A course in homological algebra

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Introduction

Homological algebra emerged from algebraic topology as a tool to translate geometric intuition into an algebraic formalism. Homological algebra shifts the focus from a direct descriptive approach of a mathematical object X in some category $\mathcal{C}$ towards a study of that object in relation to all others in the same category via the functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X)$. This leads to new invariants which in many situations describe obstructions regarding the existence and uniqueness of objects and maps between objects with certain properties.

Homological methods can be applied to a wide range of mathematical objects, and hence some basic category theoretic language is essential. The appendix contains a review of the terminology and background material from category theory which we will use from the start.

The first three sections cover three key concepts of homological algebra, namely complexes, homology, and homotopy. The focus in these sections is on methods how to manipulate chain complexes and their invariants. We will then describe how chain complexes are attached to a variety of mathematical objects, including algebras, groups, and topological spaces, and the interaction between the structure of these objects and the homological invariants attached to them.

Contents

1 Complexes

Definition 1.1. A *graded object over a category* $\mathcal{C}$ is a family $X = (X_n)_{n\in\mathbb{Z}}$ of objects X_n in $\mathcal{C}$. Given two graded objects $X = (X_n)_{n\in\mathbb{Z}}$ and $Y = (Y_n)_{n\in\mathbb{Z}}$ in $\mathcal{C}$, a *graded morphism of degree* m is a family $f = (f_n)_{n\in\mathbb{Z}}$ of morphisms $f_n : X_n \to Y_{n+m}$ in $\mathcal{C}$. The category of graded objects over $\mathcal{C}$ with graded morphisms of degree zero is denoted $\mathrm{Gr}(\mathcal{C})$.

The notion of a graded object makes sense for any category. We adopt the convention that a graded object is a family of objects indexed by $\mathbb{Z}$, but it is worth noting that here may be situations where it is useful to specify gradings indexed by $\mathbb{N}$ or more general groups and monoids.

For instance, a graded vector space over a field k is an object in the category $\mathrm{Gr}(\mathbf{Vect}(k))$, and a graded A-module is a graded object over the category $\mathrm{Mod}(A)$ of (left unital) A-modules, where A is a ring or an algebra over a commutative ring. In an additive category in which direct sums indexed by $\mathbb{Z}$ exist, such as in the categories $\mathbf{Vect}(k)$ and $\mathrm{Mod}(A)$, one writes sometimes $\oplus_{n\in\mathbb{Z}}X_n$ instead of $(X_n)_{n\in\mathbb{Z}}$, where this notation is understood to include the structural monomorphisms $X_m \to \oplus_{n\in\mathbb{Z}}X_n$.

The minimum requirement for the following definition of a (co-) chain complex over a category $\mathcal{C}$ is the notion of a *zero morphism* between any two objects X and Y in $\mathcal{C}$. The existence of zero morphisms is ensured by the existence of a *zero object*. This is an object, typically denoted 0, which is *terminal* and *initial*; that is, there are unique morphisms $X \to 0$ and $0 \to X$ for all objects X. The zero morphism $X \to Y$ is the unique morphism which factors $X \to 0 \to Y$.

Definition 1.2. A *chain complex* over a category $\mathcal{C}$ with a zero object is a pair (X, δ) consisting of a graded object X in $\mathcal{C}$ and a graded endomorphism δ of degree -1, called the *differential of the complex* , satisfying $\delta \circ \delta = 0$. Explicitly, δ is a family of morphisms $\delta_n : X_n \to X_{n-1}$ satisfying $\delta_{n-1} \circ \delta_n = 0$. Dually, a *cochain complex* over a $\mathcal{C}$ is a pair (X, δ) consisting of a graded object $X = (X^n)_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and a graded endomorphism $\delta = (\delta^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$, called *differential* of the cochain complex, of degree 1 satisfying $\delta \circ \delta = 0$, or equivalently, $\delta^{n+1} \circ \delta^n = 0$ for $n \in \mathbb{Z}$.

One can visualise a chain complex as a possibly infinite sequence of morphisms in which the composition of any two consecutive morphisms is zero.

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

For cochain complexes, the only difference is that the indices increase in the direction of the differential:

$$\cdots \longrightarrow X^{n-1} \xrightarrow{\delta^{n-1}} X^n \xrightarrow{\delta^n} X_{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

In order to distinguish between chain complexes and cochain complexes, the standard notational convention is to use subscripts for chain complexes and superscripts for cochain complexes. One can always switch from a chain complex to a cochain complex and vice versa by setting $X^n = X_{-n}$ and $\delta^n = \delta_{-n}$. Through this correspondence, any terminology in the context of chain complexes has an analogue for cochain complexes.

Example 1.3. Let $\mathcal{C}$ be a category with a zero object. Let U, V be objects in $\mathcal{C}$. Any morphism $f : U \to V$ in $\mathcal{C}$ can be regarded as a chain or cochain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{\;f\;} V \longrightarrow 0 \longrightarrow \cdots$$

with U and V in any two consecutive degrees. A special case of this example arises for A an algebra and c any element in A. Then the map $f : A \to A$ given by $f(a) = ac$ for all $a \in A$ (that is, f is given by right multiplication with c) is an A-module endomorphism, giving rise to a two-term complex of the form

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\;f\;} A \longrightarrow 0 \longrightarrow \cdots$$

Tensor products (to be defined) of complexes of this form yield Koszul complexes.

Definition 1.4. Let $\mathcal{C}$ be a category with a zero object. A *chain map* between two chain complexes (X, δ), (Y, ϵ) over $\mathcal{C}$ is a graded morphism of degree zero $f = (f_n : X_n \to Y_n)_{n \in \mathbb{Z}}$ satisfying

$$f \circ \delta = \epsilon \circ f \ ,$$

or equivalently,

$$f_{n-1} \circ \delta_n = \epsilon_n \circ f_n$$

for all $n \in \mathbb{Z}$. Cochain maps are defined similarly. The chain complexes, together with chain maps, form the category $\mathrm{Ch}(\mathcal{C})$ of *chain complexes over* $\mathcal{C}$.

A chain map can be visualised as a commutative ladder of the form

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots \longrightarrow & Y_{n+1} & \xrightarrow{\epsilon_{n+1}} & Y_n & \xrightarrow{\epsilon_n} & Y_{n-1} & \xrightarrow{\epsilon_{n-1}} & \cdots
\end{array}$$

If the differential of a complex (X, δ) is clear from the context, we adopt the notational abuse of just calling X a chain complex. We have a forgetful functor $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ mapping a chain complex (X, δ) to its underlying graded object X. We have a functor from $\mathrm{Gr}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$ sending a graded object X to the complex $(X, 0)$ with zero differential; when composed with the forgetful functor this yields the identity functor on $\mathrm{Gr}(\mathcal{C})$. Note that complexes and chain maps can be defined in any category with a zero object.

For the next concept we need slightly more structure, namely that $\mathcal{C}$ is an *additive category* (so that in particular morphism sets have an abelian group structure). For any integer i, we define the *shift automorphism* $[i]$ of $\mathrm{Ch}(\mathcal{C})$ as follows. For (X, δ) a chain complex, we define a graded object $X[i]$ by setting

$$X[i]_n = X_{n-i} \ ;$$

this becomes a chain complex together with the differential $\delta[i]$ defined by

$$\delta[i]_n = (-1)^i \delta_{n-i} \ ,$$

for $n \in \mathbb{Z}$. Note the sign convention here (this is where we need $\mathcal{C}$ to be additive, so that morphism sets between objects are abelian groups; the minus sign amounts to taking the inverse with respect to the abelian group structure of the relevant morphism space). This defines the shift functor on graded objects and on chain complexes. We need to define the shift functor on chain maps. For $f : X \to Y$ a chain map, we define $f[i] : X[i] \to Y[i]$ by setting

$$f[i]_n = f_{n-1}$$

for all $n \in \mathbb{Z}$. We define a shift functor on the category of cochain complexes analogously; that is, for (X, δ) a cochain complex, we define $(X[i], \delta[i])$ by $X[i]^n = X^{n+i}$ and $\delta[i]^n = (-1)^i \delta^{n+i}$ for all $n \in \mathbb{Z}$.

A chain complex X is called *bounded above* if $X_n = 0$ for n large enough; we denote by $\mathrm{Ch}^+(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded above chain complexes over $\mathcal{C}$. A chain complex X is called *bounded below* if $X_n = 0$ for n small enough; we denote by $\mathrm{Ch}^-(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded below chain complexes over $\mathcal{C}$. A chain complex X is called *bounded* if $X_n = 0$ for all but finitely many i; we denote by $\mathrm{Ch}^b(\mathcal{C})$ the full subcategory of $\mathrm{Ch}(\mathcal{C})$ consisting of all bounded chain complexes over $\mathcal{C}$.

Exercise 1.5. Let $\mathcal{C}$ be a category, and let X, Y, Z be graded objects over $\mathcal{C}$. Let $f : X \to Y$ be a graded map of degree i, and let $g : Y \to Z$ be a graded map of degree j. Show that $g \circ f$ is a graded map of degree $i + j$.

Exercise 1.6. Let $\mathcal{C}$ be an additive category, and let X, Y be bounded chain complexes over $\mathcal{C}$. Show that there are only finitely many integers i such that the space $\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(X, Y[i])$ of chain maps from X to $Y[i]$ is nonzero.

Exercise 1.7. Let $\mathcal{C}$ be an additive category, let i, $j \in \mathbb{Z}$. Show that for any chain complex X over $\mathcal{C}$ we have $X[i + j] = (X[i])[j]$. Show that this is an equality of functors $[i + j] = [j] \circ [i]$. Deduce that $[i]$ and $[-i]$ are inverse functors.

Exercise 1.8. Show that if $\mathcal{C}$ is an abelian category (e. g. the module category $\mathrm{Mod}(A)$ of an algebra A), then $\mathrm{Ch}(\mathcal{C})$ is again an abelian category. More precisely, show that for any chain map f from a complex (X, δ) to a complex (Y, ϵ), the differential δ restricts to a differential on the graded object $\ker(f) = (\ker(f_n : X_n \to Y_n))_{n \in \mathbb{Z}}$, and the resulting chain complex $(\ker(f), \delta|_{\ker(f)})$ is a kernel of f; similarly, show that ϵ induces a differential on the cokernel of f as graded morphism, which yields a cokernel of f in the category $\mathrm{Ch}(\mathcal{C})$. Deduce that f is a monomorphism (resp. epimorphism) in $\mathrm{Ch}(\mathcal{C})$ if and only if all f_i are monomorphisms (resp. epimorphisms) in $\mathcal{C}$. Show finally that the categories $\mathrm{Ch}^+(\mathcal{C})$, $\mathrm{Ch}^-(\mathcal{C})$, and $\mathrm{Ch}^b(\mathcal{C})$ are full abelian subcategories of $\mathrm{Ch}(\mathcal{C})$.

2 Homology

The content of this section could be formulated for arbitrary abelian categories. For expository purpose, we consider module categories, keeping in mind that this is no loss of generality, thanks to the Freyd-Mitchell embedding theorem, which says that any abelian category is equivalent to a full subcategory of a module category.

Throughout this section, A is an algebra over a commutative ring k. For a complex (X, δ) of A-modules, the condition $\delta \circ \delta = 0$ means that we have an inclusion $\mathrm{Im}(\delta) \subseteq \ker(\delta)$. The cokernel $\ker(\delta)/\mathrm{Im}(\delta)$ of this inclusion is defined to be the homology of X.

Definition 2.1. The *homology* of a chain complex (X, δ) of A-modules is the graded A-module $H_*(X, \delta) = \ker(\delta)/\mathrm{Im}(\delta)$; more explicitly,

$$H_n(X, \delta) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$$

for $n \in \mathbb{Z}$. If the differential δ is clear from the context we write $H_*(X)$ instead of $H_*(X, \delta)$. If $H_*(X) = \{0\}$ then X is called *exact* or *acyclic* . Similarly, the *cohomology* of a cochain complex (Y, ϵ) of A-modules is the graded A-module $\ker(\epsilon)/\mathrm{Im}(\epsilon)$; explicitly,

$$H^n(Y, \epsilon) = \ker(\epsilon^n)/\mathrm{Im}(\epsilon^{n-1})$$

for $n \in \mathbb{Z}$. As before, if ϵ is clear from the context we write $H^*(Y)$, and if $H^*(Y) = \{0\}$ then Y is called *exact* or *acyclic.*

The homology $H_n(X)$ in a fixed degree n of a chain complex X is a subquotient of X_n.

Exercise 2.2. Let (X, δ) be a chain complex of A-modules. Show that if $\delta = 0$, then $H_*(X, \delta) = X$, or equivalently, $H_n(X, \delta) = X_n$ for all $n \in \mathbb{Z}$.

Example 2.3. Any A-module M can be viewed as a complex 'concentrated in degree 0' by setting $X_0 = M$ and $X_n = \{0\}$ for any nonzero integer n, with the zero differential. The homology of this complex is isomorphic to the graded object X because the differential is zero.

Example 2.4. Any short exact sequence of A-modules can be viewed as bounded exact complex.

Taking homology or cohomology is functorial:

Proposition 2.5. *Let $f : (X, \delta) \to (Y, \epsilon)$ be a chain map of chain complexes of A-modules. Then f restricts to graded maps $\ker(\delta) \to \ker(\epsilon)$ and $\mathrm{Im}(\delta) \to \mathrm{Im}(\epsilon)$. In particular, f induces a graded homomorphism of graded A-modules $H_*(f) : H_*(X) \to H_*(Y)$. The assignments $X \mapsto H_*(X)$ and $f \mapsto H_*(f)$ define a functor from the category of chain complexes $\mathrm{Ch}(\mathrm{Mod}(A))$ to the category of graded A-modules $\mathrm{Gr}(\mathrm{Mod}(A))$.*

Proof. Let n be an integer. Let $x \in \ker(\delta_n)$. We have $\epsilon_n(f_n(x)) = f_{n-1}(\delta_n(x)) = 0$, hence $f(x) \in \ker(\epsilon_n)$. Thus f sends $\ker(\delta)$ to $\ker(\epsilon)$. Let $y \in \mathrm{Im}(\delta_{n+1})$. Write $y = \delta_{n+1}(z)$ for some $z \in X_{n+1}$. We have $f_n(y) = f_n(\delta_{n+1}(z)) = \epsilon_{n+1}(f_{n+1}(z)) \in \mathrm{Im}(\epsilon_{n+1})$, and hence f sends $\mathrm{Im}(\delta)$ to $\mathrm{Im}(\epsilon)$. Since $H_*(f)$ is induced by f, one verifies easily that given two composable chain maps f, g, we have $H_*(g \circ f) = H^*(g) \circ H^*(f)$. □

We have the obvious analogous statement for cohomology. The functor induced by taking homology is very different from the forgetful functor $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$.

Definition 2.6. Let $\mathcal{C}$ be an abelian category and X, Y chain complexes. A chain map $f : X \to Y$ is called a *quasi-isomorphism* if the induced map on homology $H_*(f)$ is an isomorphism $H_*(X) \cong H_*(Y)$ in $\mathrm{Gr}(\mathcal{C})$.

Note that if $f : X \to Y$ is a quasi-isomorphism, there need not be a chain map $g : Y \to X$ inducing the inverse isomorphism $H_*(Y) \cong H_*(X)$. The relation of two complexes X, Y being quasi-isomorphic is the smallest equivalence relation $\sim$ satisfying $X \sim Y$ if there is a quasi-isomorphism $X \to Y$. Similarly for cochain maps between cochain complexes.

Example 2.7. Let $0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$ be a short exact sequence of A-modules; that is, f is injective, $\mathrm{Im}(f) = \ker(g)$ and g is surjective. Then in particular $g \circ f = 0$, and hence the following diagram is commutative:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & U & \xrightarrow{f} & V & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

is commutative. This diagram represents a chain map from the top row to the bottom row, viewed as chain complexes with V, W in degree zero, with U in degree 1, and all other terms zero. This chain map is a quasi-isomorphism: the homology of the top row in degree zero is $V/\mathrm{Im}(f) = V/\ker(g) \cong \mathrm{Im}(g) = W$. In all other degrees the homology of the top row is zero (using that f is injective). The homology of the bottom row is also W in degree 0 and zero in all other degrees. Thus g induces a quasi-isomorphism. In general, there is no quasi-isomorphism from the bottom to the top row. In fact, there is such a quasi-isomorphism if and only if the exact sequence we started out with is split. To see this, observe first that any A-homomorphism $s : W \to V$ determines a chain map from the bottom to the top row (which is zero in all nonzero degrees). Such an s is a quasi-isomorphism if and only if the composition of maps $W \xrightarrow{s} V \longrightarrow V/\mathrm{Im}(f)$ is an isomorphism. This composition is injective if and only of s is injective and $\mathrm{Im}(s) \cap \mathrm{Im}(f) = \{0\}$. This composition is surjective if and only if $\mathrm{Im}(f) + \mathrm{Im}(s) = V$. Thus this is an isomorphism if and only if s is injective and if $V = \mathrm{Im}(s) \oplus \mathrm{Im}(f)$, so if and only if g is split surjective.

One might hope that if X is a subcomplex of a chain complex Y, then the homology of Y can be calculated in terms of that of X and Z. It is not quite as simple as that. One of the fundamental features of complexes over an abelian category is that short exact sequences of complexes give rise to long exact (co-)homology sequences. We state and prove this for module categories, as this will be sufficient for this course. For general abelian categories, one can either directly modify the proofs, or use Freyd's embedding theorem, saying that any abelian category can be fully embedded into a module category.

Theorem 2.8. *Let A be a k-algebra. Any short exact sequence of chain complexes of A-modules*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(g)} H_n(Z) \xrightarrow{d_n} H_{n-1}(X) \longrightarrow \cdots$$

depending functorially on the short exact sequence.

The functorial dependence in this theorem means that given a commutative diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0 \end{array}$$

we get a commutative 'ladder' of long exact sequences

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) & \xrightarrow{H_n(g)} & H_n(Z) & \xrightarrow{d_n} & H_{n-1}(X) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(Y) & \longrightarrow & \cdots \\ & & {\scriptstyle H_n(a)}\big\downarrow & & \big\downarrow{\scriptstyle H_n(b)} & & \big\downarrow{\scriptstyle H_n(c)} & & \big\downarrow{\scriptstyle H_{n-1}(a)} & & \big\downarrow{\scriptstyle H_{n-1}(b)} & & \\ \cdots & \longrightarrow & H_n(X') & \xrightarrow[H_n(f')]{} & H_n(Y') & \xrightarrow[H_n(g')]{} & H_n(Z') & \xrightarrow[d'_n]{} & H_{n-1}(X') & \xrightarrow[H_{n-1}(f')]{} & H_{n-1}(Y) & \longrightarrow & \cdots \end{array}$$

The morphism d_n is called *connecting homomorphism.* Theorem 2.8 translates verbatim to cochain complexes, except that the connecting homomorphism $d^n : H^n(Z) \to H^{n+1}(X)$ is of degree 1.

Proof of Theorem 2.8. Denote by δ, ϵ, ζ the differentials of X, Y, Z, respectively. The short exact sequence in the statement is a commutative diagram of the following form.

$$\begin{array}{ccccccccc} & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} & \xrightarrow{g_{n+1}} & Z_{n+1} & \longrightarrow & 0 \\ & & {\scriptstyle \delta_{n+1}}\big\downarrow & & {\scriptstyle \epsilon_{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{n+1}} & & \\ 0 & \longrightarrow & X_n & \xrightarrow{f_n} & Y_n & \xrightarrow{g_n} & Z_n & \longrightarrow & 0 \\ & & {\scriptstyle \delta_n}\big\downarrow & & {\scriptstyle \epsilon_n}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_n} & & \\ 0 & \longrightarrow & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Z_{n-1} & \longrightarrow & 0 \\ & & {\scriptstyle \delta_{n-1}}\big\downarrow & & {\scriptstyle \epsilon_{n-1}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_{n-1}} & & \\ 0 & \longrightarrow & X_{n-2} & \xrightarrow{f_{n-2}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Z_{n-2} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \end{array}$$

where the horizontal sequences are exact; that is, f_n is injective, g_n is surjective, and $\mathrm{Im}(f_n) = \ker(g_n)$ for all $n \in \mathbb{Z}$.

We define $d_n : H_n(Z) \to H_{n-1}(X)$ as follows. By the very definition of $H_n(Z) = \ker(\zeta_n)/\mathrm{Im}(\zeta_{n+1})$, any element in $H_n(Z)$ is of the form

$$z + \mathrm{Im}(\zeta_{n+1})$$

for some $z \in \ker(\zeta_n)$. Since g is surjective in each degree, there is $y \in Y_n$ such that

$$g_n(y) = z \ .$$

Since $\zeta_n(z) = 0$ we get that

$$g_{n-1}(\epsilon_n(y)) = \zeta_n(g_n(y)) = \zeta_n(z) = 0 \ .$$

Thus $\epsilon_n(y) \in \ker(g_{n-1}) = \mathrm{Im}(f_{n-1})$, where we use the exactness of the sequence in the statement. Thus there is $x \in X_{n-1}$ satisfying $f_{n-1}(x) = \epsilon_n(y)$. Moreover,

$$f_{n-2}(\delta_{n-1}(x)) = \epsilon_{n-1}(f_{n-1}(x)) = \epsilon_{n-1}(\epsilon_n(y)) = 0 \ .$$

Since f_{n-2} is a monomorphism, this shows that $\delta_{n-1}(x) = 0$, or equivalently, we have

$$x \in \ker(\delta_{n-1}) \ .$$

Hence $x + \mathrm{Im}(\delta_n)$ is an element in $H_{n-1}(X)$. In order to show that the assignment

$$z + \mathrm{Im}(\zeta_{n+1}) \mapsto x + \mathrm{Im}(\delta_n)$$

yields a well-defined map one needs to verify that if $z \in \mathrm{Im}(\zeta_{n+1})$ then $x \in \mathrm{Im}(\delta_n)$. Suppose that $z \in \mathrm{Im}(\zeta_{n+1})$. Write $z = \zeta_{n+1}(s)$ for some $s \in Z_{n+1}$. Since g_{n+1} is surjective there is $t \in Y_{n+1}$ such that $g_{n+1}(t) = s$. Then

$$g_n(\epsilon_{n+1}(t)) = \zeta_{n+1}(g_{n+1}(t)) = \zeta_{n+1}(s) = z = g_n(y)$$

and hence $y - \epsilon_{n+1}(t) \in \ker(g_n) = \mathrm{Im}(f_n)$. Write $y - \epsilon_{n+1}(t) = f_n(w)$ for some $w \in X_n$. We have

$$f_{n-1}(\delta_n(w)) = \epsilon_n(f_n(w)) = \epsilon_n(y - \epsilon_{n+1}(t)) = \epsilon_n(y) = f_{n-1}(x)$$

Since f_{n-1} is injective, this implies that $x = \delta_n(w)$ belongs to $\mathrm{Im}(\delta_n)$. This implies that there is indeed a well-defined map $d_n : H_n(Z) \to H_{n-1}(X)$ such that

$$d_n(z + \mathrm{Im}(\zeta_{n+1})) = x + \mathrm{Im}(\delta_n)$$

with x and z as above.

We need to show the exactness of the the sequence in three places. We start with $\mathrm{Im}(H_n(f)) = \ker(H_n(g))$. We have

$$\mathrm{Im}(H_n(f)) = \{f_n(x) + \mathrm{Im}(\epsilon_{n+1}) \mid x \in \ker(\delta_n)\}$$

$$\ker(H_n(g)) = \{y + \mathrm{Im}(\epsilon_{n+1}) \mid y \in \ker(\epsilon_n),\ g_n(y) \in \mathrm{Im}(\zeta_{n+1})\}$$

The inclusion $\mathrm{Im}(H_n(f)) \subseteq \ker(H_n(g))$ is clear because $g \circ f = 0$, hence $H_n(g) \circ H_n(f) = H_n(g \circ f) = 0$ by the functoriality of H_n. For the reverse inclusion, let $y \in \ker(\epsilon_n)$ such that $g_n(y) \in \mathrm{Im}(\zeta_{n+1})$. Thus $g_n(y) = \zeta_{n+1}(z')$ for some $z' \in Z_{n+1}$. Since g_{n+1} is surjective, there is $y' \in Y_{n+1}$ such that $g_{n+1}(y') = z'$. Then

$$g_n(y - \epsilon_{n+1}(y')) = g_n(y) - g_n(\epsilon_{n+1}(y')) = \zeta_{n+1}(z') - \zeta_{n+1}(g_{n+1}(y')) = \zeta_{n+1}(z') - \zeta_{n+1}(z') = 0$$

Thus $y - \epsilon_{n+1}(y') \in \ker(g_n) = \mathrm{Im}(f_n)$. Write $y - \epsilon_{n+1}(y') = f_n(x')$ for some $x' \in X_n$. Then $y + \mathrm{Im}(\epsilon_{n+1}) = f_n(x') + \mathrm{Im}(\epsilon_{n+1}) \in \mathrm{Im}(H_n(f))$.

