{H}^n(G, M) \to \mathrm{H}^n(L, M) \oplus \mathrm{H}^n(K, M) \to \mathrm{H}^n(H, M) \to \mathrm{H}^{n+1}(G, M) \to \cdots .$$

2. Cohomological dimension

Recall the definition of cohomological dimension from Chapter 2.1:

$$\begin{aligned}\mathrm{cd}G &= \mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}\\ &= inf\{n \mid \mathbb{Z} \text{ has a projective resolution of length } n\}\\ &= inf\{n \mid \mathrm{H}^i(G,-) = 0\ \forall I > n\}\\ &= sup\{n \mid \exists M\ s.t.\ \mathrm{H}^n(G,M) \neq 0.\}\end{aligned}$$

We have further seen that if $\mathrm{cd}G = n$ there exists a free module F such that $\mathrm{H}^n(G,F) \neq 0$.
Let us recall a few more facts:

(1) Let G be a finite group. Then $\mathrm{cd}G = \infty$. In particular groups of finite cohomological dimension are torsion-free.
(2) Let $H \leq G$. Then $\mathrm{cd}H \leq \mathrm{cd}G$.
(3) $\mathrm{cd}G = 0 \iff G = \{e\}$.
(4) Let G be a free group (in particular if G is infinite cyclic). Then $\mathrm{cd}G = 1$. The converse is also true and due to Stallings and Swan.

Lemma 4.16. *Let G be a group with $\mathrm{cd}G = n$. Then there is a free $\mathbb{Z}G$-module F such that*

$$\mathrm{H}^n(G,F) \neq 0.$$

Proposition 4.17. *Let G be a group of finite cohomological dimension and let $H \leq G$ be a subgroup of finite index. Then*

$$\mathrm{cd}G = \mathrm{cd}H.$$

One can make an even stronger statement not having to assume that G has finite cohomological dimension:

Theorem 4.18. **Serre's Lemma** *Let $|G : H| < \infty$ Then $\mathrm{cd}G = \mathrm{cd}H$.*

The proof of Serre's Lemma relies on Proposition 4.17 and on building a model for EG from a product of models for EH.

Theorem 4.19. *Let $H \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups. Then*

$$\mathrm{cd}G \leq \mathrm{cd}H + \mathrm{cd}Q.$$

This follows directly from the Hochschild-Serre spectral sequence, and we will not prove it here.

Exercise 14. Let G be a free abelian group of finite rank n. Then $\mathrm{cd}G = n$.

We have already seen that a free abelian group G of rank n has $\mathrm{cd}G \leq n$. So one only needs to prove the last part of the following theorem:

Theorem 4.20. *Let G be a free abelian group of rank n. Then*

$$\mathrm{cd}G = n,$$

for all $i < n$

$$\mathrm{H}^i(G,\mathbb{Z}G) = 0$$

and

$$\mathrm{H}^n(G,\mathbb{Z}G) \neq 0.$$

And Now we have a direct corollary to the Mayer-Vietoris Sequence in Theorem 4.15:

COROLLARY 4.21. *Let $G = K *_H L$ be a fee product with amalgamation. Then* $\mathrm{cd}G \leq sup\{\mathrm{cd}H, \mathrm{cd}K, \mathrm{cd}L\} + 1$.

3. Groups of type FP_n .

We already did see one cohomological finiteness condition, the cohomological dimension of a group. The main purpose of this chapter is a discussion of the notion of groups of type FP_n, which can be viewed as a generalisation of finite generation (at least as long as $n \geq 2$).

DEFINITION 4.22. Let R be a ring.

(1) Let M be an R-module. We say M is of type FP_n if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $i \leq n$.
(2) M is of type FP_∞ if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $n \geq 0$.
(3) M is of type FP if M is of type FP_∞ and $\mathrm{pd}_R M < \infty$.

REMARK 4.23. (1) M is of type FP_0 if and only if M is finitely generated.
(2) M is of type FP_1 if and only if M is finitely presented.
(3) Let M be of type FP_n . Then there is a free resolution $F_* \twoheadrightarrow M$ with each F_i finitely generated for all $i \leq n$.

