London Taught Course Centre

2023/24 examination

Graph Theory

Instructions to candidates

This open-book exam has 3 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed. You are also allowed to use any theorems you find, provided they are properly referenced.

1 Recall that given trees T_1 and T_2, we say T_1 is a topological minor of T_2 if we can perform a sequence of vertex deletions and degree-2 vertex suppressions on T_2 in order to obtain a tree isomorphic to T_1.

Suppose now that the vertices of each of T_1 and T_2 are labelled, with labels taken from a not necessarily finite set S, which is equipped with a well-quasi-ordering $\prec$. When we perform vertex deletions and degree-2 vertex suppressions, it does not change the labels of other vertices (in particular, if w is a degree 2 vertex whose neighbours are u and v, suppressing w means deleting w and its incident edges and adding uv, but not changing the labels of u and v).

We say T_1 is a labelled topological minor of T_2 if we can perform a sequence of vertex deletions and degree-2 vertex suppressions on T_2 in order to obtain a tree T' which is isomorphic to T_1, and furthermore there is an isomorphism $\phi : V(T') \to V(T_1)$ such that $\phi(v) \prec v$ for each $v \in V(T')$.

Prove that the class of labelled trees is well-quasi-ordered by the labelled topological minor relation.

2 (a) Show that if H is an m-vertex graph which does not contain K_3 and all of whose vertices have degree strictly greater than $\frac{2m}{5}$, then H can be properly vertex-coloured with two colours.

(b) Show that it is NP-complete to decide whether a graph G can be properly vertex-coloured with three colours.

(c) Show that for each integer C and each sufficiently large n (depending on C) there exists an n-vertex graph G which does not contain K_3, all of whose vertices have degree at least $\frac{n}{\log n}$, and which cannot be properly vertex-coloured with C colours.

3 Given a graph H, let $K_3(H)$ denote the set of copies of K_3 in H. A *fractional triangle factor* in H is a map $w : K_3(H) \to [0, 1]$ such that for each $v \in V(H)$ we have

$$\sum_{\substack{T \in K_3(H) \\ v \in T}} w(T) \leq 1\,.$$

The weight of a fractional triangle factor w is $\frac{1}{v(H)} \sum_{T \in K_3(H)} w(T)$.

(a) Prove that for any graph H, any fractional triangle factor in H has weight at most $\frac{1}{3}$.

Given a graph G and parameters $\varepsilon, d > 0$, let $R(G)$ denote the graph obtained as follows. We apply the Szemerédi Regularity Lemma to G, which returns a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ with $\varepsilon^{-1} \leq k \leq K$, where K is a constant independent of $v(G)$. We let the vertex set of $R(G)$ be $[k]$, and put an edge ij into $R(G)$ if and only if (V_i, V_j) is ε-regular of density at least d.

(b) Show that for every $d > 0$, if $\varepsilon > 0$ is sufficiently small and n is sufficiently large, the following holds for any n-vertex graph G. If $R(G)$ contains a fractional triangle factor with weight $\frac{1}{3}$, then there is a set of $0.33n$ pairwise vertex-disjoint copies of K_3 in G.