We need to show next that $\mathrm{Im}(H_n(g)) = \ker(d_n)$. We have

$$\mathrm{Im}(H_n(g)) = \{g_n(y) + \mathrm{Im}(\zeta_{n+1}) \mid y \in \ker(\epsilon_n)\}$$

For $\ker(d_n)$ we need to determine all $z \in \ker(\zeta_n)$ such that the element $x \in \ker(\delta_{n-1})$ constructed above lies actually in $\mathrm{Im}(\delta_n)$, so that x represents 0 in $H_{n-1}(X)$. Write as before $z = g_n(y)$ for some $y \in Y_n$ and $\epsilon_n(y) = f_{n-1}(x)$ for some $x \in \ker(\delta_{n-1})$, so that $x + \mathrm{Im}(\delta_n) = d_n(z + \mathrm{Im}(\epsilon_{n+1})$ as described above. Suppose that $z + \mathrm{Im}(\epsilon_{n+1}) \in \mathrm{Im}(H_n(g))$. That is, we have $z + \mathrm{Im}(\epsilon_{n+1}) = g_n(y) + \mathrm{Im}(\epsilon_{n+1})$ for some $y \in \ker(\epsilon_n)$. Then $0 = \epsilon_n(y) = f_{n-1}(x)$, which shows that $x = 0$ as f_{n-1} is injective. This shows the inclusion $\mathrm{Im}(H_n(g)) \subseteq \ker(d_n)$. Suppose conversely that $z + \mathrm{Im}(\zeta_{n+1}) \in \ker(d_n)$. This is equivalent to $x \in \mathrm{Im}(\delta_n)$. Write $x = \delta_n(x')$ for some $x' \in X_n$. Set $y' = f_n(x')$. Then

$$\epsilon(y') = \epsilon(f_n(x')) = f_{n-1}(\delta_n(x')) = f_{n-1}(x)$$

and

$$g_n(y - y') = g_n(y) - g_n(f_n(x')) = g_n(y) = z$$

The above implies

$$\epsilon_n(y - y') = \epsilon_n(y) - \epsilon_n(y') = f_{n-1}(x) - f_{n-1}(x) = 0$$

and thus $z = g_n(y - y')$ and $y - y' \in \ker(\epsilon_n)$ which means exactly that the class of z is in the image of $H_n(g)$.

The last verification for the exactness is $\mathrm{Im}(d_n) = \ker(H_{n-1}(f))$. By the construction of d_n, $\mathrm{Im}(d_n)$ consists of all classes $x + \mathrm{Im}(\delta_n)$ such that $f_{n-1}(x) = \epsilon_{(}(y)$ for some $y \in Y_n$ satisfying $g_n(y) = z \in \ker(\zeta_n)$. We have

$$\ker(H_{n-1}(f)) = \{x + \mathrm{Im}(\delta_n) \mid x \in \ker(\delta_n),\ f_{n-1}(x) \in \mathrm{Im}(\epsilon_n)\}$$

The inclusion $Im(d_n) \subseteq \ker(H_{n-1}(f))$ is clear from this description. For the converse, suppose that $f_{n-1}(x) = \epsilon_n(y)$ for some $y \in Y_n$. Consider $z = g_n(y)$. We have

$$\zeta_n(z) = \zeta_n(g_n(y)) = g_{n-1}(\epsilon_n(y)) = g_{n-1}(f_{n-1}(x)) = 0$$

and hence $z \in \ker(\zeta_n)$, which shows the equality as required.

This concludes the proof of the exactness statement. It remains to verify the naturality of the connecting homomorphisms. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain maps such that the rows are exact. We need to show that the ladder above is commutative. The squares which involve only H_n are commutative because H_n is a functor; that is, we have a commutative diagram

$$\begin{array}{ccccc} H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) & \xrightarrow{H_n(g)} & H_n(Z) \\ \downarrow{\scriptstyle H_n(a)} & & \downarrow{\scriptstyle H_n(b)} & & \downarrow{\scriptstyle H_n(c)} \\ H_n(X') & \xrightarrow[H_n(f')]{} & H_n(Y') & \xrightarrow[H_n(g')]{} & H_n(Z') \end{array}$$

It remains to show the commutativity of the diagram

$$\begin{array}{ccc} H_n(Z) & \xrightarrow{d_n} & H_{n-1}(X) \\ \downarrow{\scriptstyle H_n(c)} & & \downarrow{\scriptstyle H_{n-1}(a)} \\ H_n(Z') & \xrightarrow[d'_n]{} & H_{n-1}(X') \end{array}$$

Let x, y, z are as above in the construction of d_n; that is, $z \in \ker(\zeta_n)$, $z = g_n(y)$, and $f_{n-1}(x) = \epsilon_n(y)$. We need to show that the images $a_{n-1}(x)$, $b_n(y)$, $c_n(z)$ of x, y, z are the corresponding elements required in the construction of d'_n evaluated at the class of $c_n(z)$. That is, we need to verify that

$$g'_n(b_n(y)) = c_n(z)$$

and that

$$f'_{n-1}(a_{n-1}(x)) = \epsilon'_n(b_n(y))$$

The first equation holds because $g'_n(b_n(y)) = c_n(g_n(y)) = c_n(x)$. The second equation holds because $f'_{n-1}(a_{n-1}(x)) = b_{n-1}(f_{n-1}(x)) = b_{n-1}(\epsilon_n(y)) = \epsilon'_n(b_n(y))$. This shows the commutativity in the ladder above for the square involving d_n, completing the proof. □

Exercise 2.9. State the cohomology version of Theorem 2.8.

Corollary 2.10. *Let A be a k-algebra and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of chain complexes of A-modules.

(i) f is a quasi-isomorphism if and only if Z is acyclic.

(ii) g is a quasi-isomorphism if and only if X is acyclic.

(iii) If two of the complexes X, Y, Z are acyclic, so is the third.

Proof. The long exact homology sequence shows that if $H_{n+1}(Z) = H_n(Z) = \{0\}$, then $H_n(f)$ is an isomorphism, and if $H_n(f)$, $H_{n-1}(f)$ are isomorphisms, then the maps $H_n(g)$, d_n are zero, hence $H_n(Z) = \{0\}$. This shows (i), and the rest follows similarly. □

The following observation is used to compare the long exact homology sequences via a commutative ladder as above:

Proposition 2.11 (The 5-Lemma)**.** *Let A be a k-algebra and let*

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & \downarrow{\scriptstyle a_4} & & \downarrow{\scriptstyle a_5} \\ Y_1 & \xrightarrow[g_1]{} & Y_2 & \xrightarrow[g_2]{} & Y_3 & \xrightarrow[g_3]{} & Y_4 & \xrightarrow[g_4]{} & Y_5 \end{array}$$

be a commutative diagram of A-modules with exact rows. If a_1, a_2, a_4, a_5 are isomorphisms then a_3 is an isomorphism.

Proof. Let $x \in \ker(a_3)$. Then $a_4(f_3(x)) = g_3(a_3(x)) = 0$, hence $f_3(x) = 0$ as a_4 is an isomorphism. Thus $x \in \ker(f_3) = \mathrm{Im}(f_2)$, and so there is $y \in X_2$ such that $f_2(y) = x$. Then $g_2(a_2(y)) = a_3(f_2(y)) = a_3(x) = 0$, hence $a_2(y) \in \ker(g_2) = \mathrm{Im}(g_1)$, and so there is $z \in Y_1$ satisfying $g_1(z) = a_2(y)$. As a_1 is an isomorphism, there is $w \in X_1$ such that $a_1(w) = z$. Then $a_2(f_1(w)) = g_1(a_1(w)) = g_1(z) = a_2(y)$. Since a_2 is an isomorphism this implies that $f_1(w) = y$. But then $x = f_2(y) = f_2(f_1(w)) = 0$, and so a_3 is injective. For the surjectivity of a_3, let $y \in Y_3$. Then $g_3(y) \in Y_4$. Since a_4 is an isomorphism, there is $v \in X_4$ such that $a_4(v) = g_3(y)$. Then $a_5(f_4(v)) = g_4(a_4(v)) = g_4(g_3(y)) = 0$. Thus $f_4(v) = 0$ as a_4 is an isomorphism. It follows that $v \in \ker(f_4) = \mathrm{Im}(f_3)$. Write $v = f_3(u)$ for some $u \in X_3$. Then $g_3(a_3(u) - y) = g_3(a_3(u)) - g_3(y) = a_4(f_3(u)) - g_3(y) = a_4(v) - g_3(y) = 0$. Thus $a_3(u) - y \in \ker(g_3) = \mathrm{Im}(g_2)$. Write $a_3(u) - y = g_2(w)$ for some $w \in y_2$. Since a_2 is an isomorphism there is $r \in X_2$ such that $a_2(r) = w$. Then $a_3(v) - y = g_2(w) = g_2(a_2(r)) = a_3(f_2(r))$. This shows that $y = a_3(v - f_2(r))$, and hence that a_3 is surjective. □

Corollary 2.12. *Let A be an algebra over a commutative ring k and let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain complexes of A-modules with exact rows. If two of a, b, c are quasi-isomorphisms, so is the third.

Proof. Apply the 5-Lemma to the five terms in the commutative ladder following Theorem 2.8. □

Exercise 2.13. Describe the connecting homomorphism in the long exact cohomology sequence associated with a short exact sequence of cochain complexes.

3 Homotopy

Definition 3.1. Let $\mathcal{C}$ be an additive category and let $(X\delta)$, (Y, ϵ) be complexes over $\mathcal{C}$. A *(chain) homotopy from X to Y* is a graded morphism $h : X \to Y$ of degree 1; that is, h is a family of morphisms $h_n : X_n \to Y_{n+1}$ in $\mathcal{C}$, for any $n \in \mathbb{Z}$. Two chain morphisms $f, f' : X \to Y$ are called *homotopic* , written $f \sim f'$, if there is a homotopy $h : X \to Y$ such that

$$f - f' = h \circ \delta + \epsilon \circ h \ ,$$

or equivalently, if

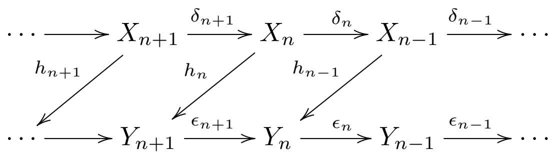
$$f_n - f'_n = h_{n-1} \circ \delta_n + \epsilon_{n+1} \circ h_n$$

for any $n \in \mathbb{Z}$.

Note that there is no requirement for a homotopy to be compatible with the differentials.

Exercise 3.2. Let (X, δ), (Y, ϵ) be chain complexes over an additive category $\mathcal{C}$. Show that if $h : X \to Y$ is a homotopy, then the graded map $f = h \circ \delta + \epsilon \circ h$ is a chain map. Deduce that the chain maps from X to Y which are homotopic to the zero chain map are exactly all chain maps of the form $h \circ \delta + \epsilon \circ h$ with h running over all homotopies from X to Y.

For cochain complexes, we define analogously a *cochain homotopy* to be a graded morphism of degree -1 (so the degree of a homotopy is always opposite to that of the differential). One can visualise a homotopy between chain complexes by a diagram of the form



Definition 3.3. Let $\mathcal{C}$ be an additive category and let (X, δ), (Y, ϵ) be complexes over $\mathcal{C}$. A chain map $f : X \to Y$ is a *homotopy equivalence* if there is a chain map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$; in that case, g is called a *homotopy inverse of* f, and the complexes X, Y are said to be *homotopy equivalent*, written $X \simeq Y$.

Proposition 3.4. *For a chain complex X over an additive category $\mathcal{C}$ we have $X \simeq 0$ (the zero complex) if and only if $\mathrm{Id}_X \sim 0$ (the zero chain map on X).*

Proof. We have $X \simeq 0$ if and only if there are maps $g : 0 \to X$ and $f : X \to 0$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_0$. Since the zero chain maps are the only chain maps between X and the zero complex 0 in either direction, this holds if and only if the zero map 0 on X is homotopic to Id_X. $\square$

Definition 3.5. Let $\mathcal{C}$ be an additive category and let $(X\delta)$ be a complex over $\mathcal{C}$. If $X \simeq 0$ (the zero complex), then X is called *contractible.*

The terminology comes from topology (complexes calculating the singular homology of a contractible topological space are contractible chain complexes).

Proposition 3.6. *Let X, Y be chain complexes over an additive category $\mathcal{C}$. The relation $\sim$ on the set of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, Y)$ of chain maps from X to Y is an equivalence relation, compatible with sums and compositions of chain maps.*

Proof. Denote by δ, ϵ the differentials of X, Y. Let f, f', f'' : $X \to Y$ be chain maps. The relation $\sim$ is reflexive: we have $f \sim f$, using the zero homotopy. The relation $\sim$ is symmetric: if $f \sim f'$ then $f' \sim f$; indeed, if h is a homotopy satisfying $f - f' = h \circ \delta + \epsilon \circ h$, then $f' - f = (-h) \circ \delta + \epsilon \circ (-h)$. Finally, the relation $\sim$ is transitive: if $f - f' = h \circ \delta + \epsilon \circ h$ for some homotopy h and $f' - f'' = k \circ \delta + \epsilon \circ k$ for some homotopy k, then $h + k$ is a homotopy from X to Y satisfying $f - f'' = f - f' + f' - f'' = (h + k) \circ \delta + \epsilon \circ (h + k)$. If g, g' : $X \to Y$ are chain maps and if $f \sim f'$ and $g \sim g'$, then $f + g \sim f' + g'$; this follows from taking sums of homotopies. If (Z, ζ) is a third chain complex and $g, g : Y \to Z$ are chain maps such that $f \sim f'$ and $g \sim g'$ via homotopies $h : X \to Y$ and $h' : Y \to Z$, then a short calculation shows that $g \circ f \sim g' \circ f'$ via the homotopy $g \circ h + h' \circ f'$. □

Exercise 3.7. Show that a direct summand (in the category of chain complexes over some additive category) of a contractibe complex is contractible.

Proposition 3.8. *Let A be an algebra over a commutative ring k and let f, f' : $(X, \delta) \to (Y, \epsilon)$ be chain maps of complexes of A-modules.*

(i) For any homotopy $h : X \to Y$, the graded morphism $h \circ \delta + \epsilon \circ h : X \to Y$ is a chain map inducing the zero morphism from $H_*(X)$ to $H_*(Y)$.

(ii) If $f \sim f'$ then $H(f) = H(f') : H_*(X) \to H_*(Y)$.

(iii) If f is a homotopy equivalence, then f is a quasi-isomorphism.

(iv) If $X \simeq 0$ then X is acyclic.

Proof. The first part of the first statement was noted in an exercise above. We have

$$\epsilon \circ (h \circ \delta + \epsilon \circ h) = \epsilon \circ h \circ \delta = (h \circ \delta + \epsilon \circ h) \circ \delta \ ,$$

hence $h \circ \delta + \epsilon \circ h$ is a chain map from X to Y. Moreover, the induced map

$$\ker(\delta) \to \ker(\epsilon)$$

by $h \circ \delta + \epsilon \circ h$ is equal to the map induced by $\epsilon \circ h$ and hence has image contained in $\mathrm{Im}(\epsilon) \subset \ker(\epsilon)$, which shows that it induces the zero map on homology. This proves (i). If $f \sim f'$ then by (i), the difference $f - f'$ induces the zero map on homology and thus $H(f) = H(f')$, which proves (ii). Suppose f has a homotopy inverse g. Then, by (ii), we have $\mathrm{Id}_{H_*(X)} = H(g \circ f) = H(g) \circ H(f)$, thus $H(g)$ and $H(f)$ are inverse, proving (iii). If $X \simeq 0$, then X is quasi-isomorphic to zero by (iii), which is equivalent to $H_*(X) = 0$, whence (iv). □

Remark 3.9. Let A, B be two algebras over a commutative ring k and let $\mathcal{F}$: $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$ a k-linear (not necessarily exact) functor. Since $\mathcal{F}$ sends A-modules to B-modules and A-homomorphisms to B-homomorphisms, it extends to a functor, denoted by the same letter, $\mathcal{F}$: $\mathrm{Ch}(\mathrm{Mod}(A)) \to \mathrm{Ch}(\mathrm{Mod}(B))$. This functor need not send a quasi-isomorphism to a quasi-isomorphism. But it sends a homotopy $h : X \to Y$ between chain complexes of A-modules X and Y to a homotopy $\mathcal{F}(h) : \mathcal{F}(X) \to \mathcal{F}(Y)$ between the chain complexes of B-modules $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. Thus homotopic chain maps from $X \to Y$ are sent to homotopic chain maps from $\mathcal{F}(X) \to \mathcal{F}(Y)$. In particular, $\mathcal{F}$ sends a homotopy equivalence $X \simeq Y$ to a homotopy equivalence $\mathcal{F}(X) \simeq \mathcal{F}(Y)$, and hence $\mathcal{F}$ sends contractible chain complexes of A-modules to contractible chain complexes of B-modules.

Exercise 3.10. Show that a chain complex over some additive category of the form

$$\cdots \longrightarrow 0 \longrightarrow U \xrightarrow{\mathrm{Id}_U} U \longrightarrow 0 \longrightarrow \cdots$$

is contractible, where the two terms equal to the same object U are in two arbitrary consecutive degrees. (One can show that any contractible chain complex is a direct sum of complexes of this form.)

Exercise 3.11. Show that a short exact sequence of A-modules is split if and only if it is contractible when regarded as a chain complex.

Definition 3.12. Let A be an algebra over a commutative ring k. The *homotopy category of complexes over* $\mathrm{Mod}(A)$ is the category $K(\mathrm{Mod}(A))$ whose objects are the complexes over $\mathrm{Mod}(A)$ and and whose morphisms are the homotopy equivalence classes

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)/\sim$$

of chain maps, for any two complexes X, Y over $\mathcal{C}$. The composition of morphisms in $K(\mathrm{Mod}(A))$ is induced by that in $\mathrm{Ch}(\mathrm{Mod}(A))$. We denote by $K^+(\mathrm{Mod}(A))$, $K^-(\mathrm{Mod}(A))$, $K^b(\mathrm{Mod}(A))$ the full subcategories of $K(\mathrm{Mod}(A))$ consisting of left bounded, right bounded, bounded complexes of A-modules, respectively.

Slightly more explicitly, the composition in $K(\mathrm{Mod}(A))$ is defined as follows. If $f : X \to Y$ and $g : Y \to Z$ are chain maps, and if we denote by $[f]$ the class of all chain maps homotopic to f, then $[f] : X \to Y$ is a morphism in the category $K(\mathrm{Mod}(A))$, and we have $[g] \circ [f] = [g \circ f]$. For this to be well-defined we need the observation from 3.6 that $\sim$ is compatible with the composition of chain maps. If f, $f' : X \to Y$ are chain maps, then the equality $[f] = [f']$ is equivalent to $f - f' \sim 0$. Thus if we denote by $\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)$ the k-submodule of all chain maps $f : X \to Y$ satisfying $f \sim 0$, or equivalently, $[f] = [0]$, then $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y)$ is the quotient space

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)/\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)\ .$$

Remark 3.13. Let A be an algebra over a commutative ring. The categories $K(\mathrm{Mod}(A))$ and $\mathrm{Ch}(\mathrm{Mod}(A))$ have the same objects, but morphisms in $K(\mathrm{Mod}(A))$ are classes of morphisms in $\mathrm{Ch}(\mathrm{Mod}(A))$. Thus we have a functor

$$\mathrm{Ch}(\mathrm{Mod}(A)) \to K(\mathrm{Mod}(A))$$

which is sends any chain complex X to itself and any chain map $f : X \to Y$ to the homotopy class $[f]$ of f. Taking homology yields a functor $H_* : \mathrm{Ch}(\mathrm{Mod}(A)) \to \mathrm{Gr}(\mathrm{Mod}(A))$. It follows from 3.8 that for f a chain map, the induced map $H(f)$ on homology depends only on the homotopy class $[f]$ of f, and therefore this functor factors through the canonical functor $\mathrm{Ch}(\mathrm{Mod}(A)) \to K(\mathrm{Mod}(A))$; that is, we have a commutative diagram of canonical functors

$$\begin{array}{ccc} \mathrm{Ch}(\mathrm{Mod}(A)) & \longrightarrow & K(\mathrm{Mod}(A)) \\ & {}_{H_*}\searrow \quad \swarrow & \\ & \mathrm{Gr}(\mathrm{Mod}(A)) & \end{array}$$

Remark 3.14. Let A be an algebra over a commutative ring. If two chain maps f, $f' : X \to Y$ of complexes of A-modules are homotopy equivalent via a homotopy $h : X \to Y$, then for any integer i, the "shifted" chain maps $f[i], f'[i] : X[i] \to Y[i]$ are homotopic via the homotopy $h[i]$ given by $h[i]_n = h_{n-i}$ for any $n \in \mathbb{Z}$. In other words, the shift automorphism $[i]$ of $\mathrm{Ch}(\mathrm{Mod}(A))$ induces an automorphism, still denoted by $[i]$, of the homotopy category $K(\mathrm{Mod}(A)))$. This automorphism preserves any of the subcategories $K^+(\mathrm{Mod}(A))$, $K^-(\mathrm{Mod}(A))$, $K^b(\mathrm{Mod}(A))$.

Remark 3.15. Let A be an algebra over a commutative ring. Two homotopic chain maps f, $f' : X \to Y$ between complexes of A-modules may have different kernels and cokernels. Therefore, in the category $K(\mathrm{Mod}(A))$ there is no well-defined notion of kernel and cokernel of a morphism. In particular, the category $K(\mathrm{Mod}(A))$ is additive but not abelian - there is no notion of exactness. The search for a replacement of short exact sequences is what led to the concept of a *triangulated category.*

The following theorem shows that although a bounded below complex of projective A-modules is not a projective object in the category of chain compelxes, it does have a lifting property with respect to quasi-isomorphisms. Similarly, bounded above complexes of injective A-modules have the extension property with respect to quasi-isomorphisms.

Theorem 3.16. *Let P be a complex of projective A-modules, I a complex of injective objects A-modules, and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of complexes of A-modules.
(i) Suppose that X is acyclic and that one of P, Y is bounded below. The map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Y) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Z)$$

given by composition with g is surjective and induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Z)\ .$$

(ii) Suppose that Z is acyclic and that one of Y, I is bounded above. The map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(Y, I) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, I)$$

given by precomposition with f induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I)\ .$$

Proof. (i) Denote by δ, ϵ, ζ, π the differentials of X, Y, Z, P, respectively. We first show the surjectivity of the map

$$\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Y) \to \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P, Z)$$

sending a chain map $p : P \to Y$ to the chain map $g \circ p : X \to Z$. Let $q : P \to Z$ be a chain map. We need to construct a chain map $p : P \to Y$ satisfying $g \circ p = q$. We construct such a chain map p inductively, by induction over the degree. In order to start an inductive argument, we will need the hypothesis that one of P or Y is bounded below. (Note that if Y is bounded below, then so is Z by the surjectivity of g.) This hypothesis ensures that we have $q_i = 0$ for all sufficiently small integers i, so we may simply take $p_i = 0$ for i sufficiently small. Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \to Y_i$ satisfying

$$g_i \circ p_i = q_i$$

$$\epsilon_i \circ p_i = p_{i-1} \circ \pi_i$$

for $i < n$. We construct p_n as follows. Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \to Y_n$ such that $g_n \circ p'_n = q_n$. That is, p'_n satisfies the first of the two conditions above, but we may have to adjust p'_n to make sure, that it is compatible with the differentials as in the second condition. We have

$$g_{n-1} \circ (\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Thus we have

$$\mathrm{Im}(\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subset \ker(g_{n-1}) = \mathrm{Im}(f_{n-1})$$

Consequently, since f_{n-1} is a monomorphism, there is a morphism $\sigma : P_n \to X_{n-1}$ such that

$$f_{n-1} \circ \sigma = \epsilon_n \circ p'_n - p_{n-1} \circ \pi_n$$

Moreover, we have

$$f_{n-2} \circ \delta_{n-1} \circ \sigma = \epsilon_{n-1} \circ f_{n-1} \circ \sigma = \epsilon_{n-1} \circ \epsilon_n \circ p'_n - \epsilon_{n-1} \circ p_{n-1} \circ \pi_n = -p_{n-2} \circ \pi_{n-1} \circ \pi_n = 0$$

and hence

$$\delta_{n-1} \circ \sigma = 0$$

as f_{n-2} is a monomorphism. Therefore we have

$$\mathrm{Im}(\sigma) \subset \ker(\delta_{n-1}) = \mathrm{Im}(\delta_n) \ ,$$

where the last equality holds as X is acylic. Since P_n is projective, the morphism $\sigma : P_n \to \mathrm{Im}(\delta_n)$ lifts to a morphism $\rho : P_n \to X_n$; that is, $\sigma = \delta_n \circ \rho$. Set $p_n = p'_n - f_n \circ \rho$. We still have

$$g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n \ ,$$

and we now also have the compatibility with the differentials

$$\epsilon_n \circ p_n = \epsilon_n \circ p'_n - \epsilon_n \circ f_n \rho = \epsilon_n \circ p'_n - f_{n-1} \circ \delta_n \circ \rho = \epsilon_n \circ p'_n - f_{n-1} \circ \sigma = \epsilon_n \circ p'_n - (\epsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = p_{n-1} \circ \pi_n$$

as required. This shows the surjectivity of the map given by composition with g.