We say a module is of type FL if there is a finite length free resolution $F_* \twoheadrightarrow M$ where all F_i are finitely generated. It is obvious that modules of type FL are of type FP but the converse is not necessarily true.

DEFINITION 4.24. A group G is said to be of type FP_n if $\mathbb{Z}$ is a $\mathbb{Z}G$-module of type FP_n .

REMARK 4.25. Every group is of type FP_0, since the augmentation map $\epsilon : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ gives the beginning of a projective resolution and $\mathbb{Z}G$ is a finitely generated $\mathbb{Z}G$-module.

PROPOSITION 4.26. *A group G is of type* FP_1 *if and only if G is finitely generated.*

The description of groups of type FP_2 is already a lot more complicated. A group is called almost finitely presented if there is an exact sequence of groups $K \hookrightarrow F \twoheadrightarrow G$ where F is finitely generated free and $K/[K, K]$ is finitely generated as a G-module. Finitely presented groups are almost finitely presented but the converse is not true in general, see the examples by Bestvina and Brady [**1**]. Bieri [**2**] has shown that the property FP_2 is equivalent to the group being almost finitely presented.

Now let's have a look at finite extensions. We cannot make any more general statements as even finite generation is in general not a subgroup-closed property.

PROPOSITION 4.27. *Let $G' \leq G$ be a subgroup of finite index. Then G is of type* FP_n *if and only if G' is of type* FP_n .

DEFINITION 4.28. (1) A group G is of type FP iff G is of type FP_∞ and $\mathrm{cd}G < \infty$.
(2) A group is of type FL if G has a finite length finitely generated free resolution.

Obviously does FL imply FP but the converse is not known. Let P be a projective module in the top dimension of a projective resolution of $\mathbb{Z}$. Suppose F is a finitely generated free module such that $P \oplus F$ is free. Then on can construct a finitely generated free resolution

$$F \hookrightarrow P \oplus F \to F_{n-1} \to ... \to F_0 \twoheadrightarrow \mathbb{Z}$$

. We say such a P is **stably free.**

PROPOSITION 4.29. *Let G be a group of type* FP *Suppose that*

$$0 \to P \to F_{n-1} \to ... \twoheadrightarrow \mathbb{Z}$$

is a finitely generated resolution with F_i finitely generated for all $i \leq n-1$. Then G is of type FL *if and only of P is stable free.*

Hence the question whether FL implies FP reduces to the question whether there are projectives that are not stably free. Over general rings the answer can be Yes. There are even examples over group rings $\mathbb{Z}G$ where $G = \mathbb{Z}_{23}$ due to Milnor [**13**, Chapter 3]. These groups, however have infinite cohomological dimension.

Let is conclude this chapter with some topological remarks. We have already defined finite type and finitelt generated G-CW-complexes. A G-CW-complex X is finitely dominated if there exists a finite complex K such that Y is a homotopy retract, i.e. there are maps $i : X \to K$ and $r : K \to X$ such that $ri \simeq id_X$.

PROPOSITION 4.30.
(1) *Let G admit a finite type model for* EG*. Then G is of type* FP_∞ .
(2) *Let G admit a cocompact model for* EG*. Then G is of type* FL .
(3) *Let G admit a finitely dominated model for* EG*. Then G is of type* FP .

The converse to the above is also true if we also assume that G is finitely presented. This is due to Eilenberg-Ganea and Wall.

We also have a direct corollary to the Mayer-Vietoris Sequence in Theorem 4.15 for FP_∞. This also relies on the fact that G is of type FP_∞ if and only if $\mathrm{H}^*(G, -)$ commutes with direct limits, see, for example [**4**, VIII,4.8]

COROLLARY 4.31. *Let $G = K *_H L$ be a fee product with amalgamation. Then*
(1) $\mathrm{cd}G \leq sup\{\mathrm{cd}H, \mathrm{cd}K, \mathrm{cd}L\} + 1$.
(2) *If H, K and L are of type* FP_∞ *or* FP *respectively, then so is G.*

EXERCISE 15. State and prove an analogous result to 4.31 (2) for H, K, L of types $\mathrm{FP}_l, \mathrm{FP}_n$ and FP_m respectively. (Use an analogue result for direct limits, due to Bieri-Eckman, see [**2**]

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