We need to show that $p \sim 0$ if and only if $q \sim 0$. If $p \sim 0$ there is a homotopy $h : P \to Y$ such that

$$p = \epsilon \circ h + h \circ \pi \ .$$

Composing with g yields

$$q = g \circ p = g \circ \epsilon \circ h + g \circ h \circ \pi = \delta \circ g \circ h + g \circ h \circ \pi \ ,$$

where we have used that g is a chain map. Thus $q \sim 0$ via the homotopy $g \circ h : P \to X$. Conversely, suppose that $q \sim 0$. We need to show that then $p \sim 0$. The first part of the argument shows that we may assume that $q = 0$. To see this, observe first that since g_{n+1} is an epimorphism, any morphism $P_n \to X_{n+1}$ lifts to a morphism $P_n \to Y_{n+1}$, and thus every homotopy $P \to X$ lifts to some homotopy $P \to Y$. This means that if $q \sim 0$, there is some chain map $p' : P \to Y$ such that $p' \sim 0$ and $g \circ p' = q$, but p' need not be equal to p. It suffices to show that $p - p' \sim 0$. Since $g \circ (p - p') = 0$, we may therefore assume that $q = 0$. Then $g \circ p = q = 0$, hence

$$\mathrm{Im}(p) \subset \ker(g) = \mathrm{Im}(f) \ .$$

This implies that there is a chain map $u : P \to X$ such that $f \circ u = p$. It suffices to show that $u \sim 0$. This is again done inductively. Given an integer n, suppose that we have morphisms $h_i : P_i \to X_{i+1}$ satisfying $u_i = \delta_{i+1} \circ h_i + h_{i-1} \circ \pi_i$ for any $i < n$. Using this equality for $i = n - 1$ we get

$$\delta_n \circ (u_n - h_{n-1} \circ \pi_n) = \delta_n \circ u_n - \delta_n \circ h_{n-1} \circ \pi_n = \delta_n \circ u_n - (u_{n-1} - h_{n-2} \circ \pi_{n-1}) \circ \pi_n = \delta_n \circ u_n - u_{n-1} \circ \pi_n = 0 \ .$$

as u is a chain map. Thus

$$\mathrm{Im}(u_n - h_{n-1} \circ \pi_n) \subset \ker(\delta_n) = \mathrm{Im}(\delta_{n+1}) \ .$$

As P_n is projective, there is $h_n : P_n \to X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

as required. This completes the proof of (i). The proof of (ii) is obtained by dualising the arguments (reversing all arrows and exchanging monomorphisms and epimorphisms). □

The above theorem holds verbatim for arbitrary abelian categories instead of module categories; the only adjustment in the proof is that inclusion maps need to replaced by canonical monomorphisms.

Remark 3.17. The condition that X is acyclic is equivalent to g being a quasi-ismorphism. In addition, in the statement of 3.16, the chain map g is surjective in each degree. One can show that for the second isomorphism in (i), the surjectivity of g is not necessary. That is, one can show that if $g : Y \to Z$ is any quasi-isomorphism, then composition with g induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Z) \ .$$

Similary, for any quasi-isomorphism $f : X \to Y$, precomposition with f induces an isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I) \ .$$

In both cases, this is played back to Theorem 3.16 by adding a contractible complex to Y which maps onto Z or which admits a degreewise injective map $X \to Y$, so that g (resp. f) can be assumed to be surjective (resp. injective) in each degree. One way to achieve this uses the *cone* of a chain complex X; this is a contractible complex which admits a degreewise injective chain map $X \to C(X)$ and a degreewise surjective chain map $C(X) \to X[1]$. See the exercises at the end of this section. We will come back to this in the last section, where we show that homotopy categories are triangulated.

Corollary 3.18. *Let P be a complex of projective A-modules, let I be a complex of injective A-modules, and let X be an acyclic complex of A-modules.*

(i) If one of X, P is bounded below, then $\mathrm{Hom}_{K(\mathcal{C})}(P, X) = \{0\}$.

(ii) If one of X, I is bounded above, then $\mathrm{Hom}_{K(\mathcal{C})}(X, I) = \{0\}$.

Corollary 3.19. *Let X be a complex over A-modules.*

(i) X is acyclic if and only if $\mathrm{Hom}_{K(\mathcal{C})}(P, X) = \{0\}$ *for any bounded below complex P of projective A-modules.*

(ii) X is acyclic if and only if $\mathrm{Hom}_{K(\mathcal{C})}(X, I) = \{0\}$ *for any bounded above complex I of injective A-modules.*

Proof. (i) If X is acyclic then $\mathrm{Hom}_{K(\mathcal{C})}(P, X) = \{0\}$ for any bounded below complex P of projective A-modules by 3.18. If X is not acyclic, there is an integer n such that $H_n(X)$ is not zero, or equivalently, such that the canonical monomorphism $\mathrm{Im}(\delta_{n+1}) \subset \ker(\delta_n)$ is not an isomorphism, where δ is the differential of X. Let P be the complex which is zero in any degree other than n and which is a projective A-module such that there is an epimorphism $\pi : P_n \to \ker(\delta_n)$. Then π defines a chain map from P to X which cannot be homotopic to zero, because π does not factor through δ_{n+1}. This shows (i). By dualising the above proof, one shows (ii). □

The following two observations describe the homology and cohomology of complexes in terms of homotopy classes of chain maps. Both of these observations are very easy - but they have an important consequence: since chain maps can be composed, the interpretation of (co-)homology in terms of chain maps introduces extra structure on (co-)homology. We will see later that the graded algebra structure of Ext-algebras is induced in this way.

Proposition 3.20. *Let X be a chain complex of A-modules and let n be an integer. There is a natural isomorphism*

$$H_n(X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X) \ ,$$

where $A[n]$ is the complex equal to A in degree n and zero in all other degrees.

Proof. A chain map $A[n] \to X$ is represented by a commutative diagram of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\
\cdots & \longrightarrow & X_{n+1} & \xrightarrow[\delta_{n+1}]{} & X_n & \xrightarrow[\delta_n]{} & X_{n-1} & \longrightarrow & \cdots
\end{array}$$

for some A-homomorphism $f : A \to X_n$ satisfying $\delta_n \circ f = 0$, where δ is the differential of X. Thus any such A-homomorphism f maps A to $\ker(\delta_n)$, and and any such A-homomorphism is uniquely determined by the the image of 1_A in $\ker(\delta_n)$. A homotopy from $A[n]$ to X is zero except in degree n, where it is a map $A \to X_{n+1}$. The chain map determined by the homomorphism f is homotopic to zero if and only if f factors through δ_{n+1}. A necessary condition for that to happen is that $\mathrm{Im}(f) \subseteq \mathrm{Im}(\delta_{n+1})$. This condition is also sufficient because A is projective as an A-module, so every A-homomorphism $A \to \mathrm{Im}(\delta_{n+1})$ lifts through the surjective map $X_{n+1} \to \mathrm{Im}(\delta_{n+1})$. Thus the map sending $x \in \ker(\delta_n)$ to the unique chain map $A[n] \to X$ sending 1_A to x induces an isomorphism as stated. The naturality statements just means that this defines an isomorphism of functors $H_n(-) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], -)$ from $\mathrm{Ch}(\mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$; this is an easy verification. The result follows. $\square$

For (X, δ) a chain complex of A-modules and V an A-module, applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to (X, δ), written as a chain of morphisms,

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \longrightarrow \cdots$$

yields a cochain complex $\mathrm{Hom}_A(X, V)$ of k-modules of the form

$$\cdots \longrightarrow \mathrm{Hom}_A(X_{n-1}, V) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_A(X_n, V) \xrightarrow{\delta^n} \mathrm{Hom}_A(X_{n+1}, V) \longrightarrow \cdots \ ;$$

that is, the cochain complex $\mathrm{Hom}_A(X, V)$ is defined by

$$\mathrm{Hom}_A(X, V)^n = \mathrm{Hom}_A(X_n, V)$$

with differential

$$\delta^n : \mathrm{Hom}_A(X_n, V) \to \mathrm{Hom}_A(X_{n+1}, V)$$

given by $\delta^n(\alpha) = \alpha \circ \delta_{n+1}$ for any $\alpha \in \mathrm{Hom}_A(X_n, V)$. The cohomology of this cochain complex is as follows.

Proposition 3.21. *Let X be a chain complex of A-modules, and let V be an A-module. For any integer n we have a natural isomorphism of k-modules*

$$H^n(\mathrm{Hom}_A(X, V)) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, V[n]) \ .$$

Proof. A chain map from X to $V[n]$ is a commutative diagram of the form

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & V & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

This is a chain map if and only if $\alpha \circ \delta_{n+1} = 0$, thus if and only of $\delta^n(\alpha) = 0$. This shows that

$$\ker(\delta^n) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, V[n]) \ .$$

Any homotopy between the above complexes is zero except possibly in degree $n-1$, where it is homomorphism $h : X_{n-1} \to V$. Thus α determines a chain map which is homotopic to zero if and only if $\alpha = h \circ \delta_n$ for some $h : X_{n-1} \to V$; that is, if and only if $\alpha \in \mathrm{Im}(\delta^{n-1})$ The naturality statements is easily verified; this means that the contravariant functors $H^n(\mathrm{Hom}_A(-, V))$ and $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(-, V[n])$ from $\mathrm{Ch}(\mathrm{Mod}(A))$ to $\mathrm{Mod}(k)$ are isomorphic. $\square$

The right side in the isomorphism in this proposition depends only on the homotopy category $K(\mathrm{Mod}(A))$, so any isomorphism in this category preserves the left side as well:

Corollary 3.22. *Let $f : X \to Y$ be a chain homotopy equivalence of chain complexes of A-modules, let n be an integer and V an A-module. Then f induces an isomorphism*

$$H^n(\mathrm{Hom}_A(Y,V)) \cong H^n(\mathrm{Hom}_A(X,V)) \ .$$

Exercise 3.23. Let (X,δ) be a chain complex of A-modules. Show the following statements.
(1) There is a chain complex, denoted $C(X)$ and called the *cone of* X, with the following properties: for any integer n, the term in degree n of $C(X)$ is given by

$$C(X)_n = X_{n-1} \oplus X_n$$

and the differential of $C(X)$ in degree n is given by

$$\Delta_n = \begin{pmatrix} -\delta_{n-1} & 0 \\ \mathrm{Id}_{X_{n-1}} & \delta_n \end{pmatrix} : X_{n-1} \oplus X_n \to X_{n-2} \oplus X_{n-1} \ ;$$

that is, $\delta_n(x,y) = (-\delta_{n-1}(x), x + \delta_n(y))$ for any $x \in X_{n-1}$ and any $y \in X_n$.
(2) The cone $C(X)$ is a contractible chain complex. (Hint: consider a homotopy which identifies the summand X_n of $C(X)_n$ with the summand X_n of $C(X)_{n+1}$.)
(3) The canonical inclusions $X_n \to X_{n-1} \oplus X_n$ define a chain map $i_X : X \to C(X)$, and the canonical projections $X_{n-1} \oplus X_n \to X_{n-1}$ define a chain map $C(X) \to X[1]$ such that the sequence of chain maps

$$0 \longrightarrow X \xrightarrow{i_X} C(X) \xrightarrow{p_X} X[1] \longrightarrow 0$$

is exact.
(4) Use the previous exercise and Theorem 3.16 to prove the statements in Remark 3.17.
(5) The forgetful functor $\mathrm{Ch}(\mathrm{Mod}(A)) \to \mathrm{Gr}(\mathrm{Mod}(A))$ sends the exact sequence in **(3)** of chain complexes to a split exact sequence of graded A-modules.
(6) Give an example where the exact sequence in **(3)** does not split (as a sequence of chain complexes).
(7) The exact sequence in Exercise **(3)** is split, as a sequence of chain complexes, if and only if X is contractible.

Remark 3.24. The sign of $-\delta_{n-1}$ in the definition of the differential of $C(X)$ ensures that Δ is indeed a differential; that is, $\Delta_{n-1} \circ \Delta_n = 0$. Thus, for p_X to be a chain map, we need an earlier sign convention by which the differential of the complex $X[1]$ is the negative of the shifted differential of X.

Exercise 3.25. Use the previous exercise to show that a chain complex of A-modules is projective as an object of the category of chain complexes $\mathrm{Ch}(\mathrm{Mod}(A))$ if and only if it is a contractible complex of projective A-modules.

4 Ext and Tor

Let A be an algebra over a commutative ring k. Informally, a bounded below chain complex of A-modules of the form

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi} U \longrightarrow 0$$

is called a *projective resolution of* U if it is exact and all P_i are projective. An exact complex as above can be viewed as a chain map obtained from 'bending down' the map π and viewing U as a chain complex concentrated in degree zero:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{\delta_2} & P_1 & \xrightarrow{\delta_1} & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & U & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

The exactness of the above sequence is equivalent to this chain map being a quasi-isomorphism because the homology of both rows is concentrated in degree 0, where it is isomorphic to $P_0/\mathrm{Im}(\delta_0) = P_0/\ker(\pi) \cong U$. For convenience, we will denote the A-homomorphism $P_0 \to U$ and the induced chain map $P \to U$ by the same letter, if no confusion arises. The formal definition of a projective resolution is as follows.

Definition 4.1. A *projective resolution of an A-module U* is a pair (P, μ) consisting of a complex P of projective A-modules such that $P_i = 0$ for $i < 0$ and a quasi-isomorphism $\mu : P \to U$.

If μ is clear from the context or not needed in a particular statement, we suppress it and simply say 'Let P be a projective resolution of U..', implicitly assuming that there is such a quasi-isomorphism μ. Similarly, the informal version of an *injective resolution of* U is an exact bounded below cochain complex of the form

$$0 \longrightarrow U \xrightarrow{\iota} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} I^2 \xrightarrow{\delta^2} \cdots$$

where the modules I^i are injective. As before, we view the A-homomorphism ι as a quasi-isomorphism of cochain complexes, again denoted by the same letter whenever convenient,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & U & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & {\scriptstyle \iota}\downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{\delta^0} & I^1 & \xrightarrow{\delta^1} & \cdots \end{array}$$

and this is the formal definition of an injective resolution.

Definition 4.2. An *injective resolution of an A-module U* is a pair (I, ι) consisting of a cochain complex I of injective A-modules such that $I^i = 0$ for $i < 0$ and a quasi-isomorphism $\iota : U \to I$.

Every A-module has a projective resolution P; in fact, the resolution can be taken to be *free*; that is, all P_i are free A-modules. This follows from the fact that every A-module is a quotient of a free A-module. Thus there is a free A-module P_0 such there exists a surjective A-homomorphism $\mu : P_0 \to U$. Applied to $\ker(\mu)$, there exists a free A-module P_1 and a surjective A-homomorphism $\delta_1 : P_1 \to \ker(\mu)$. One constructs P inductively by taking for P_n a free A-module and for δ_n a surjective A-homomorphism $P_n \to \ker(\delta_{n-1})$ composed with the inclusion $\ker(\delta_{n-1}) \subseteq P_{n-1}$, where $n \geq 2$. Every A-module also has an injective resolution. This follows from the fact that every A-module is a submodule of an injective A-module, and hence dualising the construction of a projective resolution one can construct an injective resolution inductively.

Examples 4.3.
(1) Let n be a positive integer. We have an obvious exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a \mapsto an} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Thus a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is the pair consisting of the 2-term complex $\mathbb{Z} \xrightarrow{a \mapsto an} \mathbb{Z}$ with nonzero differential given by multiplication with n, together with the canonical map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ starting at the second term of this complex.
(2) An abelian group A is called *divisible* if for any $a \in A$ and any positive integer n there is $b \in A$ such that $nb = a$. One can show that the divisible abelian groups are exactly the injective $\mathbb{Z}$-modules. In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules. The obvious short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

shows that an injective resolution of $\mathbb{Z}$ is the pair consisting of the 2-term complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, together with the inclusion $\mathbb{Z} \to \mathbb{Q}$. For n a positive integer, the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by $\frac{1}{n} + \mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The group $\mathbb{Q}/\mathbb{Z}$ is the colimit of these subgroups, and every element in $\mathbb{Q}/\mathbb{Z}$ has finite order; that is, $\mathbb{Q}/\mathbb{Z}$ is a torsion abelian group. By contrast, the torsion subgroup of $\mathbb{Q}$ is trivial.
(3) Hilbert's Syzygy Theorem shows that every module U over a polynomial algebra $k[x_1, x_2, .., x_n]$ in $n \geq 1$ indeterminates over a field k has a projective resolution of length at most n (that is, with at most $n + 1$ nonzero terms). For $n = 1$ this follows from tensoring the obvious short exact sequence

$$0 \longrightarrow k[x] \xrightarrow{f \mapsto xf} k[x] \longrightarrow k \longrightarrow 0$$

by $- \otimes_k U$. This yields a short exact sequence in which the first two terms are the free $k[x]$-modules $k[x] \otimes_k U$, and the third term is $k \otimes_k U \cong U$. For $n > 1$, one way to see this is to note that $k[x_1, x_2, .., x_n]$ is isomorphic to the tensor product of the algebras $k[x_i]$ and then tensor the above two-term complexes together for $1 \leq i \leq n$; this yields a complex with $n + 1$ terms. (One needs to extend the tensor product to complexes for this argument).
(4) Here is an example of a projective resolution of infinite length. Set $A = k[x]/(x^2)$. That is, A has a k-basis $\{1, \bar{x}\}$ such that $\bar{x}^2 = 0$. Note that $A\bar{x}$ is an ideal in A and that $A/A\bar{x} \cong k$, viewed as an A-module with $\bar{x}$ acting as zero on k. Multiplication by $\bar{x}$ on A is an endomorphism of A with image $A\bar{x}$ and also kernel $A\bar{x}$, since $\bar{x}^2 = 0$. Moreover, we have $A\bar{x} \cong k$. Thus we get an infinite projective resolution of the form

$$\cdots \longrightarrow A \xrightarrow{\bar{x}} A \xrightarrow{\bar{x}} k \longrightarrow 0$$

where the superscript $\bar{x}$ means multiplication by $\bar{x}$. One can in fact show that any projective resolution of k is infinite.

Definition 4.4. Let U, V be A-modules. For any nonnegative integer n we define a k-module $\mathrm{Ext}^n_A(U,V)$ as follows. Let P be a projective resolution of U with differential π. Applying $\mathrm{Hom}_A(-,V)$ yields a cochain complex $\mathrm{Hom}_A(P,V)$

$$\mathrm{Hom}_A(P_0,V) \xrightarrow{\pi^0} \mathrm{Hom}_A(P_1,V) \xrightarrow{\pi^1} \mathrm{Hom}_A(P_2,V) \longrightarrow \cdots$$

that is, $\mathrm{Hom}_A(P,V)$ is in degree $n \geq 0$ equal to $\mathrm{Hom}_A(P_n,V)$ with differential $\pi^n : \mathrm{Hom}_A(P_n,V) \to \mathrm{Hom}_A(P_{n+1};V)$ given by $\pi^n(\alpha) = \alpha \circ \pi_{n+1}$ for $n \geq 0$. We set

$$\mathrm{Ext}^n_A(U,V) = H^n(P,V) \ .$$

Proposition 4.5. *Let U, V be A-modules, P a projective resolution of U, and let n be an integer. we have a natural isomorphism of k-modules*

$$\mathrm{Ext}^n_A(U,V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P,V[n]) \ .$$

Proof. This is a special case of Proposition 3.21. □

We will use Proposition 4.5 to show that $\mathrm{Ext}^n_A(U,V)$ does not depend on the choice of P and that $\mathrm{Ext}^n_A(U,V)$ is contravariant functorial in U and covariantly functorial in V.

Proposition 4.6. *Let P, Q be projective resolutions of A-modules U, V, respectively. We have canonical isomorphisms*

$$\mathrm{Hom}_A(U,V) \cong \mathrm{Ext}^0_A(U,V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P,Q) \ .$$

This isomorphism sends $\alpha : U \to V$ to the homotopy class of a chain map $\varphi : P \to Q$ such that $\alpha \circ \mu \sim \nu \circ \varphi$ as chain maps from P to V.

Proof. With the notation of Definition 4.4, we have $\mathrm{Ext}^0_A(U,V) = \ker(\pi^0)$. This is the space of all A-homomorphisms $\alpha : P_0 \to V$ such that $\alpha \circ \pi_1 = 0$, that is, all A-homomorphisms α such that $\mathrm{Im}(\pi_1) \subseteq \ker(\alpha)$. Any such homomorphism factors uniquely through the canonical surjection $P_0 \to P_0/\mathrm{Im}(\pi_1)$. Denote by $\mu : P \to U$ a quasi-isomorphism; that is, μ is determined by a surjective A-homomorphism (still denoted μ) from P_0 to V such that $\ker(\mu) = \mathrm{Im}(\pi_1)$. Thus $\ker(\pi^0)$ can be canonically identified with the space of A-homomorphisms from $P/\ker(\mu) = U$ to V. This shows the first isomorphism. Since the projective resolution Q comes with a quasi-isomorphism $\nu : Q \to V$, composing with ν induces by Theorem 3.16 an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P,Q) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P,V) = \mathrm{Ext}^0_A(U,V)$ The compatibility with μ and ν follows from the explicit descriptions of these two isomorphisms. □

There is a structural way to look at this result: attaching a projective resolution to an A-module is a functorial construction through which the module category $\mathrm{Mod}(A)$ gets fully embedded into the chain homotopy category $K^-(\mathrm{Mod}(A))$ of bounded below chain complexes.

Corollary 4.7. *For any two projective resolutions (P,μ), (P',μ') of an A-module U there is a homotopy equivalence $\beta : P \simeq P'$ such that $\mu' \circ \beta = \mu$. Moreover, β is unique up to homotopy.*

Proof. Applying 4.6 with $U = V$ and $P' = Q$ shows that Id_U corresponds to the homotopy class of a chain map $\beta : P \to P'$ satisfying $\mu' \circ \beta \sim \mathrm{Id}_U \circ \mu = \mu$. Exchanging the roles of P and P' yields a chain map, unique up to homotopy, $\gamma : P' \to P$ satisfying $\mu \circ \gamma = \mu'$. Thus $\mu \circ \gamma \circ \beta = \mu$. But also $\mu \circ \mathrm{Id}_P = \mu$. Since, again by Theorem 3.16, composition with μ induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, P) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, U)$, it follows that $\gamma \circ \beta \sim \mathrm{Id}_P$. A similar argument shows that $\beta \circ \gamma \sim \mathrm{Id}_{P'}$, and hence that $P \simeq P'$ as stated. □

This Corollary is more precise than merely stating that two projective resolutions P, P' of U are homotopy equivalent as chain complexes; it says that there is a homotopy equivalence $P \simeq P'$ which is unique up to homotopy with the property that it is compatible with the second component of what makes a projective resolution, namely the quasi-isomorphisms μ and μ'. That is, in an appropriate category of pairs consisting of a chain complex and a chain map from this complex to U, the pairs (P, μ) and (P', μ') are uniquely isomorphic.

Theorem 4.8. *Let U, V be A-modules with projective resolutions P, Q, iand injective resoluitions I, J, respectively. Let $n \geq 0$ be an integer. We have a natural k-linear isomorphism*

$$\begin{aligned} \mathrm{Ext}^n_A(U, V) &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, J[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(U, J[n]) \\ &\cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(I, J[n]) \end{aligned}$$

Proof. The first isomorphism is from 4.5. Let $\nu : Q \to V$ be a quasi-isomorphism, and ν is surjective. Then $\nu[n] : Q[n] \to V[n]$ is a quasi-isomorphism, hence induces by Theorem 3.16 an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n]) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n])$. This shows the second isomorphism. The third isomorphism is induced by the quasi-isomorphism $V[n] \to J[n]$, the fourth isomorphism is induced by the quasi-isomorphism $P \to U$, and the last isomorphism is induced by the quasi-isomorphism $U \to I$. The naturality is an easy verification. □

The isomorphisms in this theorem are determined by the quasi-isomorphisms $P \to U \to I$ and $Q \to V \to J$, which are a structural part of the data which make up projective and injective resolutions. The interpretation of Ext in terms of homotopy classes of chain maps has one important consequence: we can compose chain maps, and this introduces ring and module structures on Ext-spaces as follows.

Proposition 4.9. *Let U, V be A-modules. Then the graded k-module*

$$\mathrm{Ext}^*_A(U, U) = \oplus_{n \geq 0} \mathrm{Ext}^n_A(U, U)$$

is a graded unital associative k-algebra, and the graded k-module

$$\mathrm{Ext}^*_A(U, V) = \oplus_{n \geq 0} \mathrm{Ext}^n_A(U, V)$$

is an $\mathrm{Ext}^*_A(V, V)$-$\mathrm{Ext}^*_A(U, U)$-bimodule, through composition of chain maps.

Proof. Let P, Q be projective resolutions of U, V, respectively. For

$$\zeta \in \mathrm{Ext}^n_A(U,V) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n])$$

$$\tau \in \mathrm{Ext}^m_A(U,U) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, P[m])$$

define the product $\zeta \cup \tau$ in $\mathrm{Ext}^{m+n}_A(U,V)$ by

$$\zeta \cup \tau = \zeta[m] \circ \tau \ \in \mathrm{Ext}^{n+m}_A(U,V) = \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Q[n+m]) \ .$$

For the special case $V = U$ and $Q = P$ this defines a graded product on $\mathrm{Ext}^*_A(U,U)$ which is associative because it is induced by composition in a category (which is always associative as part of the definition of a category). The map Id_U, viewed as an element of $\mathrm{Ext}^0_A(U,U)$, is the unit element of this multiplication. For arbitrary V this defines a right module structure of $\mathrm{Ext}^*_A(U,U)$ on $\mathrm{Ext}^*_A(U,V)$, and there is an obvious analogue of this argument which defines a left $\mathrm{End}^*_A(V,V)$-module structure on $\mathrm{Ext}^*_A(U,V)$. □

The product in $\mathrm{Ext}^*_A(U,U)$ is called *cup product*.

Remark 4.10. Theorem 4.8 shows that in the definition of $\mathrm{Ext}^n_A(U,V)$ we could have used injective resolutions of V instead of projective resolutions of U and would have ended up with the same concept. In some circumstances, calculating an injective resolution may be easier than calculating a projective resolution. There are cases - such as in the category of sheaves - where every object has an injective resolution but not a projective resolution.

The bifunctors Ext^n_A, for $n \in \mathbb{Z}$, are defined using the bifunctor $\mathrm{Hom}_A(-,-)$ applied to appropriate resolutions and taking cohomology. A similar construction, using the bifunctor $- \otimes_A -$, yields the bifunctors $\mathrm{Tor}^A_n(-,-)$.

Definition 4.11. Let V be an A-module and W a right A-module. Let Q be a projective resolution of V. For $n \geq 0$ we set

$$\mathrm{Tor}^A_n(W,V) = H_n(W \otimes_A Q) \ .$$

That is, $\mathrm{Tor}^A_n(W,V)$ is the homology in degree n of the chain complex $W \otimes_A Q$ obtained from applying the covariant functor $W \otimes_A -$ to the projective resolution Q of V. Since any two projective resolutions of V are homotopy equivalent and since any functor maps homotopy equivalent complexes to homotopy equivalent complexes, it follows as in the case of Ext that $\mathrm{Tor}^A_n(W,V)$ does not depend on the choice of Q. One can show that $\mathrm{Tor}^A_n(W,V)$ is also isomorphic to the homology in degree n of the chain complex $R \otimes_A V$ obtained from applying the functor $- \otimes_A V$ to a projective resolution of the right A-module W. The use of the notation 'Tor' for this concept comes from the following fact (stated without proof).

Theorem 4.12. *Let A be an abelian group. Then $\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A)$ is isomorphic to the torsion subgroup of A.*

Definition 4.13. Let U be an A-module. The *projective dimension of* U, denoted by $\mathrm{pdim}(U)$, is the smallest nonnegative integer n such that U has a projective resolution P satisfying $P_i = 0$ for $i > n$, with the convention $\mathrm{pdim}(U) = \infty$ if every projective resolution of U is unbounbded. The *injective dimension of* U, denoted by $\mathrm{idim}(U)$, is the smallest nonnegative integer n such that U

has an injective resolution I satisfying $I^i = 0$ for $i > n$, with the convention $\operatorname{idim}(U) = \infty$ if every injective resolution of U is unbounbded. The *global dimension of A* is equal to

$$\operatorname{gldim}(A) = \sup\{\operatorname{pdim}(U) \mid U \in \operatorname{Mod}(A)\}$$

Thus $\operatorname{gldim}(A) = \infty$ unless every A-module U has a finite projective dimension which bounded by some fixed integer, independent of U. Note that $\operatorname{gldim}(A)$ is defined in terms of left A-module; there is an obvious analogue of a right global dimension. Left and right global dimension of an algebra need not coincide. The following result shows that we could have defined the global dimension using injective dimensions of modules.

Theorem 4.14. *We have*

$$\begin{aligned}\operatorname{gldim}(A) &= \sup\{\operatorname{idim}(U) \mid U \in \operatorname{Mod}(A)\} \\ &= \sup\{d \in \mathbb{N} \mid \operatorname{Ext}^d_A(U,V) \neq 0 \textit{ for some } U, V \in \operatorname{Mod}(A)\}\end{aligned}$$

Proof. Let $d \geq 0$ and U, $V \in \operatorname{Mod}(A)$ such that $\operatorname{Ext}^d_A(U,V) \neq 0$. Let P be a projective resolution of U. Since $\operatorname{Ext}^d_A(U,V) \cong \operatorname{Hom}_{K(\operatorname{Mod}(A))}(P, V[d]) \neq 0$, there must be a nonzero A-homomorphism $P_d \to V$. In particular, $P_d \neq 0$. This shows $\operatorname{pdim}(U) \geq d$, hence $\operatorname{gldim}(A)$ is greater or equal to the supremum of all such d. Let I be an injective resolution of V. Since $\operatorname{Ext}^d_A(U,V) \cong \operatorname{Hom}_{K(\operatorname{Mod}(A))}(U, I[d]) \neq 0$, there must be a nonzero A-homomorphism $U \to I^d$. In particular, $I^d \neq 0$. This shows $\operatorname{idim}(U) \geq d$, hence $\sup\{\operatorname{idim}(U) \mid U \in \operatorname{Mod}(A)\}$ is greater or equal than the supremum of all such d. For the converse inequality, suppose that $d \geq 0$ satisfies $\operatorname{Ext}^d_A(U,V) = 0$ for all U, $V \in \operatorname{Mod}(A)$. We need to show that $\operatorname{pdim}(U)$ and $\operatorname{idim}(U)$ are both bounded by d. We do this for $\operatorname{pdim}(U)$; the argument for $\operatorname{idim}(U)$ is similar. Let P be a projective resolution of U, with differential π. Set $V = \operatorname{Im}(\pi_d)$; this is a submodule of P_{d-1}. Consider the map $\pi_d : P_d \to V$ as a chain map $P \to V[d]$. Since $0 = \operatorname{Ext}^d_A(U,V) = \operatorname{Hom}_{K(\operatorname{Mod}(A))}(P, V[d])$ this chain map is homotopic to zero. Since a homotopy from P to $V[d]$ is zero except possible in degree $d-1$, this is equivalent to the existence of an A-homomorphism $\varphi : P_{d-1} \to V$ satisfying $\pi_d = \varphi \circ \pi_d$.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{d+1} & \xrightarrow{\pi_{d+1}} & P_d & \xrightarrow{\pi_d} & P_{d-1} & \longrightarrow & \cdots \longrightarrow & P_0 & \longrightarrow 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_d} & \swarrow{\scriptstyle \varphi} & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & V & \longrightarrow & 0 & \longrightarrow & \cdots & &
\end{array}$$

Thus

$$\varphi \circ \pi_d = \pi_d = \operatorname{Id}_V \circ \pi_d$$

That means that φ restricts to the identity map on $V = \operatorname{Im}(\pi_d)$. Thus V is a direct summand of P_{d-1}, hence P_{d-1}/V is projective. Note that $V = \operatorname{Im}(\pi_{d-1}) = \ker(\pi_{d-1})$, and hence π_{d-1} induces an injective map $\iota : P_{d-1}/V \to P_{d-2}$. It follows that there is a projective resolution of U of the form

$$\cdots \longrightarrow 0 \longrightarrow P_{d-1}/V \xrightarrow{\iota} P_{d-2} \xrightarrow{\pi_{d-2}} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0$$

where ι is the injective map induced by π_{d-1}. Thus $\operatorname{pdim}(U) \leq d$ as required. $\square$

Examples 4.15. We have $\mathrm{gldim}(\mathbb{Z}) = 1$ and $\mathrm{gldim}(k[x]) = 1$, where k is a field. More generally, any principal ideal domain which is not a field has global dimension 1. A field k has global dimension 0 because any k module is a k-vector space, hence has a basis, or equivalently, is free. Thus any k-module U is its own projective resolution concentrated in degree 0, together with the identity map Id_U. By earlier calculations, we have $\mathrm{gldim}(k[x]/(x^2) = \infty$.

Remark 4.16. Work of B. Osofsky in the 1970s explores surprising interactions between set theory and global dimensions of rings. For instance, a consequence of Osofsky's work is that the algebra $\mathbb{R}(x, y, z)$ of real rational functions in three variables viewed as a module over the polynomial subalgebra $\mathbb{R}[x, y, z]$ has global dimension 2 if and only if the continuum hypothesis holds. It was conjectured by J. H. C. Whitehead that for A an abelian group, $\mathrm{Ext}^1_{\mathbb{Z}}(A, Z)$ is zero if and only if A is a free abelian group. Work of S. Shelah from the 1970s shows that the truth of this conjecture depends on the set theory being used.

5 Hochschild cohomology

Let A be an algebra over a commutative ring k. In what follows we will work with the category of A-A-bimodules. This category can be identified with the category of modules over the algebra $A \otimes_k A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A. That is, $A^{\mathrm{op}} = A$ as a k-module, with product $a \cdot b = ba$, where the expression $a \cdot b$ is the product in A^{op} and ba is the product in A. More precisely, an A-A-bimodule M can be viewed as an $A \otimes_k A^{\mathrm{op}}$-module via

$$(a \otimes b) \cdot m = amb \ ,$$

where a, $b \in A$, $m \in M$. Given a left $A \otimes_k A^{\mathrm{op}}$-module M, the same equation can be used to define a bimodule structure on M. The advantage of working with $A \otimes_k A^{\mathrm{op}}$-modules is that it makes all the module theoretic machinery available for bimodules - such as projective and injective resolutions, Ext and Tor, for instance. Note that A is itself an A-A-bimodule, hence an $A \otimes A^{\mathrm{op}}$-module, through left and right multiplication by A on itself. The tensor product $A \otimes_k A$ endowed with left multiplcation by A on the first copy of A and right multiplication on the second copy of A becomes in this way an A-A-bimodule. When viewed as an $A \otimes_k A^{\mathrm{op}}$-module, this is then equal to the free $A \otimes_k A^{\mathrm{op}}$-module of rank 1 because the right action of A on the second copy of A is the same as the left action of A^{op} on A^{op}. We start by observing that there is a canonical chain complex (X, d) of $A \otimes_k A^{\mathrm{op}}$-modules of the form

$$\cdots \longrightarrow A \otimes_k A \otimes_k A \xrightarrow{d_1} A \otimes_k A \xrightarrow{d_0} A \longrightarrow 0$$

with differential d constructed as follows. Multiplication in A induces a surjective homomorphism of A-A-bimodules

$$d_0 : A \otimes_k A \to A, \ a \otimes b \mapsto ab, \ (a, b \in A)$$

Tensoring this map on the right and left by A and taking the difference of the two resulting maps yields an A-A-bimodule homomorphism

$$d_1 : A \otimes_k A \otimes_k A \to A \otimes_k A \ , \quad a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc$$

The sign of the right term ensures that the image of this map is equal to the kernel of the first map given by multiplication in A. We can iterate this construction, and we will show that we obtain in the process a resolution of A as an A-A-bimodule of the form

$$\cdots \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow 0$$

together with the quasi-isomorphism given by the multiplication map $A^{\otimes 2} \to A$. Here the notation is

$$A^{\otimes n} = A \otimes_k A \otimes_k A \otimes_k \cdots \otimes_k A \ ,$$

where we tensor $n \geq 1$ copies of A over k. For later use we adopt the convention $A^{\otimes 0} = k$. For $n \geq 1$ we regard $A^{\otimes n}$ as an A-A-bimodule in such a way that A acts on the left by left multiplication on the first copy of A in this tensor product and A acts on the right by right multiplication on the last copy of A. The intermediate copies of A matter for this bimodule structure only as far as their k-module structure is concerned.

Proposition 5.1. *For $n \geq -1$ set $X_n = A^{\otimes n+2}$ and for $n \geq 0$ denote by $d_n : X_n \to X_{n-1}$ the $A \otimes_k A^{\mathrm{op}}$-homomorphism given by*

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \ .$$

Set $X_n = 0$ for $n \leq -2$ and $d_n = 0$ for $n \leq -1$. Then $X = (X_n, d_n)_{n \in \mathbb{Z}}$ is an acyclic complex as $A \otimes_k A^{\mathrm{op}}$-modules. More precisely, (X_n, d_n) is contractible as a complex of left A-modules and as a complex of right A-modules.

Proof. For $n \geq -1$ and i satisfying $0 \leq i \leq n$ define the $A \otimes_k A^{\mathrm{op}}$-homomorphism $d_{n,i} : A^{\otimes(n+2)} \to A^{\otimes(n+1)}$ by setting

$$d_{n,i}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \ .$$

Then $d_n = \sum_{i=0}^{n} (-1)^i d_{n,i}$. The maps $d_{n,i}$ and hence d_n are bimodule homomorphisms. For $n \geq 0$ we have

$$d_{n-1} \circ d_n = \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} d_{n-1,j} \circ d_{n,i} \ .$$

We show that the terms in this sum can be paired with opposite signs. If $j \geq i$, then $d_{n-1,j} \circ d_{n,i} = d_{n-1,i} \circ d_{n,j+1}$. If $j < i$, then $d_{n-1,j} \circ d_{n,i} = d_{n-1,i-1} \circ d_{n,j}$. Thus pairing the summand indexed (i,j) with that indexed by $(j+1,i)$ if $j \geq i$ and with $(j, i-1)$ if $j < i$ shows that all summands cancel. This shows that (X_n, d_n) is a chain complex of $A \otimes A^{\mathrm{op}}$-modules.

Define homomorphisms of right A-modules $h_n : X_n \to X_{n+1}$ by

$$h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$$

for $n \geq -1$ and $h_n = 0$ for $n \leq -2$. We will shows that (X_n, d_n) is contractible as a complex of right A-modules with the homotopy h. We need to show that

$$\mathrm{Id}_{X_n} = d_{n+1} \circ h_n + h_{n-1} \circ d_n$$

for all $n \in \mathbb{Z}$. We have

$$(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) =$$

$$= h_{n-1}(\sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \; + \; d_{n+1}(1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) =$$

$$= \sum_{i=0}^{n} (-1)^i 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \; +$$

$$a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} + \sum_{i=0}^{n} (-1)^{i+1} 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} =$$

$$= a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \; .$$

This shows that (X_n, d_n) is contractible as a complex of right A-modules. In particular, this complex is acyclic. A similar argument, using a homotopy of left A-modules sending $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$ to $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1$ shows that this complex is also contractible as a complex of left A-modules. □

Definition 5.2. Suppose that A is projective as a k-module. Let M be an A-A-bimodule. The *Hochschild cohomology of A with coefficients in M* is the graded k-module

$$HH^*(A; M) = \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, M)$$

and the *Hochschild cohomology of A* is the graded k-algebra

$$HH^*(A) = HH^*(A; A) = \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, A)$$

By earlier results, $HH^*(A; M)$ is a graded right $HH^*(A)$-module. In order to calculate Hochschild cohomology, we will need a projective resolution P of A as an $A \otimes_k A^{\mathrm{op}}$-module, to which we then apply the functor $\mathrm{Hom}_{A \otimes_k A^{\mathrm{op}}}(-, M)$ and take the cohomology of the resulting cochain complex. The complex (X_n, d_n) is not always a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. We will need the extra condition on A in the definition to ensure this: we will need to assume that A is projective as a k-module. Then $A^{\otimes n}$ is projective as a k-module (because the tensor product of two free k-modules is again free). Thus $A^{\otimes n+2} = A \otimes_k A^{\otimes n} \otimes_k A$ is a projective $A \otimes_k A^{\mathrm{op}}$-module. It follows that the chain complex P with $P_n = X_n = A^{\otimes n+2}$ for $n \geq 0$ with differential $d_n : P_n \to P_{n-1}$ for $n > 0$ together with the quasi-isomorphism $d_0 : P_0 \to A$ given by multiplication in A is a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. This resolution is called the *bar resolution of A* and can be used to calculate the Hochschild cohomology of A with coefficients in any $A \otimes_k A^{\mathrm{op}}$-module M as

$$HH^n(A; M) = H^n(\mathrm{Hom}_{A \otimes_k A^{\mathrm{op}}}(P, M))$$

We will describe the cochain complex $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(P, M)$ more explicitly; this will in particular yield interpretations of low degree Hochschild cohomology. The term in degree $n \geq 0$ of the cochain complex $\mathrm{Hom}_{A \otimes_k A^{\mathrm{op}}}(P, M)$ is equal to $\mathrm{Hom}_{A \otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M)$. This can be simplified as follows. We make use of the convention $A^{\otimes 0} = k$.

Lemma 5.3. *For $n \geq 0$ we have a canonical isomorphism*

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)\ .$$

This isomorphism sends an $A \otimes_k A^{\mathrm{op}}$-homomorphism $\zeta : A^{\otimes n+2} \to M$ to the unique k-linear map $\tau : A^{\otimes n} \to M$ defined for $n > 0$ by $\tau(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \zeta(1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)$ and for $n = 0$ by $\tau(1) = \zeta(1 \otimes 1)$.

Proof. To show that this is an isomorphism, we describe explicitly an inverse map as follows. Let $\tau : A^{\otimes n} \to M$ be a k-linear map. Define $\zeta : A^{\otimes n+2} \to M$ by setting $\zeta(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes a_{n+1}) = a_0\tau(a_1 \otimes a_2 \otimes \cdots \otimes a_n)a_{n+1}$; this expression is well-defined: the element in the middle belongs to M, and since M can be regarded as a bimodule, we can multiply this element on the left by a_0 and on the right by a_{n+1}. A trivial verification shows that ζ defined this way is an $A \otimes_k A^{\mathrm{op}}$-module homomorphism and that the given assignment is inverse to that described in the statement. □

The above Lemma is a special case of the Tensor-Hom adjunction. In particular, the degree zero term of the complex $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M)$ can be identified as

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A \otimes_k A, M) \cong \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(k, M) \cong M\ ,$$

where the first isomorphism is from Lemma 5.3, and the second isomorphism sends a linear map $\tau : k \to M$ to $\tau(1)$. The composition of these two isomorphisms sends a bimodule homomorphism $\zeta : A \otimes_k A \to M$ to the element $\zeta(1 \otimes 1)$ in M. With this identification, we can describe the cochain complex $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, M)$ as follows.

Theorem 5.4. *Let M be an $A \otimes_k A^{\mathrm{op}}$-module. Define k-modules $C^n(A; M)$ for $n \geq 0$ by setting*

$$C^n(A, M) = \mathrm{Hom}_k(A^{\otimes n}, M)$$

Define maps $\delta^n : \mathrm{Hom}_k(A^{\otimes n}; M) \to \mathrm{Hom}_k(A^{\otimes n+1}; M)$ for $n \geq 0$ by by setting

$$\delta^n(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) =$$

$$= a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n) + (-1)^{n+1} f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})a_n$$

for any $f \in Hom_k(A^{\otimes n}, M)$. Then $(C^n(A; M), \delta^n)$ is a cochain complex which is isomorphic to $\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(P; M)$, where P is the bar resolution of A as before. In particular, the cohomology of this cochain complex is the Hochschild cohomology of A.

Proof. By Lemma 5.3, $C^n(A; M)$ is isomorphic to the term in degree n of $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P; M)$. One needs to chase the differential through these isomorphisms; this yields the maps as described. For $n \geq 0$, we have from 5.3 an isomorphism

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)\ .$$

Let $f \in \mathrm{Hom}_k(A^{\otimes n}, M)$. Through the previous isomorphism, this corresponds to the homomorphism $\hat{f} \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes n+2}, M)$ given by the formula

$$\hat{f}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 f(a_1 \otimes \cdots \otimes a_n)a_{n+1}\ .$$

By the definition of δ^n, we have $\delta^n(\hat{f}) = \hat{f} \circ \delta_{n+1} \in \mathrm{Hom}_{A \otimes_k A^{\mathrm{op}}}(A^{\otimes n+3}; M)$, and $\delta^n(f)$ is then obtained from restricting $\hat{f} \circ \delta_{n+1}$ to $1 \otimes A^{\otimes n+1} \otimes 1$. So we need to calculate

$$(\hat{f} \circ \delta_{n+1})(1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \otimes 1)$$

and this is the expression as in the statement of the theorem. Note that in the expression $1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \otimes 1$ the term a_i is now in the component $i+1$ because of the first component equal to 1; this is the reason why in the sum, the term indexed by i multiplies $a_{i-1}a_i$ and not $a_i a_{i+1}$. □

Remark 5.5. With the notation of the previous theorem, we have $C^0(A; M) = \mathrm{Hom}_k(k, m) \cong M$, because a k-linear map $\tau : k \to M$ is uniquely determined by its value $\tau(1_k)$. If we identify $C^0(k; M)$, then the differential $\delta^0 : M \to \mathrm{Hom}_k(A, M)$ sends $m \in M$ to the map $\delta^0(m) : A \to M$ given by $\delta^0(m)(a) = am - ma$. Thus the cochain complex calculating Hochschild cohomology takes the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A \otimes_k A, M) \longrightarrow \cdots$$

We can use this cochain complex to calculate Hochschild cohomology in low degrees. For M an A-A-bimodule we set $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. Note that M^A is a left and right $Z(A)$-submodule of M and that $Z(A) = A^A$.

Proposition 5.6. *We have a canonical isomorphism $HH^0(A; M) = M^A$. In particular, we have $HH^0(A) \cong Z(A)$.*

Proof. With the notation of 5.4 and the identification $C^0(A; M) = M$, we have

$$HH^0(A; M) = \ker(\delta^0) = \{m \in M \mid am - ma = 0 \text{ for all } a \in A\} = M^A \ .$$

Since $A^A = Z(A)$, the second statement follows. □

The Hochschild cohomology in degree 1 has an interpretation in terms of derivations.

Definition 5.7. For M an A-A-bimodule, a k-linear map $f : A \to M$ is called a *derivation* if $f(ab) = af(b) + f(a)b$ for all a, $b \in A$.

The set $\mathrm{Der}(A; M)$ of derivations $A \to M$ is a k-subspace of $\mathrm{Hom}_k(A, M)$.

Exercise 5.8. Let M be an A-A-bimodule. Show that for a fixed element $m \in M$, the map $[-, m]$ sending $a \in A$ to the additive commutator $[a, m] = am - ma$ is a derivation.

Definition 5.9. Any derivation $A \to M$ which is equal to $[-, m]$ for some $m \in M$ is called an *inner derivation.*

The set $\mathrm{IDer}(A)$ of inner derivations from A to M is a subspace of $\mathrm{Der}(A; M)$. If $A = M$ we set $\mathrm{IDer}(A) = \mathrm{IDer}(A; A)$ and $\mathrm{Der}(A) = \mathrm{Der}(A; A)$.

Proposition 5.10. *We have a canonical isomorphism $HH^1(A; M) \cong \mathrm{Der}(A; M)/\mathrm{IDer}(A; M)$.*

Proof. With the notation of 5.4, we have

$$HH^1(A;M) = \ker(\delta^1)/\mathrm{Im}(\delta^0)\ .$$

It suffices to verify that

$$\ker(\delta^1) = \mathrm{Der}(A;M)$$

and

$$\mathrm{Im}(\delta^0) = \mathrm{IDer}(A;M)\ .$$

Let $f \in \mathrm{Hom}_k(A;M)$. Then $\delta^1(f) \in \mathrm{Hom}_k(A \otimes_k A; M)$ is defined by $\delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$. Thus f belongs to $\ker(\delta^1)$ if and only if

$$f(ab) = af(b) + f(a)b$$

for all a, b in A, hence if and only if f is a derivation. This shows that $\ker(\delta^1) = \mathrm{Der}(A;M)$. We have $f \in \mathrm{Im}(\delta^0)$ if and only if there exists $m \in M$ such that $f = \delta^0(m)$, that is, if and only if $f(a) = am - ma$ for all $a \in A$, which is equivalent to $f \in \mathrm{IDer}(A;M)$. This shows $\mathrm{Im}(\delta^0) = \mathrm{IDer}(A;M)$, whence the result. □

Exercise 5.11. Let f, $g : A \to A$ be derivations. Show that $[f,g] = f \circ g - g \circ f$ is a derivation. Show that if $c \in A$ and $g = [-,c]$, then $[f,g] = [-,f(c)]$.

Remark 5.12. There is a lot more structure on Hochschild cohomology. The previous exercise shows that the space of derivations $\mathrm{Der}(A)$ on A is a Lie subalgebra of $\mathrm{End}_k(A)$ with the bracket $[f,g] = f \circ g - g \circ f$, and that $\mathrm{IDer}(A)$ is an ideal in the Lie algebra $\mathrm{Der}(A)$. Thus $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$ is a Lie algebra. It was shown by Gerstenhaber that this Lie algebra structure extends to a graded Lie algebra structure of degree -1 on $HH^*(A)$ If A is a finite-dimensional algebra over $\mathbb{C}$, then its outer automorphism group $\mathrm{Out}(A)$ is an algebraic group. The tangent space of the connected component of this algebraic group is the Lie algebra $HH^1(A)$. See for instance Keller [5] for an application and references for this point of view. For an algebra A over an algebraically closed field of prime characteristic p it is still true that $\mathrm{Out}(A)$ is an algebraic group, but its tangent space need not be $HH^1(A)$.

One can use any projective resolution P of A as an $A \otimes_k A^{\mathrm{op}}$-module to obtain projective resolutions of an arbitrary A-module U simply by tensoring with U over A.

Proposition 5.13. *Let (P,μ) be a projective resolution of A as an $A \otimes_k A^{\mathrm{op}}$-module. Regard P as a complex of A-A-bimodules. Let U be an A-module which is projective as a k-module. Then $(P \otimes_A U, \mu \otimes \mathrm{Id}_U)$ is a projective resolution of U. The map sending a chain map $\zeta : P \to P[n]$ to the chain map $\zeta \otimes \mathrm{Id}_U : P \otimes_A U \to (P \otimes U)[n]$ induces a homomorphism of graded algebras*

$$HH^*(A) \to \mathrm{Ext}_A(U,U)$$

Proof. We first note that the statement makes sense. Applying the functor $- \otimes_A U$ to P yields a chain complex $P \otimes_A U$. The terms of P are projective as bimodules, hence direct summands of free bimodules, so direct summands of sums of copies of the free rank one bimodule $A \otimes_k A$. Thus the terms of $P \otimes_A U$ are direct summands of sums of copies of the left A-module $A \otimes_k U$. Now U is projective as a k-module, to isomorphic to a direct summand of a free k-module, hence a direct summand of a sum of copies of k. Thus, as a left A-module, $A \otimes_k U$ is a direct summand of a sum of copies of $A \otimes_k k \cong A$. This shows that the terms of $P \otimes_k U$ are all projective as A-modules. The quasi-isomorphism μ is the surjective $A \otimes_k A^{\mathrm{op}}$-homomorphism $A \otimes_k A \to A$ given by multiplication in A. Thus $\mu \otimes \mathrm{Id}_U$ is a surjective A-homomorphism $A \otimes_k A \otimes_A U \to A \otimes_A U$. After identifying $A \otimes_A U = U$, this is the surjective A-homomorphism $\nu : A \otimes_k U \to U$ sending $a \otimes u$ to au, where $a \in A$ and $u \in U$. If we restrict attention to the right A-module structure on P, then (P, μ) remains a projective resolution of A as a right A-module. Since A is projective (even free of rank 1) as a right A-module, A itself with the identity Id_A is its own projective resolution as a right A-module. Since projective resolutions are unique up to homotopy, it follows that then map μ, when considered as a homomorphism of right A-modules, induces a homotopy equivalence $P \simeq A$ as chain complexes of right A-modules. But then tensoring by $- \otimes U$ implies that ν induces a homotopy equivalence $P \otimes_A U \simeq A \otimes_A U \cong U$ as complexes of k-modules. Bringing back the left A-module structure, this shows that ν is indeed a quasi-isomorphism $P \otimes_A U \to U$ as required. □

The k-module $HH^2(A; M)$ parametrises associative algebra structures on $A \oplus M$ such that the canonical projection $A \oplus M \to A$ is an algebra homomorphism and such that M becomes an ideal which squares to zero. That is, the multiplication in $A \oplus M$ is given by

$$(a, m)(b, n) = (ab, an + mb + \alpha(a, b))$$

where a, $b \in A$, m, $n \in M$ and $\alpha(a, b) \in M$. This defines a bilinear map $\alpha : A \times A \to M$. A short verification shows that the associativity of this multiplication is equivalent to

$$\alpha(a, b)c + \alpha(ab, c) = a\alpha(b, c) + \alpha(a, bc)$$

for all a, b, $c \in A$. If we extend α If we extend α to the unique linear map $\alpha : A \otimes_k A \to M$ and bring all terms in the previous equality to one side, then this reads

$$a\alpha(b \otimes c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

which is equivalent to $\alpha \in \ker(\delta^2)$. Thus a linear map α yields an associative multiplication on $A \oplus M$ as above if and only if $\alpha \in \ker(\delta^2)$. We denote this algebra by $T_\alpha(A; M)$.

Exercise 5.14. With the notation above, let α, $\alpha' \in \ker(\delta^2)$. Show that there is an isomorphism of algebras

$$T_\alpha(A \oplus M) \cong T_{\alpha'}(A \oplus M)$$

which induces the identity on the ideals M and on the quotients A if and only if α and α' determine the same class in $HH^2(A; M)$.

The zero class in $HH^2(A; M)$ corresponds to what is called the *trivial extension algebra* $T(A \oplus M) = A \oplus M$, with multiplication given by $(a, m)(b, n) = (ab, an + mb)$.

Theorem 5.15 (Gerstenhaber). *The algebra $HH^*(A)$ is graded-commutative; that is, for integers m, $n \geq 0$ and $\zeta \in HH^m(A)$, $\eta \in HH^n(A)$, we have $\eta\zeta = (-1)^{mn}\zeta\eta$.*

Thus if one of m, n is even, then η and ζ commute, and if m is odd, then $\zeta^2 = -\zeta^2$, so $\zeta^2 = 0$ unless A is an algebra over a field of characteristic 2. In particular, the even part $HH^{\mathrm{ev}}(A) = \oplus_{i \geq 0} HH^{2i}(A)$ of Hochschild cohomology is strictly commutative, and if A is an algebra over a field of charatceristic 2, then $HH^*(A)$ is commutative.

6 Cohomology of groups

Let k be a commutative ring and G a group. The *group algebra* kG of G over k is the algebra which is the free k-module having the set of elements of G as a k-basis, endowed with the unique k-bilinear multiplication induced by the group multiplication of G. More explicitly, the elements of kG are the formal sums $\sum_{x\in G} \lambda_x x$, where $\lambda_x \in k$ for all $x \in G$, with only finitely many of the coefficients λ_x nonzero. The sum in kG is given componentwise by the formula

$$(\sum_{x\in G} \lambda_x x) + (\sum_{x\in G} \mu_x x) = \sum_{x\in G} (\lambda_x + \mu_x)x \ ,$$

the scalar multiplication in kG is given by

$$\lambda(\sum_{x\in G} \lambda_x x) = \sum_{x\in G} (\lambda\lambda_x)x,$$

and the product in kG is given by the formula

$$(\sum_{x\in G} \lambda_x x)(\sum_{x\in G} \mu_x x) = \sum_{x,y\in G} \lambda_x\mu_y xy = \sum_{z\in G} (\sum_{x,y\in G,xy=z} \lambda_x\mu_y)z \ ,$$

where as before the coefficients $\lambda_x, \mu_x, \lambda$ are in k, with only finitely many of the λ_x and the μ_x nonzero. The unit element 1_{kG} of the algebra kG is the image in kG of the unit element 1_G of the group G. The images in kG of the elements of the group G become invertible in the algebra kG in such a way that the image in kG of the inverse x^{-1} in G of an element $x \in G$ is the inverse of the image of x in kG. We tend not to notationally distinguish the elements of G from their images in kG unless this is needed to avoid confusion. The associativity of the product in G implies that the multiplication in kG is associative.

The ring k has a trivial kG-module structure, with all group elements acting as identity. This does not mean that all elements of kG act as identity: the action of an element $\sum_{x\in G} \lambda_x x$ on k is given by multiplication with the scalar $\sum_{x\in G} \lambda_x$. This is well-defined, as only finitely many of the λ_x are nonzero. We call k endowed with this module structure the *trivial kG-module*, and denote it again by k, if no confusion arises. The structural homomorphism $\eta : kG \to k$ determined by the trivial module sends an element $\sum_{x\in G} \lambda_x x$ in kG to the scalar $\sum_{x\in G} \lambda_x$ in k. This is a surjective algebra homomorphism, called the *augmentation homomorphism.* Its kernel, denoted $I(kG)$, is the *augmentation ideal in kG*.

Definition 6.1. The cohomology in degree $n \geq 0$ of G with coefficients in a kG-module M is defined as

$$H^n(G; M) = \mathrm{Ext}^n_{kG}(k, M)$$

The cohomology in degree $n \geq 0$ of G with coefficients in an abelian group A is defined as

$$H^n(G; A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}; A)$$

More explicitly, $H^n(G, M) = H^n(\mathrm{Hom}_{kG}(P, M))$, where P is a projective resolution of the trivial kG-module k. Thus, in order to calculate group cohomology, we need to describe a projective resolution of the trivial kG-module k. As in the case of Hochschild cohomology, there is a canonical projective resolution, called the *bar resolution of the trivial kG-module.* Applying the functor $\mathrm{Hom}_{kG}(-, M)$ to this resolution yields a cochain complex whose cohomology is then cohomology of G with coefficients in M. Hochschild cohomology offers a shortcut to this programme. Thanks to Proposition 5.13, tensoring the bimodule bar resolution of kG by $-\otimes_{kG} k$ yields a projective resolution of k. This can be described explicitly as follows.

Theorem 6.2. *Let G be a group and M a kG-module which is projective as a k-module. For $n \geq 0$ set*

$$C^n(G; M) = \{\alpha : G^n \to M\}$$

where G^n is the direct product of n copies of G, with the convention $C^0(G; M) = M$. For $n \geq 0$ define a k-linear map

$$\delta^n : C^n(G; M) \to C^{n+1}(G; M)$$

by setting

$$\delta^n(\alpha)(x_0, x_1, \dots, x_n) =$$

$$x_0\alpha(x_1, \dots, x_n) + \sum_{i=1}^{n}(-1)^i\alpha(x_0, .., x_{i-1}x_i, .., x_n) + (-1)^{n+1}\alpha(x_0, .., x_{n-1})$$

with the convention $\delta^0(m)(x) = xm - m$ Here x and the x_i are elements in G.

Proof. The multiplication map $\mu : kG \otimes_k kG \to kG$ has the property that upon tensoring it with $-\otimes_{kG} k$, it yields the augmentation map. Indeed, after identifying $kG \otimes_{kG} k = k$, we get that $\mu \otimes \mathrm{Id}_{kG} : kG \otimes_k k = kG \to k$ is equal to the augmentation map η. Consider the bar resolution P of kG. Applying $-\otimes_{kG} k$ yields a projective resolution $P \otimes_{kG} k$ of the trivial kG-module k. The term in degree n of this resolution is $\mathrm{Hom}_{kG}((kG)^{\otimes n+1}, M)$, where we use the identification $kG \otimes_{kG} k = k$. Just as in Lemma 5.3, we have an isomorphism

$$\mathrm{Hom}_{kG}((kG)^{\otimes n+1}, M) \cong \mathrm{Hom}_k((kG)^{\otimes n}, M)$$

where the passage from the right side gto the left side sends a linear map $\tau : (kG)^{\otimes n} \to M$ to the kG-homomrophism $\alpha : (kG)^{\otimes n+1} \to M$ given by $\alpha(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = x_0\tau(x_1 \otimes \cdots \otimes x_n)$. The space $\mathrm{Hom}_k((kG)^{\otimes n}, M)$ is clearly isomorphic to $C^n(G; M)$, since any map $\alpha : G^n \to M$ extends uniquely to a linear map $(kG)^{\otimes n} \to M$. Through these identifications, the differential is as stated in the theorem. □

The last term $\alpha(x_0, .., x_{n-1})$ in the differential of 6.2 is seemingly different from the corresponding last term in the differential of the Hochschild cohomology $f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})a_n$ in 5.4. This is because $x \otimes 1$ and $1 \otimes 1$ are equal in $kG \otimes_k k$ for any $x \in G$, and hence the right multiplication by a_n in 5.4 gets 'absorbed' by tensoring with $-\otimes_{kG} k$.

Remark 6.3. Theorem 6.2 describes group cohomology $H^*(G; M)$ as the cohomology of a cochain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\delta^0} C^1(G; M) \xrightarrow{\delta^1} C^2(G; M) \longrightarrow \cdots$$

The first few differentials are explicitly given by

$$\delta^0(m)(x) = xm - m,$$

$$\delta^1(\beta)(x, y) = x\beta(y) - \beta(xy) + \beta(x),$$

$$\delta^2(\alpha)(x, y, z) = x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y),$$

where x, y, $z \in G$, $m \in M$, and $\beta : G \to M$ and $\alpha : G \times G \to M$ are maps. This allows us to calculate low degree group cohomology and give it interpretations.

Exercise 6.4. Let G be a group and M a kG-module. Show that

$$H^0(G; M) = M^G = \{m \in M \mid xm = m \text{ for all } x \in G\}$$

$$H^1(G; M) = \ker(\delta^1)/\mathrm{Im}(\delta^0)$$

where

$$\ker(\delta^1) = \{\beta : G \to M \mid \beta(xy) = x\beta(y) + \beta(x) \text{ for all } x, y \in G\}$$

$$\mathrm{Im}(\delta^0) = \{\beta : G \to M \mid \text{there exists } m \in M \text{ such that} \beta(x) = xm - m \text{ for all } x \in G\}$$

In particular, show that if G acts trivially on M, then $H^1(G, M) = \mathrm{Hom}(G, M)$, the set of all group homomorphisms from G to the additive group $(M, +)$; that is, the group of all maps $\beta : G \to M$ satisfying $\beta(xy) = \beta(x) + \beta(y)$ for all x, $y \in G$.

Let A be an abelian group acted upon by a group G. That is, A is a $\mathbb{Z}G$-module. We write A and G both multiplicatively, and we write the action of $x \in G$ on $a \in A$ by ${}^x a$. Any short exact sequence of groups of the form

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

gives rise to an action of G on A as follows. For $x \in G$, choose an element $\hat{x} \in H$ such that $f(\hat{x}) = x$. Note that $\hat{x}$ is unique up to multiplication by an element in A. Since A is abelian, the conjugation action of $a \in A$ on A is trivial, and hence the conjugation action of $\hat{x}$ and $\hat{x}a$ on A is the same. That is, we have a well-defined action of G on A by setting ${}^x a = \hat{x}a\hat{x}^{-1}$. Let now x, $y \in G$. Then $\widehat{xy}$ and $\hat{x}\hat{y}$ are two elements which satisfy $f(\widehat{xy}) = xy = f(\hat{x}\hat{y})$, and therefore there is an element $\alpha(x, y) \in A$ such that

$$\hat{x}\hat{y} = \widehat{xy}\alpha(x, y) \ .$$

Using the associativity $(\hat{x}\hat{y})\hat{z} = \hat{x}(\hat{y}\hat{z})$ for all x, y, $z \in G$, a short calculation shows that

$$\alpha(x, y)\alpha(xy, z) = {}^x\alpha(y, z)\alpha(x, yz)$$

and that is exactly saying that $\alpha \in \ker(\delta^2)$; that is, α represents a class in $H^2(G; A)$. For a given fixed action of G on A, we consider the set of equivalence classes of group extensions of the form

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

with two such extensions

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

$$1 \longrightarrow A \longrightarrow H' \xrightarrow{f'} G \longrightarrow 1$$

being equivalent if there is a group isomorphism $\varphi : H \cong H'$ making the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{f} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}_G} & & \\ 1 & \longrightarrow & A & \longrightarrow & H' & \xrightarrow{f'} & G & \longrightarrow & 1 \end{array}$$

commutative.

Exercise 6.5. With the notation above, show that there is a bijection between the equivalence classes of group extensions

$$1 \longrightarrow A \longrightarrow H \xrightarrow{f} G \longrightarrow 1$$

of G by A for a given action of G on A and the set of classes $H^2(G; A)$. Show that the zero class in $H^2(G; A)$ corresponds to the split extension where $H = A \rtimes G$.

Definition 6.6. Let G be a finite group. The *Schur multiplier of* G is the abelian group $H^2(G; \mathbb{C}^\times) = \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}; \mathbb{C}^\times)$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is the the multiplicative group of nonzero complex numbers considered with the trivial action of G.

The Schur multiplier $H^2(G; \mathbb{C}^\times)$ of a finite group G parametrises central extensions of G. Moreover, $H^2(G; \mathbb{C}^\times)$ is a finite abelian group. Slightly more precisely, we have the following result.

Theorem 6.7. *Let G be a finite group and k an algebraically closed field. Then the abelian group $H^2(G; k^\times)$ is finite, where we consider the multiplicative group $k^\times = k \setminus \{0\}$ with G acting trivially. Moreover, $|G|$ annihilates $H^2(G; k^\times)$.*

Proof. From the description of group cohomology in Theorem 6.2 written multiplicatively, we get that $H^2(G; k^\times) = Z/B$, where

$$Z = \{\alpha : G \times G \to k^\times \mid \alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) \text{ for all } x, y, z \in G\}$$

$$B = \{\alpha : G \times G \to k^\times \mid \text{there exists } \beta : G \to k^\times \text{ such that } \alpha(x, y) = \beta(x)\beta(y)\beta(xy)^{-1} \text{ for all } x, y \in G\}$$

Let $\alpha \in Z$. Define a map $\mu : G \to k^\times$ by setting

$$\mu(x) = \prod_{y \in G} \alpha(x, y)$$

for all $x \in G$. For x, y, $z \in G$, we have the 2-cocycle identity

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z) \ .$$

Taking the product over all $z \in G$ yields the identity

$$\mu(xy)\alpha(x, y)^{|G|} = \mu(x)\mu(y) \ .$$

Thus

$$\alpha(x,y)^{|G|} = \mu(x)\mu(y)\mu(xy)^{-1}$$

showing that $\alpha^{|G|} \in B$, and hence $|G|$ annihilates $Z/B = H^2(G;k^\times)$. Since k is algebraically closed, every element in $k^\times$ has a $|G|$-th root, and hence there is a map $\delta : G \to k^\times$ such that $\delta(x)^{|G|} = \mu(x)$ for all $x \in G$. Define a map $\beta : G \times G \to k^\times$ by setting

$$\beta(x,y) = \alpha(x,y)\delta(x)^{-1}\delta(y)^{-1}\delta(xy)$$

for all x, $y \in G$. Then α and β represent the same class in $H^2(G;k^\times)$ because they differ by an element in the subgroup B of Z. And now we also have $\beta(x,y)^{|G|} = 1$ for all x, $y \in G$. Thus β is a map from $G \times G$ to the finite subgroup of $|G|$-th roots of unity in $k^\times$, so there are only finitely many such maps β. Thus $H^2(G;k^\times)$ is a quotient of the finite subgroup

$$U = \{\beta : G \times G \to k^\times \mid \beta(x,y)^{|G|} = 1 \text{ for all } x,y \in G\}$$

of Z, and the result follows. □

Proposition 6.8. *Let G be a finite group. The functor $- \otimes_{kG} k$ induces a homormophism of graded algebras $HH^*(kG) \to H^*(G;k)$ which is split surjective.*

Proof. The splitting is constructed as follows: consider the diagonal subgroup $\Delta G = \{(x,x) \mid x \in G\}$ of $G \times G$. Then the induced module $\mathrm{Ind}_{\Delta G}^{G\times G}(k) = k(G \times G) \otimes_{k\Delta G} k$ is isomorphic to kG viewed as a $k(G \times G)$ module with $(x,y) \in G \times G$ acting on $z \in kG$ by xzy^{-1}. Through the isomorphism $k(G \times G) \cong kG \otimes_k (kG)^{\mathrm{op}}$ sending (x,y) to $x \otimes y^{-1}$, this is kG viewed as a kG-kG-bimodule. The induction functor $\mathrm{Ind}_{\Delta G}^{G\times G}$ is exact and maps projective kG-modules to projective $k(G \times G)$-modules, hence sends a projective resolution of the trivial kG-module to a projective resolution of the bimodule kG. It follows that this functor induces a map $H^*(G;k) \to HH^*(kG)$, andone checks that this is a splitting as stated. □

Proposition 6.9. *Let G be a finite group. We have an isomorphism of graded k-modules*

$$H^*(kG) = \oplus_x \; H^*(C_G(x);k) \;,$$

where x runs over a set of representatives of the conjugacy classes in G.

This isomorphism is not an isomorphism as graded algebras. The product in $HH^*(kG)$ can be expressed explicitly in terms of the cohomology rings of centralisers of elements; this is due to Siegel and Witherspoon [8].

Remark 6.10. Using the above additive decomposition of $HH^*(kG)$ and the classification of finite simple groups, one can show that $HH^1(kG)$ is nonzero whenever k is a field of prime characteristic dividing the order of G.

7 Singular cohomology of topological spaces

Definition 7.1. For n a non negative integer, the *standard topological n-simplex* is the compact subspace of $\mathbb{R}^{n+1}$ defined by

$$\Delta_n = \{(x_0, x_1, .., x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ \sum_{i=0}^{n} x_i = 1\}$$

For $0 \leq i \leq n$ the *i-th face map* is the map

$$d_i : \Delta_{n-1} \to \Delta_n$$

sending $(x_0, x_1, .., x_{n-1}) \in \Delta_{n-1}$ to $(x_0, x_1, .., x_{i-1}, 0, x_i, .., x_{n-1}) \in \Delta_n$.

Thus if X is a topological space and n a positive integer, then precomposing with any of the $n+1$ different face maps sends any continuous map $\Delta_n \to X$ to a continuous map $\Delta_{n-1} \to X$. Assembling these maps yields singular homology.

Definition 7.2. Let X be a topological space. For $n \geq 0$ denote by $S_n(X)$ the free $\mathbb{Z}$-module having as a basis the set of all continuous maps $f : \Delta_n \to X$. For $n \geq 1$ define a k-linear map $\delta_n : S_n(X) \to S_{n-1}(X)$ by setting

$$\delta_n(f) = \sum_{i=0}^{n-1} (-1)^i f \circ d_i$$

for any continuous map $f : \Delta_n \to X$, where $d_i : \Delta_{n-1} \to \Delta_n$ is the face map defined above. We set $\delta_0 = 0$ for notational convenience. One checks that $\delta_n \circ \delta_{n+1} = 0$. We denote by $S_*(X)$ the chain complex thus obtained. The n-th *singular homology of X* is the $\mathbb{Z}$-module

$$H_n(X) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$$

For A an abelian group, we set $S^n(X; A) = \mathrm{Hom}_{\mathbb{Z}}(S_n(X), A)$ and denote by $\delta^n : S^n(X; A) \to S^{n+1}(X; A)$ the map induced by precomposition with δ_{n+1}, with the notational conovention $\delta^{-1} = 0$. The n-th *singular cohomology of X with coefficients in A* is the $\mathbb{Z}$-module

$$H^n(X; k) = \ker(\delta^n)/\mathrm{Im}(\delta^{n-1})$$

Exercise 7.3. Let $X = \{*\}$ be a single point. Show that $S_n(X) = \mathbb{Z}$ for $n \geq 0$, and that $\delta_n = \mathrm{Id}_{\mathbb{Z}}$ for n even and $\delta_n = 0$ for n odd. Show that there is a chain homotopy equivalence $S_*(X) \simeq \mathbb{Z}$, where $\mathbb{Z}$ is regarded as the chain complex concentrated in degree 0. Deduce that $H_n(X)$ is zero for $n > 0$ and $\mathbb{Z}$ for $n = 0$.

Definition 7.4. Let X, Y be topological spaces and let f, $g : X \to Y$ be continuous maps. We say that f and g are homotopic and write $f \sim g$ if there is a continuous map $F : [0,1] \times X \to Y$, such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$.

A continuous map $F : X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. In that case we say that X and Y are homotopy equivalent, and we write $X \simeq Y$. A space X is called *contractible* if it is homotopy equivalent to a point $\{*\}$. The following two theorems, stated here without proof (which can be found in many standard sources on algebraic topology), collect some of the fundamental properties of singular homology.

Theorem 7.5.

(1) *Singular homology is functorial; that is, for $n \geq 0$, any continuous map $f : X \to Y$ induces a canonical chain map $S(f) : S_*(X) \to S_*(Y)$ by composition with f and hence a map $H_n(f) : H_n(X) \to H_n(Y)$ both of which are functorial.*

(2) *Singular homology is compatible with homotopy; that is, if $f, g : X \to Y$ are homotopic continuous maps, then the induced chain maps $S(f), S(g) : S_*(X) \to S_*(Y)$ are homotopic chain maps, and we have $H_n(f) = H_n(g)$ for all $n \geq 0$.*

(3) *If $f : X \to Y$ is a homotopy equivalence, then $S(f) : S_*(X) \to S_*(Y)$ is a chain homotopy equivalence. In particular, if X is contractible, then $S_*(X) \simeq \mathbb{Z}$.*

For X a topological space and A a subspace, we denote by A° the set of interior points in A; this is the set of all elements $a \in A$ which have an open neighbourhood in X which is contained in A.

Theorem 7.6 (Mayer-Vietoris)**.** *Let X be a topological space and A, B subspaces of X such that $A^\circ \cup B^\circ = X$. There is a long exact singular homology sequence*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

ending at the map $H_0(X) \to 0$.

Given a sphere $S^n = \{(x_0, x_1, .., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\}$, we can use the above results to calculate its cohomology.

Theorem 7.7. *For $n \geq 0$ and $i > 0$ such that $i \neq n$ we have $H_i(S^n) = 0$. For $n > 0$ we have $H_n(S^n) \cong H_0(S^n) \cong \mathbb{Z}$, and we have $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Since $S^0 = \{-1, 1\}$ is the disjoint unsion of two one-point spaces, we have $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_n(S^0) =)$ for $n > 0$. Whenever we puncture a sphere S^n, $n > 0$, by removing a single point, we obtain a contractible space. Denote by A_n the subspace of S^n obtained by removing the 'south pole' $(0, 0, .., 0, -1)$ and by B_n the space obtained by removing the 'north pole' $(0, 0, .., 0, 1)$ of S^n. For $n = 1$, the Mayer-Vietoris sequence takes the following form

$$0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0$$

The map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is easily seen to be nonzero but not injective, so has a kernel and cokernel isomorphic to $\mathbb{Z}$, which shows that $H_1(S^1) \cong H_0(S^1) \cong \mathbb{Z}$. For $n \geq 1$, one shows that the intersection $A_n \cap B_n$ is homotopic to S^{n-1}. Thus using the Mayer-Vietoris repeatedly proves the theorem. □

We illustrate some of the concepts and methods with an example describing the use of homology to prove Brouwer's fixpoint theorem.

Theorem 7.8 (Brouwer's fixpoint theorem)**.** *Let m be a positive integer. Any continuous map $f : D^m \to D^m$ has a fixpoint (that is, there is $x \in D^m$ such that $f(x) = x$).*

Proof. For $m = 1$ this is an easy application of the intermediate value theorem. Suppose that $m \geq 2$. Suppose the theorem is not true; that is, there is a continuous map $f : D^m \to D^m$ such that $f(x) \neq x$ for any $x \in D^m$. Since $f(x)$ is a point different from x in the disc D^m, there is a unique line starting from $f(x)$ and passing through x. This line intersects the boundary S^{m-1} of D^m in a point which we denote by $g(x)$. One verifies that the map $x \mapsto g(x)$ is a continuous map $g : D^m \to S^{m-1}$. If x is on that boundary S^{m-1}, then clearly $g(x) = x$. That is, $g : D^m \to S^{m-1}$ restricts to the identity map on S^{m-1}. In other words, if we denote by $a : S^{m-1} \to D^m$ the inclusion map, then $g \circ a = \mathrm{Id}_{S^{m-1}}$. Applying H_n for any $n \geq 0$ and using the functoriality properties of H_n yields $H_n(g) \circ H_n(a) = \mathrm{Id}_{H_n(S^{m-1})}$. For $n = m - 1$ this yields a contradiction: the right side is the identity map on the nonzero abelian group $H_{m-1}(S^{m-1})$, but the map $H_{m-1}(a)$ is zero, because $H_{m-1}(D^m)$ is zero, and so $H_n(g) \circ H_n(a)$ is zero as well. □

There are other statements that can be proved in a similar way: for $n > 0$, the sphere S^{n-1} is not contractible, two spheres S^{n-1}, S^{m-1} for positive integers m, n are isomorphic if and only if $n = m$, and $\mathbb{R}^n$, R^m are isomorphic as topological spaces if and only if $n = m$.

8 Triangulated categories

A triangulated category is an additive category with an additional structure of *exact triangles*, which should be thought of as a replacement for short exact sequences. This concept, which we will introduce in the present section, has been developed independently by J.L. Verdier, and, in a topological context, by D. Puppe. We will show in the next section that homotopy categories of chain complexes are triangulated.

Given an additive category $\mathcal{C}$ and an additive functor $\Sigma : \mathcal{C} \to \mathcal{C}$ on $\mathcal{C}$, we call a *triangle in* $\mathcal{C}$ a sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

where X, Y, Z are objects in $\mathcal{C}$ and f, g, h are morphisms in $\mathcal{C}$. The triangles in $\mathcal{C}$ form the objects of a category: a *morphism of triangles* is a triple (u, v, w) of morphisms in $\mathcal{C}$ making the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma(u)} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma(X')
\end{array}$$

commutative, where the two rows are triangles in $\mathcal{C}$.

Definition 8.1. A *triangulated category* is a triple $(\mathcal{C}, \Sigma, \mathcal{T})$ consisting of an additive category $\mathcal{C}$, a covariant additive self equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$ and a class $\mathcal{T}$ of triangles in $\mathcal{C}$ - called *exact* or sometimes also *distinguished triangles* in $\mathcal{C}$ - fulfilling the axioms T1, T2, T3, T4 below.

T1: For any object X in $\mathcal{C}$, the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0$ is exact (i.e., belongs to the class $\mathcal{T}$), for any morphism $f : X \to Y$ in $\mathcal{C}$ there is an exact triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

for some object Z in $\mathcal{C}$ and some morphisms g, h, any triangle in $\mathcal{C}$ which is isomorphic to an exact triangle is itself exact (i.e., the class $\mathcal{T}$ is closed under isomorphisms).

T2: Any commutative diagram in $\mathcal{C}$ of the form

$$\begin{CD}
X @>{f}>> Y @>{g}>> Z @>{h}>> \Sigma(X) \\
@V{u}VV @VV{v}V @. @. \\
X' @>>{f'}> Y' @>>{g'}> Z' @>>{h'}> \Sigma(X')
\end{CD}$$

whose rows are exact triangles, can be completed to a commutative diagram

$$\begin{CD}
X @>{f}>> Y @>{g}>> Z @>{h}>> \Sigma(X) \\
@V{u}VV @VV{v}V @VV{w}V @VV{\Sigma(u)}V \\
X' @>>{f'}> Y' @>>{g'}> Z' @>>{h'}> \Sigma(X')
\end{CD}$$

for some morphism w.

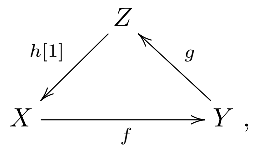
T3: If the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in $\mathcal{C}$ is exact, so is the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y)$$

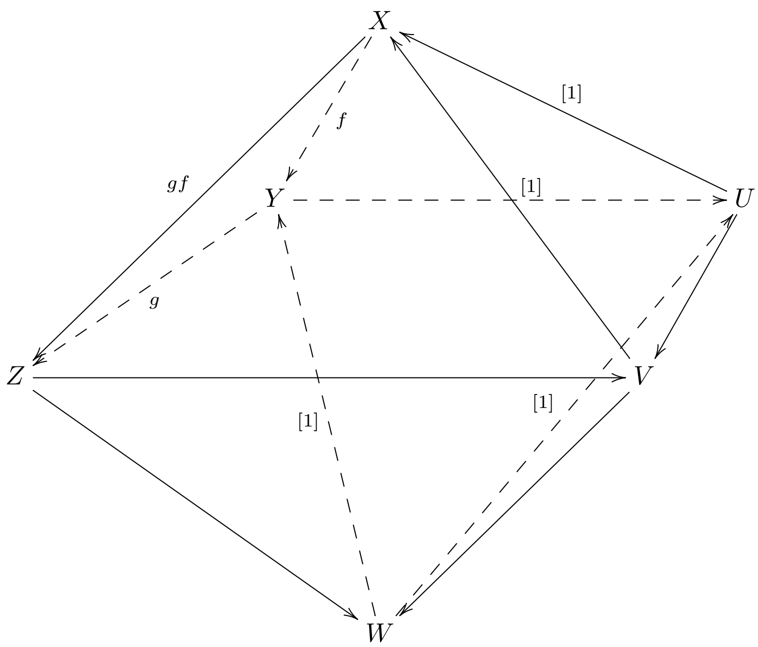
T4: Given any sequence of two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ there is a commutative diagram in $\mathcal{C}$ whose first two rows and middle two columns are exact triangles:

$$\begin{CD}
X @>{f}>> Y @>{k}>> U @>>> \Sigma(X) \\
@| @V{g}VV @VV{u}V @| \\
X @>>{gf}> Z @>>> V @>>> \Sigma(X) \\
@. @VVV @V{v}VV @VV{\Sigma(f)}V \\
@. W @= W @>>{w}> \Sigma Y \\
@. @VV{w}V @VVV @. \\
@. \Sigma(Y) @>>{\Sigma(k)}> \Sigma(U) @.
\end{CD}$$

Remark 8.2. The axiom T4 describes in which way the three triangles over f, g, $g \circ f$ are connected. This axiom is called the *octahedral axiom* for the following reason: if we rewrite a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ in the form



where [1] means that h "is of degree 1", then the diagram in T4 takes the following form:



Proposition 8.3. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$ *be an exact triangle in a triangulated category* $\mathcal{C}$*, and let* U *be an object in* $\mathcal{C}$*.*

(i) We have $g \circ f = 0$ *and* $h \circ g = 0$*.*

(ii) Given any morphism $j : Y \to U$ *there is a morphism* $i : Z \to U$ *satisfying* $i \circ g = j$ *if and only if* $j \circ f = 0$*.*

(iii) Given any morphism $q : U \longrightarrow Y$ *there is a morphism* $p : U \to X$ *satisfying* $f \circ p = q$ *if and only if* $g \circ q = 0$*.*

Proof. By T1 the triangle $0 \longrightarrow X = X \longrightarrow 0$ is exact, and hence, by T3, the triangle $X = X \longrightarrow 0 \longrightarrow \Sigma(X)$ is exact. Applying T2 yields the existence of a commutative diagram

$$\begin{CD} X @= X @>>> 0 @>>> \Sigma(X) \\ @| @VV f V @VVV @| \\ X @>>f> Y @>>g> Z @>>> \Sigma(X) \end{CD}$$

which shows that $g \circ f = 0$. The same argument, after turning the triangle by means of T3, shows that $h \circ g = 0$, whence (i). If $j \circ f = 0$, then it follows from T1 and T2 that we have a commutative diagram

$$\begin{CD} X @>f>> Y @>g>> Z @>h>> \Sigma(X) \\ @VVV @V j VV @V i VV @VVV \\ 0 @>>> U @= U @>>> 0 \end{CD}$$

for some morphism i, which means that $i \circ g = j$. Conversely, if there is a morphism $i : Z \to U$ such that $i \circ g = j$, then $j \circ f = i \circ g \circ f = 0$, since $g \circ f = 0$ by (i). This proves (ii). The last statement is proved by applying a dual argument to the exact triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$. □

Lemma 8.4. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category, and let*

$$\begin{CD} @. Y @>g>> Z \\ @. @V v VV @VV 0 V \\ X' @>f'>> Y' @>g'>> Z' @>h'>> \Sigma(X') \\ @V 0 VV @VV v' V \\ X'' @>>f''> Y'' \end{CD}$$

be a commutative diagram in $\mathcal{C}$ whose middle row is an exact triangle. We have $v' \circ v = 0$.

Proof. Since $g' \circ v = 0$ there is, by 8.3, a morphism $p : Y \to X'$ such that $f' \circ p = v$. Similiarly, since $v' \circ f' = 0$ there is a morphism $i : Z' \to Y''$ such that $v' = i \circ g'$. Together we obtain $v' \circ v = i \circ g' \circ f \circ p = 0$, since $g' \circ f' = 0$ by 8.3. □

Proposition 8.5. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let*

$$\begin{CD} X @>f>> Y @>g>> Z @>h>> \Sigma(X) \\ @V u VV @V v VV @V w VV @VV \Sigma(u) V \\ X' @>>f'> Y' @>>g'> Z' @>>h'> \Sigma(X') \end{CD}$$

be a commutative diagram in $\mathcal{C}$ whose rows are exact triangles. If u, v are isomorphisms, so is w.

Proof. By applying T2 to u^{-1} and v^{-1} we may assume that $X = X'$, $Y = Y'$, $Z = Z'$, $f = f'$, $g = g'$, $h = h'$, $u = \mathrm{Id}_X$, and $v = \mathrm{Id}_Y$. Thus we are down to considering the endomorphism $(\mathrm{Id}_X, \mathrm{Id}_Y, w)$ of the exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$, and we have to show that w is an automorphism. Clearly $(\mathrm{Id}_X, \mathrm{Id}_Y, \mathrm{Id}_Z)$ is an endomorphism of this triangle, too, thus taking the difference of the two endomorphisms yields an endomorphism $(0, 0, \mathrm{Id}_Z - w)$. Using T3 and 8.4 shows that $(\mathrm{Id}_Z - w)^2 = 0$, or equivalently, $\mathrm{Id}_Z = w \circ (2\mathrm{Id}_Z - w)$, which implies that w is invertible with inverse $2\mathrm{Id}_Z - w$. □

Corollary 8.6. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category. If the triangle*

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y)$$

is exact in $\mathcal{C}$, so is the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

Proof. By T1, there is an exact triangle in $\mathcal{C}$ of the form $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X)$. We turn this triangle three times, and the first triangle in the statement twice; this yields two exact triangles in $\mathcal{C}$

$$\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g'} \Sigma Z' \xrightarrow{-\Sigma h'} \Sigma^2(X)$$

$$\Sigma(X) \xrightarrow{-\Sigma f} \Sigma(Y) \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2(X)$$

and by Proposition 8.5, these two triangles are isomorphic. Since Σ is an equivalence, it follows that the triangles

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X)$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} \Sigma(X)$$

are isomorphic, and as the first one is exact, so is the second by T1. □

Corollary 8.7. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be an exact triangle in $\mathcal{C}$.

(i) f is an isomorphism if and only if $Z = 0$.

(ii) g is an isomorphism if and only if $X = 0$.

(iii) h is an isomorphism if and only if $Y = 0$.

Proof. If f is an isomorphism, then, by 8.5 and T1, the given exact triangle is isomorphic to the exact triangle $X = X \longrightarrow 0 \longrightarrow \Sigma(X)$, thus $Z = 0$. Conversely, if $Z = 0$, turning the triangle by T3 shows that $\Sigma(f)$ is an isomorphism, and hence f is so, too, as Σ is an equivalence. This shows (i), and the other statements follow from (i) with T3 and the fact, that Σ is an equivalence. $\square$

Corollary 8.8. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be an exact triangle in $\mathcal{C}$. For any object U in $\mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(U, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, U)$ induce long exact sequences of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(X)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(Y)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(Z)) \to \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^{n+1}(X)) \to \cdots$$

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^{n+1}(X), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(Z), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(Y), U) \to \mathrm{Hom}_{\mathcal{C}}(\Sigma^n(X), U) \to \cdots$$

Proof. By 8.3, the sequence $\mathrm{Hom}_{\mathcal{C}}(U, X) \to \mathrm{Hom}_{\mathcal{C}}(U, Y) \to \mathrm{Hom}_{\mathcal{C}}(U, Z)$ is exact. Turning the triangle by means of T3 and its converse 8.6 yields the first of the two long exact sequences. An analogous argument shows the exactness of the second sequence. $\square$

Proposition 8.9. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a triangulated category and let $f : X \to Y$ be a morphism in $\mathcal{C}$.*

(i) f is an epimorphism if and only if f has a right inverse.

(ii) f is a monomorphism if and only if f has a left inverse.

(iii) f is an isomorphism if and only if f is both an epimorphism and a monomorphism.

Proof. If f has a right inverse, f is trivially an epimorphism. Conversely, suppose that f is an epimorphism. Consider an exact triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

Since $g \circ f = 0$ and since f is an epimorphism, we have $g = 0$. Thus $g \circ \mathrm{Id}_Y = 0$. Thus there is $r : Y \to X$ such that $f \circ r = \mathrm{Id}_Y$, which shows that r is a right inverse of f. This shows (i), and a dual argument shows statement (ii). If f is both an epimorphism and a monomorphism, it has a right inverse r and a left inverse l, and then $l = l \circ f \circ r = r$, which shows that f is an isomorphism. The converse in (iii) is trivial. $\square$

This shows that epimorphisms and monomorphisms in a triangulated category are all split. As a consequence, a triangulated category is abelian if and only if it is semisimple. There are, however, nontrivial examples of proper abelian subcategories of triangulated categories arising in the context of central p-extensions of finite groups. See [7] for a systematic treatment of triangulated categories.

9 Homotopy categories are triangulated

If $\mathcal{C}$ is an additive category, then the chain homotopy category $K(\mathcal{C})$ together with the shift functor [1] is triangulated. To see this we need to define a suitable class of exact triangles.

Definition 9.1. Let $\mathcal{C}$ be an additive category and let $f : (X, \delta) \to (Y, \epsilon)$ be a chain map of complexes over $\mathcal{C}$. The *mapping cone of f* is the complex $C(f)$ over $\mathcal{C}$ given by $C(f)_n = X_{n-1} \oplus Y_n$ with differential Δ

$$\Delta_n = \begin{pmatrix} -\delta_{n-1} & 0 \\ f_{n-1} & \epsilon_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X_{n-2} \oplus Y_{n-1}$$

for any integer n. The mapping cone comes along with canonical chain maps $i(f) : Y \to C(f)$ given by the canonical monomorphims $Y_n \hookrightarrow X_{n-1} \oplus Y_n$ and $p(f) : C(f) \to X[1]$ given by the canonical projections $X_{n-1} \oplus Y_n \twoheadrightarrow X_{n-1} = X[1]_n$ for any integer n.

One checks that $\Delta \circ \Delta = 0$. The *cone $C(X)$ of a chain complex X* is the mapping cone of the identity chain map Id_X.

Associated with any chain map $f : X \to Y$ we have a triangle in $(\mathrm{Ch}(\mathcal{C}), [1])$ given by the "mapping cone sequence"

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{p(f)} X[1]$$

and we denote by $\mathcal{T}$ the class of all triangles in $K(\mathcal{C})$ isomorphic to the image of a triangle in $\mathrm{Ch}(\mathcal{C})$ of this form. We are going to show that this yields a structure of a triangulated category for $K(\mathcal{C})$.

Theorem 9.2. *Let $\mathcal{C}$ be an additive category. Then $K(\mathcal{C})$, endowed with the shift functor* [1] *and the class $\mathcal{T}$ of triangles induced by mapping cone sequences, is a triangulated category. Moreover, the categories $K^+(\mathcal{C})$, $K^-(\mathcal{C})$, $K^b(\mathcal{C})$ are full triangulated subcategories in $K(\mathcal{C})$.*

Axiom T1 holds trivially: by definition, any chain map is part of a mapping cone sequence, and the mapping cone of the zero map $0 \to X$ is just X again, so the mapping cone sequence degenerates to $0 \to X \to X \to 0$. The following Proposition shows, that T2 holds:

Proposition 9.3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

be a commutative diagram of chain complexes over an additive category $\mathcal{C}$. Then there is a chain map $w : C(f) \to C(f')$ making the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{p(f)} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[i(f')]{} & C(f') & \xrightarrow[i(f')]{} & X'[1] \end{array}$$

homotopy commutative.

Proof. Since $v \circ f \sim f' \circ u$, there is a homotopy $h : X \to Y'$ such that $v \circ f - f' \circ u = \epsilon' \circ h + h \circ \delta$, where δ and ϵ' are the differentials of X and Y', respectively. Set

$$w_n = \begin{pmatrix} u_{n-1} & 0 \\ h_{n-1} & v_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X'_{n-1} \oplus Y'_n$$

for any $n \in \mathbb{Z}$. A straightforward verification shows that $w = (w_n)_{n \in \mathbb{Z}}$ is a chain map from $C(f)$ to $C(f')$ which makes the middle and right square in the above diagram commutative. □

The next Proposition describes in which way the mapping cones $C(f)$, $C(i(f))$, $C(p(f))$ are connected, implying in particular, that the axiom T3 holds for the class $\mathcal{T}$ in $K(\mathcal{C})$:

Proposition 9.4. *Let $f : X \to Y$ be a chain map of chain complexes over an additive category $\mathcal{C}$. Denote by $q(f) : C(i(f)) \to X[1]$ the graded map given by the canonical projections $q(f)_n = (0, \mathrm{Id}_{X_{n-1}}, 0) : Y_{n-1} \oplus (X_{n-1} \oplus Y_n) \to X_{n-1}$, for any $n \in \mathbb{Z}$. Denote by $s(f) : C(p(f)) \to Y[1]$ the graded map given by $s(f)_n = (0, \mathrm{Id}_{Y_{n-1}}, f_{n-1}) : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \to Y_{n-1}$, for any $n \in \mathbb{Z}$. Then $q(f)$ and $s(f)$ are homotopy equivalences making the following diagram of chain complexes homotopy commutative:*

$$\begin{array}{ccccccccccc}
 & & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{i(i(f))} & C(i(f)) & \xrightarrow{p(i(f))} & Y[1] & & \\
 & & \| & & \| & & \downarrow{\scriptstyle q(f)} & & \| & & \\
X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{p(f)} & X[1] & \xrightarrow{-f[1]} & Y[1] & \xrightarrow{-i(f)[1]} & C(f)[1] \\
 & & & & \| & & \| & & \uparrow{\scriptstyle -s(f)} & & \| \\
 & & & & C(f) & \xrightarrow[p(f)]{} & X[1] & \xrightarrow[i(p(f))]{} & C(p(f)) & \xrightarrow[p(p(f))]{} & C(f)[1]
\end{array}$$

Proof. The verification, that both $q(f)$, $s(f)$ are chain maps, is straightforward. We construct homotopy inverses $r(f)$, $t(f)$ of $q(f)$, $s(f)$, respectively, as follows. Set

$$r(f)_n = \begin{pmatrix} -f_{n-1} \\ \mathrm{Id}_{X_{n-1}} \\ 0 \end{pmatrix} : X_{n-1} \to Y_{n-1} \oplus X_{n-1} \oplus Y_n$$

for any integer n. Then $r(f)$ is a chain map satisfying $q(f) \circ r(f) = \mathrm{Id}_{X[1]}$. In order to show that $r(f) \circ q(f) \sim \mathrm{Id}_{C(i(f))}$, we define the homotopy h on $C(i(f))$ by

$$h_n = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{Y_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : Y_{n-1} \oplus X_{n-1} \oplus Y_n \to Y_n \oplus X_n \oplus Y_{n+1}$$

for any integer n. If Δ denotes the differential of $C(i(f))$, we have $\Delta \circ h + h \circ \Delta = \mathrm{Id}_{C(i(f))} - r(f) \circ q(f)$. This shows also the homotopy commutativity of the upper part of the diagram, since $q(f) \circ i(i(f)) = p(f)$ and $p(i(f)) \circ r(f) = -f[1]$. We proceed similarly for $t(f)$. Set

$$t(f)_n = \begin{pmatrix} 0 \\ \mathrm{Id}_{Y_{n-1}} \\ 0 \end{pmatrix} : Y_{n-1} \to X_{n-2} \oplus Y_{n-1} \oplus X_{n-1}$$

for any integer n. Clearly $t(f)$ is a chain map satisfying $s(f) \circ t(f) = \mathrm{Id}_{Y[1]}$. In order to show that $t(f) \circ s(f) \sim \mathrm{Id}_{C(p(f))}$, we define the homotopy k on $C(p(f))$ by

$$k_n = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{X_{n-1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \to X_{n-1} \oplus Y_n \oplus X_n$$

for any integer n. If Π denotes the differential of $C(p(f))$, we have $\Pi \circ k + k \circ \Pi = \mathrm{Id}_{C(p(f))} - t(f) \circ s(f)$. This shows also the homotopy commutativity of the lower part of the diagram, since $s(f) \circ p(f) = f[1]$ and $p(p(f)) \circ t(f) = i(f)[1]$. □

It remains to show, that the octahedral axiom T4 holds:

Proposition 9.5. *Given two composable chain maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *of complexes over an additive category* $\mathcal{C}$, *there are chain maps* u, v *making the following diagram of chain complexes commutative*

$$\begin{CD}
X @>f>> Y @>i(f)>> C(f) @>p(f)>> X[1]\\
@| @VgVV @VuVV @|\\
X @>>gf> Z @>>i(gf)> C(gf) @>>p(gf)> X[1]\\
@. @Vi(g)VV @VvVV @VVf[1]V\\
@. C(g) @= C(g) @>>p(g)> Y[1]\\
@. @Vp(g)VV @VVwV\\
@. Y[1] @>>i(f)[1]> C(f)[1]
\end{CD}$$

and there is a homotpy equivalence $t(u) : C(g) \to C(u)$ *such that the diagram*

$$\begin{CD}
C(f) @>u>> C(gf) @>v>> C(g) @>w>> C(f)[1]\\
@| @| @VVt(u)V @|\\
C(f) @>>u> C(gf) @>>i(u)> C(u) @>>p(u)> C(f)[1]
\end{CD}$$

is homotopy commutative.

Proof. For any $n \in \mathbb{Z}$ set

$$u_n = \begin{pmatrix} \mathrm{Id}_{X_{n-1}} & 0 \\ 0 & g_n \end{pmatrix} : X_{n-1} \oplus Y_n \to X_{n-1} \oplus Z_n \quad ,$$

$$v_n = \begin{pmatrix} f_{n-1} & 0 \\ 0 & \mathrm{Id}_{Z_n} \end{pmatrix} : X_{n-1} \oplus Z_n \to Y_{n-1} \oplus Z_n \quad ,$$

$$w_n = \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{Y_{n-1}} & 0 \end{pmatrix} : Y_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1} \quad .$$

A straightforward verification shows that $u = (u_n)_{n\in\mathbb{Z}}$, $v = (v_n)_{n\in\mathbb{Z}}$ and $w = (w_n)_{n\in\mathbb{Z}}$ are chain maps which make the first diagram in the statement commutative. For $t(u)$ we take the morphism given by the obvious split monomorphisms

$$Y_{n-1} \oplus Z_n \longrightarrow (X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n)$$

and we define a morphism $s(u) : C(u) \to C(g)$ given by the projections

$$(X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n) \longrightarrow Y_{n-1} \oplus Z_n$$

for any $n \in \mathbb{Z}$. Then $s(u) \circ t(u) = \mathrm{Id}_{C(g)}$, and it remains to show that $t(u) \circ s(u) \sim \mathrm{Id}_{C(u)}$. For this we consider on $C(u)$ the homotopy h given, for any $n \in \mathbb{Z}$, by the map

$$h_n : (X_{n-2} \oplus Y_{n-1}) \oplus (X_{n-1} \oplus Z_n) \to (X_{n-1} \oplus Y_n) \oplus (X_n \oplus Z_{n+1}) \quad ,$$

where h_n is zero on the summands X_{n-2}, Y_{n-1}, Z_n, and h_n maps X_{n-1} identically to its canonical image in $C(u)_{n+1}$. Then in the second diagram in the statement, the left and middle square are commutative. Clearly $p(v) \circ s(u) = p(u)$; thus the right square is homotopy commutative, as $s(u)$ is a homotopy inverse to $t(u)$. □

This completes the proof of Theorem 9.2. We note some immediate consequences.

Corollary 9.6. *Let $\mathcal{C}$ be an additive category, let $f : X \to Y$ be a chain map of complexes over $\mathcal{C}$, and consider the mapping cone sequence* $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{p(f)} X[1]$.

(i) We have $i(f) \circ f \sim 0$.

(ii) f is a homotopy equivalence if and only if $C(f) \simeq 0$.

(iii) $i(f)$ is a homotopy equivalence if and only if $X \simeq 0$.

(iv) $p(f)$ is a homotopy equivalence if and only if $Y \simeq 0$.

(v) If two of X, Y, $C(f)$ are homotopic to zero, so is the third.

Proof. Statement (i) follows from 8.3 (i), but one can see this also directly: the canonical monomorphisms $X_n \hookrightarrow X_n \oplus Y_{n+1}$ define a homotopy $h : X \longrightarrow C(f)$ through which $i(f) \circ f$ becomes homotopic to the zero map. The statements (ii), (iii), (iv) are all particular cases of 8.7. Finally, (v) follows from (ii) and (iii). □

Corollary 9.7. *Let $\mathcal{C}$ be an additive category, $f : X \to Y$ a chain map of complexes over $\mathcal{C}$, and let U be a complex over $\mathcal{C}$.*

(i) The covariant functor $\mathrm{Hom}_{K(\mathcal{C})}(U, -)$ induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\mathcal{C})}(U, X[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, Y[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, C(f)[n]) \to \mathrm{Hom}_{K(\mathcal{C})}(U, X[n{+}1]) \to \cdots$$

(ii) The contravariant functor $\mathrm{Hom}_{K(\mathcal{C})}(-, U)$ induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{K(\mathcal{C})}(X[n{+}1], U) \to \mathrm{Hom}_{K(\mathcal{C})}(C(f)[n], U) \to \mathrm{Hom}_{K(\mathcal{C})}(Y[n], U) \to \mathrm{Hom}_{K(\mathcal{C})}(X[n], U) \to \cdots$$

Proof. This is a particular case of 8.8. □

Corollary 9.8. *Let A be an algebra over a commutative ring k and let $f : X \to Y$ be a chain map of complexes of A-modules. Taking homology induces a long exact sequence of A-modules*

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(C(f)) \to H_{n-1}(X) \to \cdots$$

Proof. By 3.20 we have $H_n(X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A, X[-n])$, thus the statement follows from the first of the two long exact sequences in the previous corollary applied to $U = A$. □

Corollary 9.9. *Let A be an algebra over a commutative ring k and let $f : X \to Y$ be a chain map of complexes of A-modules. The following are equivalent.*
(i) f is a quasi-isomorphism.
(ii) $C(f)$ is acyclic.
(iii) For any bounded below complex P of projective A-modules, the map f induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, Y)$.
(iv) For any bounded above complex I of injective A-modules, the map f induces an isomorphism $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(Y, I) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, I)$.

Proof. The equivalence of (i), (ii) follows from the long exact homology sequence in the previous corollary, and the equivalence with (iii), (iv) follows then from the long exact sequences in 9.7, together with the characterisation 3.19 of acyclic complexes, using the fact that $\mathrm{Mod}(A)$ has enough projective and injective objects. □

We have two ways of producing long exact sequences: via mapping cone sequences and via short exact sequences of complexes. Both approaches are equivalent in the sense that we can view mapping cone sequences as being induced by short exact sequences of complexes in the same way we defined triangles in a stable module category using short exact sequences of modules.

Theorem 9.10. *Let A be an algebra over a commutative ring k and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of chain complexes A-modules. The maps $s_n = (0, g_n) : X_{n-1} \oplus Y_n \to Z_n$ induce a quasi-isomorphism $s : C(f) \to Z$ making the diagram of chain complexes of A-modules

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & & \\
 & & \| & & \| & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0
\end{array}$$

commutative, and we have an isomorphism of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H_n(X) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(C(f)) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(Y) & \longrightarrow \cdots \\
 & \| & & \| & & \downarrow{\scriptstyle H_n(s)} & & \| & & \| & \\
\cdots \longrightarrow & H_n(X) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Z) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(Y) & \longrightarrow \cdots
\end{array}$$

where the first row is from 9.8 and the second row from 2.8. Moreover, if the first exact sequence of chain complexes is degreewise split, then s is a homotopy equivalence.

Proof. The commutativity of the diagram is a straightforward verification. One can verify either directly that s is a quasi-isomorphism (which, effectively, would yield another proof of Theorem 2.8), or use the 5-Lemma. Denote by δ, ϵ, ζ, Δ the differentials of X, Y, Z, $C(f)$, respectively. Suppose now that the first exact sequence of chain complexes in the statement is degree wise split; that is, there are graded morphisms $u : Y \to X$ and $v : Z \to Y$ satisfying $\mathrm{Id}_Y = f \circ u + v \circ g$. Note that then $f = f \circ u \circ f$, hence $u \circ f = \mathrm{Id}_X$ as f is a monomorphism; in particular, u is a retraction for f. Similarly, $g \circ v = \mathrm{Id}_Z$; in particular, v is a section for g. The morphism $v \circ \zeta - \epsilon \circ v : Z \to Y$ is graded of degree -1, and it this a chain map from Z to $Y[1]$, since

$$(-\epsilon) \circ (v \circ \zeta - \epsilon \circ v) = -\epsilon \circ v \circ \zeta = (v \circ \zeta - \epsilon \circ v) \circ \zeta$$

This chain map satisfies $g \circ (v \circ \zeta - \epsilon \circ v) = g \circ v \circ \zeta - \zeta \circ g \circ v = 0$, as $g \circ v = \mathrm{Id}_Z$. Whence this map factors through f. Let $r : Z \to X$ be the graded morphism of degree -1 such that $f \circ r = v \circ \zeta - \epsilon \circ v$. Since the right side is a chain map and f is a monomorphism, r itself can be viewed as a chain map from $Z[-1]$ to X. Consider the associated triangle

$$Z[-1] \xrightarrow{r} X \xrightarrow{i(r)} C(r) \xrightarrow{p(r)} Z$$

A straightforward verification shows that $C(r) \cong Y$ via the inverse chain maps given by the morphisms $(v_n, f_n) : Z_n \oplus X_n \to Y_n$ and $\begin{pmatrix} g_n \\ u_n \end{pmatrix} : Y_n \to Z_n \oplus X_n$ for any integer n. Thus $C(i(r)) \cong C(f)$. By 9.4, we have also a homotopy equivalence $q(r) : C(i(r)) \to Z$. Together, we obtain a homotopy equivalence $C(f) \simeq Z$, and this is easily seen to be the chain map s as defined. □

Corollary 9.11. *With the notation and hypotheses of 9.10, if $Y \simeq 0$ then $Z \simeq X[1]$.*

10 Spectral sequences

Spectral sequences, introduced by Jean Leray, are a sophisticated tool to calculate the (co-)homology of (co-)chain complexes in terms of a filtration by subcomplexes. If X is a subcomplex of a cochain complex Y, then the cohomology of Y is related to that of X and Y/X via the long exact homology sequence 2.8. More generally, if we have a filtration of a cochain complex X by subcomplexes F^iX such that F^iX is a subcomplex of $F^{i-1}X$, then the cohomology of X can be approximated in terms of the cohomology of the quotiens F^iX/F^jX, with $j > i$. A spectral sequence organises the data coming from such a filtration in a way which leads to calculating a filtration of the cohomology of X in terms of the given filtration of X itself. We have, of course, as always the dual version for the homology of chain complexes. We describe here cohomology spectral sequences and leave the translation of this material to homology sequences as an exercise. We fix an algebra A over a commutative ring k and describe spectral sequences of cochain complexes of A-modules; the adaptation to arbitrary abelian categories is straightforward.

Definition 10.1. A *cohomology spectral sequence starting at the page E_a* for some integer a is a triple graded family of A-modules $E = (E_r^{p,q})$, where $r \geq a$ and p, $q \in \mathbb{Z}$, together with A-homomorphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all r, p, q as before, together with an isomorphism between

$$E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1})$$

For a fixed r, the bigraded A-module $E_r^{*,*} = (E_r^{p,q})_{p,q\in\mathbb{Z}}$ together with the differentials $d_r^{p,q}$ is called the *E_r-page of the spectral sequence*. The number $p+q$ is called the *total degree* of the A-module $E_r^{p,q}$ in the spectral sequence. The cohomology spectral sequences starting at a fixed page E_a are the objects of a category in which a morphism of spectral sequences $(E_r^{p,q}) \to (F_r^{p,q})$ is a triple graded A-homomorphism $(f_r^{p,q} : E_r^{p,q} \to F_r^{p,q})$ which commute with the differentials and the isomorphisms $E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1})$.

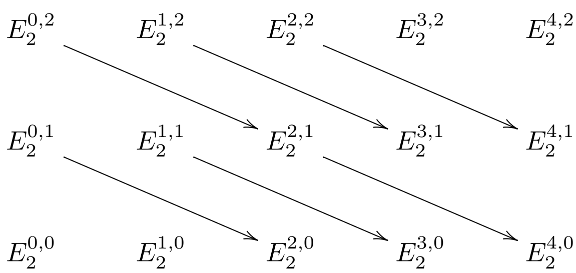
The E_r-page of a spectral sequence is a family of complexes with differentials of bidegree $(r, -r+1)$, and the passage from the E_r-page to the E_{r+1}-page is made by taking the cohomology of the involved complexes. Note that d^{p,q_r} goes from $E_r^{p,q}$ with total degree $p+q$ to $E_r^{p+r,q-r+1}$ with total degree $p+q+1$. For E a spectral sequence starting at the E_0 page, the pages E_0, E_1, E_2 can be visualised as lattices of A-modules together with their differentials as follows. The bidegree of the differential of E_0 is $(0,1)$.

$$\begin{array}{ccc}
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,0} & E_0^{1,0} & E_0^{2,0}
\end{array}$$

Any two vertical maps compose to zero, and the resulting cohomology is yields the terms of the next page E_1, whose differential has bidegree $(1,0)$.

$$\begin{array}{ccccc}
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} \\
 & & & & \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} \\
 & & & & \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0}
\end{array}$$

Again, any two consecutive horizontal maps compose to zero, and the resulting cohomology yields the terms of the next page E_2, whose differential has bidegree $(2, -1)$.



Observation: if for some fixed p, q we have $E_r^{p,q} = 0$, then also $E_s^{p,q}$ for all $s \geq r$. That is, any zero entry in a given page remains zero in all subsequent pages. More generally, the module $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$. If for any $(p,q) \in \mathbb{Z}^2$ the process of taking subquotients stabilises for some sufficiently large r, then we say that the spectral sequence converges. More precisely:

Definition 10.2. A cohomology spectral sequence $(E_r^{p,q})$ with start page E_a *converges* if for any $(p,q) \in \mathbb{Z}^2$ there exists $r \geq a$ such that $E_s^{p,q} = E_r^{p,q}$ for all $s \geq r$. In that case, we write $E_r^{p,q} = E_\infty^{p,q}$.

For a spectral sequence to be useful, convergence is a key property, and so quite some effort goes into developing sufficient criteria for a spectral sequence to converge. We consider the following two cases.

Definition 10.3. A spectral sequence $(E_r^{p,q})$ starting at the page E_a is called *bounded* if for any $n \in \mathbb{Z}$ there are only finitely many $(p,q) \in \mathbb{Z}$ such that $p + q = n$ and $E_a^{p,q} \neq 0$.

As remarked earlier, this implies that for all $r \geq a$, the page E_r satisfies the same boundedness property as the page E_a.

Definition 10.4. A spectral sequence $(E_r^{p,q})$ starting at the page E_a is called a *first quadrant spectral sequence* if $E_a^{p,q} = 0$ for all $(p,q) \in \mathbb{Z}$ such that $p < 0$ or $q < 0$.

The term 'first quadrant spectral sequence' is self-explanatory: a first quadrant spectral sequence $(E_r^{p,q})$ has in any page nonzero terms $E_r^{p,q}$ only if $p \geq 0$ and $q \geq 0$, that is, only if (p,q) belongs to the first quadrant.

Exercise 10.5. Let $E = (E_r^{p,q})$ be a first quadrant spectral sequence starting at the page E_0. Show that for any integer $n \geq 0$ and any $(p,q) \in \mathbb{Z}^2$ such that $p + q = n$ we have $E_\infty^{p,q} = E_{n+2}^{p,q}$.

Proposition 10.6.

(i) A first quadrant spectral sequence is bounded.

(ii) A bounded spectral sequence converges.

Proof. For any integer n there are at most finitely many pairs $(p,q) \in \mathbb{Z}^2$ such that $p \geq 0$, $q \geq 0$, and $p + q = n$. Thus a first quadrant spectral sequence is bounded, whence (i). For (ii), let $(E_r^{p,q})$

be a bounded spectral sequence. Let $n \in \mathbb{Z}$ and $(p,q) \in \mathbb{Z}^2$ such that $p+q=n$. Consider the sequence of the two differentials in the E_r-page starting and ending at $E_r^{p,q}$,

$$E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

Regardless of r, the total degree of the left term $E_r^{p-r,q+r-1}$ is $n-1$, and the total degree of the right term $E_r^{p+r,q-r+1}$ is $n+1$. Since the spectral sequence is bounded, there are only finitely many values of integers r such that at least one of $E_r^{p-r,q+r-1}$, $E_r^{p+r,q-r+1}$ is nonzero. Thus for r large enough, we have

$$E_r^{p-r,q+r-1} = 0 = E_r^{p+r,q-r+1} .$$

But then the differentials ending and starting at $E_r^{p,q}$ are zero, so passing to cohomology yields $E_r^{p,q} = E_{r+1}^{p,q}$ for all large enough integers r, which proves that the spectral sequence converges. $\square$

Definition 10.7. Let $(E_r^{p,q})$ be a bounded spectral sequence of A-modules starting at the E_a-page for some integer a. Let $H^* = (H^n)_{n\in\mathbb{Z}}$ be a graded A-module (think: the cohomology of some cochain complex). We say that the spectral sequence $(E_r^{p,q})$ *converges to* H^* and write

$$E_a^{p,q} \Rightarrow H^{p+q}$$

if there exists a filtration of H^* by graded submodules F^pH^* such that $F^{p+1}H^n \subseteq F^pH^n$ for all integers p, n, and such that

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$$

for all $(p,q) \in \mathbb{Z}^2$.

Remark 10.8. The convergence of a spectral sequence $E_a^{p,q} \Rightarrow H^{p+q}$ does not mean that we can determine H^n outright; what this says is that there is a filtration of H^n whose subquotients are isomorphic to the modules $E_\infty^{p,q}$, with (p,q) running over the set of all pairs of integers such that $p+q=n$. By the boundedness assumption, there are only finitely many such pairs. For instance, if one of the $E_\infty^{p,q}$ with $p+q=n$ is nonzero, then H^n is nonzero, and if k is a field such that the $E_\infty^{p,q}$ are finite-dimensional, then $\dim_k(H^n) = \sum_{p+q=n} \dim_k(E_\infty^{p,q})$.

Example 10.9. Let X be a cochain complex and Y a subcomplex of X. Consider X with the filtration $F^0X = X$, $F^1X = Y$, and $F^2X = \{0\}$ (the zero subcomplex of X). Consider $H^n = H^n(X)$ filtered with $F^0H^n = H^n$, $F^1H^n = \mathrm{Im}(H^n(Y) \to H^n(X))$, and $F^2H^n = \{0\}$. The connecting homomorphism $d^n : H^n(X/Y) \to H^{n+1}(Y)$ can be regarded as the differential of the E_1-page of a spectral sequence of the form

$$0 \longrightarrow H^n(X/Y) \xrightarrow{d^n} H^{n+1}(Y) \longrightarrow 0$$

$$0 \longrightarrow H^{n-1}(X/Y) \xrightarrow{d^{n-1}} H^n(Y) \longrightarrow 0$$

$$0 \longrightarrow H^{n-2}(X/Y) \xrightarrow{d^{n-2}} H^{n-1}(Y) \longrightarrow 0$$

with the two nonzero columns in degree 0 and 1. The E_2-page is obtained from passing to cohomology, thus of the form

$$\begin{array}{cc} \ker(d^n) & H^{n+1}(Y)/\mathrm{Im}(d^n) \\ \\ \ker(d^{n-1}) & H^n(Y)/\mathrm{Im}(d^{n-1}) \\ \\ \ker(d^{n-2}) & H^{n-1}(Y)/\mathrm{Im}(d^{n-2}) \end{array}$$

with zero differential; that is, $E_2 = E_\infty$. The fact that this spectral sequence converges to $H^* = H^*(X)$ is equivalent to the long exact cohomology sequence. Indeed, the convergence of this spectral sequence to H^* means that H^n is filtered by the the term $H^n(Y)/\mathrm{Im}(d^{n-1})$ in position $(1, n-1)$ and $\ker(d^n)$ in position $(0, n)$. But $\ker(d^n)$ is equal to the image of $H^n(X) \to H^n(X/Y)$ by the exactness of long cohomology sequence, and the map $H^n(Y) \to H^n(X)$ has image $H^n(Y)/\mathrm{Im}(d^{n-1})$.

Any filtered complex gives rise to a spectral sequence starting at the E_1-page. A filtration of a cochain complex X by subcomplexes F^pX satisfying $F^{p+1}X \subseteq F^pX$ for $p \in \mathbb{Z}$ is called *bounded* if for any $n \in \mathbb{Z}$ there exist integers p and q such that $F^pX^n = X^n$ and $F^qX^n = \{0\}$; that is, in each fixed degree, the filtration induced by the subcomplexes F^pX is finite.

Theorem 10.10. *Let X be a cochain complex of A-modules with a bounded filtration by subcomplexes F^pX such that $F^{p+1}X \subseteq F^pX$ for $p \in \mathbb{Z}$. There is a bounded spectral sequence*

$$E_1^{p,q} \Rightarrow H^{p+q}(X)$$

with

$$E_1^{p,q} = H^{p+q}(F^pX/F^{p+1}X)$$

for any p ,$q \in \mathbb{Z}$, and

$$E_\infty^{p,q} = F^pH^{p+q}(X)/F^{p+1}H^{p+q}(X)$$

where $F^pH^n(X) = \mathrm{Im}(H^n(F^pX) \to H^n(X))$ with the map being induced by the inclusion $F^pX \subseteq X$, for any n, p, $q \in \mathbb{Z}$.

Proof. Denote by $\delta = (\delta^n)_{n\in\mathbb{Z}}$ the differential of X. We define for p, q, $r \in \mathbb{Z}$, $r \geq 1$, the following submodules of X^{p+q}.

$$Z_r^{p,q} = \{x \in F^pX^{p+q} \mid \delta^{p+q}(x) \in F^{p+r}X^{p+q+1}\}$$

In other words, $Z_r^{p,q}$ is the inverse image in $X^{p,q}$ of the differential in degree $p+q$ of the complex $F^pX/F^{p+r}X$.

$$B_r^{p,q} = \delta^{p+q-1}(F^{p-r}X^{p+q})$$

$$Z_\infty^{p,q} = \ker(\delta^{p+q}) \cap F^pX^{p+q}$$

$$B_\infty^{p,q} = \mathrm{Im}(\delta^{p+q-1}) \cap F^pX^{p+q} \ .$$

One verifies the following inclusions:

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdot \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} \subseteq X^{p+q} \ .$$

We set

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) \ ,$$

where p, q, $r \in \mathbb{Z}$ such that $r \geq 1$. The rest of the proof is a verification of the following statements:

(1) The differential δ of X induce a maps $E_r^{p,q} \to E_r^{p+r,q-r+1}$.

(2) Taking cohomology sends the page E_r^{**} to the subsequent page E_{r+1}^{**}.

(3) We have $E_1^{p,q} \cong H^{p+q}(F^pX/F^{p+1}X)$.

(4) We have $E_\infty^{p,q} \cong F^pH^{p+q}(X)/F^{p+1}H^{p+q}(X)$. □

We describe next one of the major construction principles which yield filtered complexes: the total complex of a double complex comes with two filtrations.

Definition 10.11. A double cochain complex of A-modules is a triple (X, δ, ϵ) consisting of

(a) a bigraded A-module $X^{**} = (X^{p,q})_{p,q\in\mathbb{Z}}$,

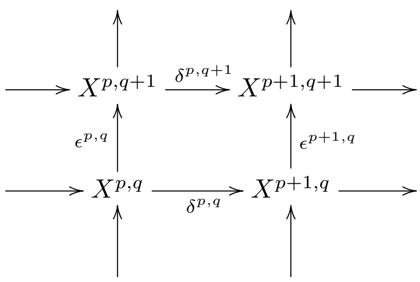
(b) a *horizontal differential* $\delta : X^{p,q} \to X^{p+1,q}$ of *bidegree* $(1,0)$ satisfying $\delta^{p+1,q} \circ \delta^{p,q} = 0$ for all p, $q \in \mathbb{Z}$,

(c) a *vertical differential* $\epsilon : X^{p,q} \to X^{p,q+1}$ of *bidegree* $(0,1)$ satisfying $\epsilon^{p,q+1} \circ \epsilon^{p,q} = 0$ for all p, $q \in \mathbb{Z}$,

with the property that

$$\epsilon^{p+1,q} \circ \delta^{p,q} = -\delta^{p,q+1} \circ \epsilon^{p,q} \ ;$$

that is, the squares



anticommute for all p, $q \in \mathbb{Z}$.

That is, a double complex can be regarded as a sequence of horizontal cochain complexes (with differentials given by δ), such that the vertical maps ϵ are 'cochain maps up to signs'. One verifies that then ϵ preserves $\ker(\delta)$ and $\mathrm{Im}(\delta)$, hence induces a vertical graded map ϵ on the 'horizontal' cohomology of the complexes with differential δ. Similarly, a double complex can be regarded as a sequence of vertical cochain complexes such that the horizontal maps are 'cochain maps up to signs'.

Definition 10.12. Let (X, δ, ϵ) be a double cochain complex of A-modules. The *total complex of* X is the cochain complex, denoted $\mathrm{tot}(X)$, defined by

$$\mathrm{tot}(X)^n = \oplus_{p+q=n} X^{p,q} = \oplus_{p\in\mathbb{Z}} X^{p,n-p} = \oplus_{q\in\mathbb{Z}} X^{n-q,q}$$

for $n \in \mathbb{Z}$, where in the first sum (p, q) runs over all $(p, q) \in \mathbb{Z}^2$ such that $p+q = n$, with differential

$$\Delta = \delta + \epsilon \ ;$$

explicitly, Δ^n is the sum of the maps $\delta^{p,q}$ and $\epsilon^{p,q}$, the sum taken over all $(p, q) \in \mathbb{Z}^2$ such that $p + q = n$.

One verifies that $\Delta \circ \Delta = 0$; this makes use of the anticommutativity of the differentials in the definition of double complexes. If for some $n \in \mathbb{Z}$ there are infinitely many pairs of integers (p, q) satisfying $p + q = n$, then it would make a difference whether we define $\mathrm{tot}(X)^n$ as direct sum or as direct product of the $X^{p,q}$ with $p + q = n$, and both versions may be useful, depending on circumstances. We will keep the focus on bounded spectral sequences, so this issue will not arise here. We describe next the two filtrations of the total complex of a double complex.

Definition 10.13. Let (X, δ, ϵ) be a double cochain complex. Set $Y = \mathrm{tot}(X)$. Define

$$F_I^p Y^n = \oplus_{r\geq p} X^{r,n-r} \ ,$$

$$F_{II}^p Y^n = \oplus_{r\geq p} X^{n-r,r} \ ,$$

for any n, $p \in \mathbb{Z}$.

An easy verification shows that $F_I^p Y$ and $F_{II}^p Y$ are subcomplexes of $Y = \mathrm{tot}(X)$, for any $p \in \mathbb{Z}$. As mentioned before, we can take the cohomology in a double complex (X, δ, ϵ) in two ways: either horizontally with respect to δ, or vertically, with respect to ϵ. We use the following notation.

Definition 10.14. Let (X, δ, ϵ) be a double cochain complex of A-modules. For any $(p, q) \in \mathbb{Z}^2$ we set

$$H_I^{p,q}(X) = \ker(\delta^{p,q})/\mathrm{Im}(\delta^{p-1,q}) \ ,$$

$$H_{II}^{p,q}(X) = \ker(\epsilon^{p,q})/\mathrm{Im}(\epsilon^{p,q-1}) \ .$$

We regard $H_I^{p,q}(X)$ again as a double complex, with horizontal differential now zero, while the vertical differential is induced by ϵ, and will be denoted by $\bar{\epsilon}$. Similarly, we regard $H_{II}^{p,q}(X)$ again as a double complex, with vertical differential zero and horizontal differential $\bar{\delta}$ induced by δ. Thus we can apply taking horizontal and vertical cohomology again; this leads to considering the bigraded objects (that is, double complexes in which both differentials are zero) of the form $H_{II}H_I(X)$ and $H_I H_{II}(X)$.

Theorem 10.15. *Let X be a first quadrant double cochain complex of A-modules; that is, $X^{p,q} = \{0\}$ if at least one of p or q is negative. There are two first quadrant spectral sequences of the form*

$${}_IE_2^{p,q} = H_I^{p,q} H_{II}(X) \Rightarrow H^{p+q}(X)$$

$${}_{II}E_2^{p,q} = H_{II}^{p,q} H_I(X) \Rightarrow H^{p+q}(X)$$

Proof. The two filtrations F_I and F_{II} of $Y = \mathrm{tot}(X)$ described in 10.13 give rise, by Theorem 10.10, to two spectral sequences starting at the E_1-page, of the form

$$_IE_1^{p,q} \Rightarrow H^{p+q}(Y)$$

with

$$_IE_1^{p,q} = H^{p+q}(F_I^pY/F_I^{p+1}Y)$$

and similarly for F_{II}. One verifies that

$$_IE_1^{p,q} = H_{II}^{p,q}(X)$$

and that the differential at the page $_IE_1$ is equal to the differential $\bar{\delta}$ induced by δ. Thus taking cohomology with respect to $\bar{\delta}$ yields the page $_IE_2$, and that is by our notation the same as applying H_I. This yields the first spectral sequence, and the analogous argument with F_{II} yields the second spectral sequence. □

Theorem 10.16. *Let (X, δ) be a cochain complex of A-modules such that $X^n = \{0\}$ for $n < 0$. There exists a double cochain complex of the form*

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,2} & \xrightarrow{\delta^{0,2}} & I^{1,2} & \xrightarrow{\delta^{1,2}} & I^{2,2} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,1} & \xrightarrow{\delta^{0,1}} & I^{1,1} & \xrightarrow{\delta^{1,1}} & I^{2,1} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,0} & \xrightarrow{\delta^{0,0}} & I^{1,0} & \xrightarrow{\delta^{1,0}} & I^{2,0} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
X^0 & \xrightarrow[\delta^0]{} & X^1 & \xrightarrow[\delta^1]{} & X^2 & \longrightarrow &
\end{array}$$

such that the following hold.

(i) $I^{p,q}$ *is injective for all* $p, q \geq 0$.

(ii) $\mathrm{Im}(\delta^{n,*})$ *is an injective resolution of* $\mathrm{Im}(\delta^n)$, *for* $n \geq 0$.

(iii) $\ker(\delta^{n,*})/\mathrm{Im}(\delta^{n-1,*})$ *is an injective resolution of* $\ker(\delta^n)/\mathrm{Im}(\delta^{n-1}) = H^n(X)$, *for* $n \geq 0$.

(iv) If $X^n = \{0\}$ *for some integer* $n \geq 0$, *then* $I^{n,q} = \{0\}$ *for all* $q \geq 0$.

Proof. The proof uses the horseshoe lemma and constructs this resolution inductively. □

The double complex $I^{*,*}$ obtained from removing the bottom row, is called a *Cartan-Eilenberg resolution of* X. An easy argument using long exact sequences shows that then $\ker(\delta^{n,*})$ is also an injective resolution of $\ker(\delta^n)$.

Theorem 10.17 (Lyndon-Hochschild-Serre spectral sequence)**.** *Suppose that k is a field. Let G be a finite group, N a normal subgroup, and U a kG-module. There is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(G/N; H^q(N; U)) \Rightarrow H^{p+q}(G; U)$$

Proof. We have $H^n(G; U) = \mathrm{Ext}^n_{kG}(k; U)$. This can be calculated by choosing a projective resolution of k followed by applying the functor $\mathrm{Hom}_{kG}(-, U)$ to this resolution and then taking cohomology in degree n. It can also be calculated by choosing an injective resolution of U followed by applying the functor $\mathrm{Hom}_{kG}(k, -)$ to this resolution and then taking cohomology in degree n. It is a specialty of finite group algebras that their classes of projective and injective modules coincide (this follows from the fact that finite group algebras are *symmetric*, hence *self-injective*). Note that the functor $\mathrm{Hom}_{kG}(k, -)$ applied to the kG-module is the same as taking G-fixed points U^G in U; indeed, we have a natural isomorphism

$$\mathrm{Hom}_{kG}(k, U) \cong U^G$$

sending a kG-homomorphism $\tau : k \to U$ to $\tau(1)$. Applied to N instead of G, we have $\mathrm{Hom}_{kN}(k, U) \cong U^N$. Since N is normal in G, the action of G on U preserves U^N, and so U^N is again a kG-module. Moreover, N acts by definition trivially on U^N, so U^N inherits as kG/N-module structure, so that it makes sense to take G/N-fixed points in U^N. We clearly have

$$U^G = (U^N)^{G/N}$$

and this should be understood as the composition of two functors from $\mathrm{Mod}(kG)$ to $\mathrm{Mod}(k)$, namely

$$\mathrm{Hom}_{kG}(k, -) = \mathrm{Hom}_{kG/N}(k, -) \circ \mathrm{Hom}_{kN}(k, -)$$

The first of these two functors, sending a kG-module U to the kG/N-module U^N, has an important structural property: it preserves injectives, or equivalently, it preserves projectives. Indeed, the N-fixed points in the free kG-module kG of rank 1 are easily seen to be equal to $(\sum_{y \in N} y)kG \cong kG/N$.

With the preliminary observations, take an injective resolution of U; this yields an exact cochain complex

$$0 \longrightarrow U \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$$

To calculate $H^q(N; U)$, we need to apply the fixed point functor $\mathrm{Hom}_{kN}(k, -)$. This yields a cochain complex of kG/N-modules of the form

$$0 \longrightarrow U^N \longrightarrow (J^0)^N \longrightarrow (J^1)^N \longrightarrow \cdots$$

Note that all modules apart from U^N are again injective kG/N-modules. Construct a Cartan-Eilenberg resolution of this complex. Then apply the G/N-fixed point functor $\mathrm{Hom}_{kG/N}(k, -)$ to the entire Cartan-Eilenberg resolution. One ends up with a double cochain complex. This double complex yields two spectral sequences. One shows that one of these collapses (using the injectivity of the $(J^i)^N$), and uses this to show that the other of these converges to $H^{p+q}(G; U)$ as stated. □

The Lyndon-Hochschild-Serre spectral sequence is a crucial ingredient in the proof of the following result:

Theorem 10.18 (Evens-Venkov)**.** *Let G be a finite group and k a field. Then $H^*(G;k)$ is a finitely generated graded-commutative k-algebra.*

The construction of the Lyndon-Hochschild-Serre spectral sequence is a special case of what is known as a *Grothendieck spectral sequence.* These are obtained from playing of the derived functors of two composable functors and those of the composition of the two functors.

Definition 10.19. Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor and $n \geq 0$. The n-the *right derived functor* $R^n\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is defined by

$$R^n\mathcal{F}(U) = H^n(\mathcal{F}(I)),$$

where I is an injective resolution of U. The n-the *left derived functor* $L_n\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is defined by

$$L_n\mathcal{F}(U) = H_n(\mathcal{F}(P)),$$

where P is an projective resolution of U.

We have analogous definitions for contravariant functors.

Exercise 10.20. With the notation of 10.19, show that if $\mathcal{F}$ is left exact, then $R^0\mathcal{F} \cong \mathcal{F}$, and if $\mathcal{F}$ is right exact, then $L_0\mathcal{F} \cong \mathcal{F}$. Show that the functors $L_n\mathcal{F}$, $R^n\mathcal{F}$ are independent, up to unique isomorphism of functors, of the choices of resolutions.

Example 10.21. Let U, V be A-modules. The fact that $\mathrm{Ext}^n_A(U, V)$ can be calculated by using either a projective resolution of U or an injective resolution of V translates to the equality

$$\mathrm{Ext}^n_A(U, V) = L_n(\mathrm{Hom}_A(-, V))(U) = R^n(\mathrm{Hom}_A(U, -))(V) \ .$$

In particular, if G is a group and U a kG-module, then

$$H^n(G; U) = L_n(\mathrm{Hom}_{kG}(-, U))(k) = R^n(\mathrm{Hom}_{kG}(k, -))(U)$$

Definition 10.22. Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor. An A-module U is called *$\mathcal{F}$-acyclic* if $R^n\mathcal{F}(U) = \{0\}$ for all $n > 0$.

Any injective module is $\mathcal{F}$-acyclic for any functor $\mathcal{F}$. The point of this definition is that one can use resolutions by $\mathcal{F}$-acyclic modules rather than injective modules to calculate the derived functors of $\mathcal{F}$.

Theorem 10.23. *Let A, B be algebras over a commutative ring k. Let $\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ be a covariant functor. Let J be a resolution of an A-module U such that all terms of J are $\mathcal{F}$-acyclic. Then $R^n\mathcal{F}(U) \cong H^n(\mathcal{F}(I))$ for $n \geq 0$.*

This allows more flexibility when it comes to calculating derived functors - for instance, in sheaf theory, a *flabby sheaf* on a space X is $\Gamma(X, -)$-acyclic, and hence, in order to calculate sheaf cohomology, one may use resolutions by flabby sheaves rather than injective sheaves. The following result, describing *Grothendieck spectral sequences*, takes this into account.

Theorem 10.24 (Grothendieck, [4]). *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be left exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects and that $\mathcal{F}$ sends injective objects in $\mathcal{A}$ to $\mathcal{G}$-acyclic objects in $\mathcal{B}$. There is a first quadrant spectral sequence*

$$E_2^{p,q} = (R^p\mathcal{G})(R^q\mathcal{F}(X)) \Rightarrow R^{p+q}(\mathcal{G} \circ \mathcal{F})(X)$$

for any object X in $\mathcal{A}$.

Proof. This follows the pattern we have aleady encountered in the proof of the Lyndon-Hochschild-Serre spectral sequence. We start with an injective resolution I of X. We apply the functor $\mathcal{F}$ to this resolution. Note that the terms of $\mathcal{F}(I)$ are $\mathcal{G}$-acyclic. We then consider a Cartan-Eilenberg resolution of this complex in $\mathcal{B}$, and we apply the functor $\mathcal{G}$. This yields a double complex in $\mathcal{C}$. One of the two spectral sequences associated with this double complex collapses (this is where we use that the terms of $\mathcal{F}(I)$ are $\mathcal{G}$-acyclic), and the other takes the form as in the statement. □

The Lyndon-Hochschild-Serre spectral sequence is a Grothendieck spectral sequence with $\mathcal{A} = \mathrm{Mod}(kG)$, $\mathcal{B} = \mathrm{Mod}(kG/N)$, $\mathcal{C} = \mathrm{Mod}(k)$, $\mathcal{F} = \mathrm{Hom}_{kN}(k, -)$, and $\mathcal{G} = \mathrm{Hom}_{kG/N}(k, -)$. So is the following spectral sequence in sheaf cohomology, due to Leray, which is fundamental in algebraic geometry.

Theorem 10.25 (Leray). *Let X, Y be topological spaces and $f : X \to Y$ a continous map. Let $\mathcal{F}$ be a sheaf on X. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y; R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X; \mathcal{F})$$

where $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the direct image functor.

Proof. This is a Grothendieck spectral sequence with $\mathcal{A} = \mathrm{Sh}(X)$, $\mathcal{B} = \mathrm{Sh}(Y)$, $\mathcal{C} = \mathbf{Ab}$ (the category of abelian groups), $\mathcal{F} = f_*$, $\mathcal{G} = \Gamma(Y; -)$ (the global section functor on Y), using the fact that $\Gamma(X; -) = \Gamma(Y; -) \circ f_*$ and that f_* preserves injectives. □

A Appendix: Category theory theoretic background

Category theory considers mathematical objects systematically together with the structure preserving maps between them, providing a unifying language for many different mathematical concepts to which homological methods can be applied. We review in this section the basic category theoretic vocabulary: *category, functor, natural transformation*, and *adjunction.*

A category $\mathcal{C}$ consists of three types of data: an *object class*, a *morphism class*, and information on how to compose morphisms, with a short list of properties one would expect any reasonable category of mathematical objects to have.

Definition A.1. A category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$, called the *class of objects of* $\mathcal{C}$, for any X, $Y \in \mathrm{Ob}(\mathcal{C})$ a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, called the *class of morphisms from* X *to* Y *in* $\mathcal{C}$, and, for any X, Y, $Z \in \mathcal{C}$ a map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto g \circ f \ ,$$

called the *composition map*, subject to the following properties.

(1) The classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, with X, $Y \in \mathcal{C}$, are pairwise disjoint. Equivalently, any morphism f in $\mathcal{C}$ determines uniquely a pair (X, Y) of objects in $\mathcal{C}$ such that $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(2) (Identity morphisms) For any object $X \in \mathrm{Ob}(\mathcal{C})$, there is a distinguished morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called *identity morphism of* X, such that for any object $Y \in \mathcal{C}$, any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and any $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ we have $f \circ \mathrm{Id}_X = f$ and $\mathrm{Id}_X \circ g = g$.

(3) (Associativity) For any X, Y, Z, $W \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$; this is an equality of morphisms in $\mathrm{Hom}_{\mathcal{C}}(X, W)$.

The objects of a category form in general a *class*, not necessarily a set. A category whose object and morphism classes are sets is called a *small category.* A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ between two objects X, Y in a category $\mathcal{C}$ is typically denoted by $f : X \to Y$ or by $X \xrightarrow{f} Y$. Morphisms are also called *maps*, although one should note that the morphisms of a category may be abstractly defined and do not necessarily induce any maps in a set theoretic sense. We write $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$, and call the morphisms in $\mathrm{End}_{\mathcal{C}}(X)$ the *endomorphisms of* X. The set $\mathrm{End}_{\mathcal{C}}(X)$ together with the composition of morphisms is a monoid with unit element Id_X.

Examples A.2.

(1) We denote by **Sets** the category of sets, having as objects the class of sets and as morphisms arbitrary maps between sets. This is a large category - considering the set of all sets leads to what is known as *Russell's paradox.*

(2) For k a field, we denote by $\mathbf{Vect}(k)$ the category of k-vector spaces; that is, the objects of $\mathbf{Vect}(k)$ are the k-vector spaces, and the morphisms are k-linear transformations between k-vector spaces. For U, V two k-vector spaces, we write $\mathrm{Hom}_k(U, V)$ instead of $\mathrm{Hom}_{\mathbf{Vect}(k)}(U, V)$ for the space of k-linear transformations from U to V, and we write $\mathrm{End}_k(U) = \mathrm{Hom}_k(U, U)$. Note that the sets $\mathrm{Hom}_k(U, V)$ are again k-vector spaces, not just sets, and that the composition maps are k-bilinear.

(3) We denote by **Grps** the category of groups, with groups as objects and group homomorphisms as morphisms.

(4) We denote by **Top** the category of topological spaces, with linear maps as morphisms.

(5) If $\mathcal{C}$ is a small category with a single object E, then $\mathrm{Hom}_{\mathcal{C}}(E, E)$ is a monoid. Conversely, if M is a monoid, we can consider M as a the morphism set of a category $\mathbf{M}$ with a single object $*$, such that the morphism set in $\mathbf{M}$ from $*$ to $*$ is equal to M, and such that composition of morphisms in $\mathbf{M}$ is equal to the product in M.

(6) We denote by $\mathbf{Alg}(k)$ the category of k-algebras, with algebra homomorphisms as morphisms.

(7) For A an algebra over a commutative ring k, we denote by $\mathrm{Mod}(A)$ the category of left A-modules; that is, the objects of $\mathrm{Mod}(A)$ are the left A-modules, and morphisms are A-module homomorphisms. For U, V two A-modules we write $\mathrm{Hom}_A(U, V)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(U, V)$ for the set of A-homomorphisms from U to V. Similarly, we write $\mathrm{End}_A(U)$ instead of $\mathrm{End}_{\mathrm{Mod}(A)}(U)$. Note that if k is a field, then $\mathbf{Vect}(k) = \mathrm{Mod}(k)$ and $\mathbf{vect}(k) = \mathrm{mod}(k)$.

Definition A.3. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for all X, $Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, with composition $g \bullet f$ in $\mathcal{C}^{\mathrm{op}}$ defined by $g \bullet f = f \circ g$, for any X, Y, $Z \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$, $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z) = \mathrm{Hom}_{\mathcal{C}}(Z, Y)$, and where $f \circ g$ is the composition in $\mathcal{C}$.

Definition A.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if $\mathrm{Ob}(\mathcal{D})$ is a subclass of $\mathrm{Ob}(\mathcal{C})$, and if for any X, Y in $\mathrm{Ob}(\mathcal{D})$, the class $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is a subclass of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that for any X, Y, $Z \in \mathrm{Ob}(\mathcal{D})$, the composition map $\mathrm{Hom}_{\mathcal{D}}(X, Y) \times \mathrm{Hom}_{\mathcal{D}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(X, Z)$ in $\mathcal{D}$ is the restriction of the composition map in $\mathcal{C}$. We say that the subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *full subcategory,* if for any X, $Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Examples A.5.

(1) For k a field we denote by $\mathbf{vect}(k)$ the full subcategory of $\mathbf{Vect}(k)$ consisting of all finite-dimensional k-vector spaces.

(2) The category **grps** of finite groups is a full subcategory of the category of all groups **Grps**.

(3) The category of finitely generated left A-modules, denoted $\mathrm{mod}(A)$, is a full subcategory of $\mathrm{Mod}(A)$.

Morphisms in a category are abstract mathematical objects and need not be maps between sets. One of the challenges is to extend to morphisms some standard notions of maps such as the property of being injective or surjective, without referring to elements in objects. The category theoretic version of surjective and injective maps are as follows.

Definition A.6. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. The morphism f is called an *epimorphism* if for any two morphisms g, g' from Y to any other object Z satisfying $g \circ f = g' \circ f$ we have $g = g'$. The morphism f is called a *monomorphism* if for any two morphisms g, g' from any other object Z to X satisfying $f \circ g = f \circ g'$ we have $g = g'$. The morphism f is called an *isomorphism* if there exists a morphism $h : Y \to X$ satisfying $h \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. An isomorphism which is an endomorphism of an object X is called an *automorphism* of X.

There are various ways to reformulate this definition. For instance, $f : X \to Y$ is an epimorphism, if and only if for any object Z the map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ sending $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ to $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is injective. Similarly, $f : X \to Y$ is a monomorphism, if and only if for any object W the map $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$ sending $g \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ to $f \circ g \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ is injective.

Exercise A.7. Show that in the category of sets, the monomorphisms are the injective maps and the epimorphisms are the surjective maps. Show that in the category $\mathrm{Mod}(A)$ of modules over an algebra A the monomorphisms are the injective A-module homomorphisms and the epimorphisms are the surjective A-module homomorphisms. Show that in the category $\mathrm{Ch}(\mathrm{Mod}(A))$ the monomorphisms (resp. epimorphisms) are the chain maps which are injective (resp. surjective) A-homomorphisms in each degree.

Exercise A.8. Show that the composition of two monomorphisms in a category is a monomorphism, and that the composition of two epimorphisms is an epimorphism.

Exercise A.9. Show that if $f : X \to Y$ is an isomorphism in a category $\mathcal{C}$, then there is a *unique* morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ satisfying $h \circ f = \mathrm{Id}_X$ and $f \circ h = \mathrm{Id}_Y$. The morphism h is then called the *inverse of* f and denoted by f^{-1}. Show that if $f : X \to Y$ and $g : Y \to Z$ are isomorphisms, then $g \circ f$ is an isomorphism with inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Show that the automorphisms of X in $\mathcal{C}$ form a subgroup $\mathrm{Aut}_{\mathcal{C}}(X)$ of the monoid $\mathrm{End}_{\mathcal{C}}(X)$.

Exercise A.10. Show that a morphism f in a category $\mathcal{C}$ is a monomorphism if and only if f is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Exercise A.11. Show that if f is an isomorphism in a category $\mathcal{C}$, then f is both a monomorphism and an epimorphism in $\mathcal{C}$.

There are examples of categories in which a morphism is both an epimorphism and a monomorphism but not an isomorphism.

Exercise A.12. Let $\mathcal{C}$ be the category having the abelian group $\mathbb{Z}$ as unique object, with all abelian group endomorphisms of $\mathbb{Z}$ as morphisms. Show that every nonzero endomorphism of $\mathbb{Z}$ is both a monomorphism and an epimorphism in $\mathcal{C}$. Use this to give an example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

Definition A.13. Let $\mathcal{C}$ be a category. An object P in $\mathcal{C}$ is called *projective* if for any epimorphism $h : X \to Y$ and any morphism $g : P \to Y$ there is a morphism $f : P \to X$ such that $h \circ f = g$. An object I in $\mathcal{C}$ is called *injective* if for any monomorphism $h : X \to Y$ and any morphism $g : X \to I$ there is a morphism $f : Y \to I$ such that $f \circ h = g$.

Exercise A.14. Deduce that an object P in a category $\mathcal{C}$ is projective (resp. injective) if and only it is injective (resp. projective) as an object in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Exercise A.15. . Let A be an algebra over some commutative ring and P an A-module. Show that P is projective if and only if for any surjective A-homomorphism $\varphi : U \to V$ the induced k-linear map $\mathrm{Hom}_A(P, U) \to \mathrm{Hom}_A(P, V)$ sending $\alpha \in \mathrm{Hom}_A(P, U)$ to $\varphi \circ \alpha$ is surjective.

Let A be a algebra over a commutative ring k and F and A-module, where we adopt the convention that a module is a unital left module, unless stated otherwise. A subset X of F is called a *basis of* F, if every element in F can be written uniquely in the form $\sum_{x \in X} a_x x$ with elements $a_x \in A$ of which only finitely many are nonzero. An A-module F is called *free* if it has a basis. If F is a free A-module and X a basis of F, then $F = \oplus_{x \in X} Ax$, and $Ax \cong A$ as a left module, for each $x \in A$. In other words, an A-module F is free if and only if F is isomorphic to a direct sum of (possibly infinitely many) copies of A. Traditionally, projective modules are defined as modules which are isomorphic to a direct summand of a free module. Since we have already defined the notion of projective objects in an arbitrary category, we will show that these definitions coincide for module categories.

Exercise A.16. Let A be an algebra over some commutative ring. Show that any free A-module is projective that and that any direct summand of a projective A-module is projective.

Exercise A.17. Let i be an idempotent in a ring A; that is, $i^2 = i \neq 0$. Show that the left ideal Ai generated by i is a projective A-module. (*Hint*: show that $A(1-i)$ is a complement of Ai in A as a left A-module).

Projective modules of an algebra can be characterised as direct summands of free modules.

Theorem A.18. *Let A be a algebra over a commutative ring k, and let P be an A-module. The following are equivalent.*

(i) The A-module P is a projective object in the category $\mathrm{Mod}(A)$ of A-modules.

(ii) Any surjective A-homomorphism $\pi : U \to P$ from some A-module U to P splits; that is, there is an A-homomorphism $\sigma : P \to U$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(iii) The functor $\mathrm{Hom}_A(P, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ is exact.

(iv) The module P is isomorphic to a direct summand of a free A-module.

Proof. We will use the fact from Exercise A.7 that the epimorphisms in $\mathrm{Mod}(A)$ are the surjective A-homomorphisms and that the monomorphisms in $\mathrm{Mod}(A)$ are the injective A-homomorphisms. Suppose that (i) holds; that is, P is projective in $\mathrm{Mod}(A)$. Let $\pi : U \to P$ be a surjective A-homomorphism, where U is an A-module. Then in particular the identity map Id_P lifts through the surjective map π; that is, there is an A-homomorphism $\sigma : P \to U$ satsifying $\pi \circ \sigma = \mathrm{Id}_P$. Thus π splits. This shows that (i) implies (ii). Suppose that (ii) holds. Let X be any subset of P which generates P; that is, every element in P can be written in the form $\sum_{x \in X} a_x x$ for some $a_x \in A$ of which only finitely many are nonzero. Take for F a free A-module having a basis $\{e_x \mid x \in X\}$ indexed by X. That is, $F = \oplus_{x \in X} Ae_x$. Since F is free, there is a unique A-module homomorphism $\pi : F \to P$ sending e_x to x. This homomorphism is surjective since X generates P as an A-module. Thus, if (ii) holds, then π splits, showing that P is isomorphic to a direct summand of F. This shows that (ii) implies (iv). The equivalence of (i) and (iii) follows easily using the Exercise A.15. It follows from Exercise A.16 that (iv) implies (i). □

Exercise A.19. Let A be an algebra over some commutative ring. Show that every A-module is isomorphic to a quotient of a free A-module.

Except for the characterisation of projective modules as direct summands of free modules, we have a similar result for injective modules.

Theorem A.20. *Let A be a k-algebra and let I be an A-module. The following are equivalent:*

(i) The A-module I is an injective object in $\mathrm{Mod}(A)$.

(ii) Any injective A-homomorphism $\iota : I \to V$ from I to some A-module V splits; that is, there is an A-homomorphism $\kappa : V \to I$ such that $\kappa \circ \iota = \mathrm{Id}_I$.

(iii) The contravariant functor $\mathrm{Hom}_A(-, I) : \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ is exact.

Exercise A.21. Give the details of the proof of Theorem A.20.

Exercise A.22. . Let A be an algebra over some commutative ring and I an A-module. Show that I is injective if and only if for any injective A-homomorphism $\varphi : U \to V$ the induced k-linear map $\mathrm{Hom}_A(V, I) \to \mathrm{Hom}_A(U, I)$ sending $\alpha \in \mathrm{Hom}_A(V, I)$ to $\alpha \circ \varphi$ is surjective.

Exercise A.23. Show that the additive group of rational numbers $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Show that the additive quotient $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

Definition A.24. Let $\mathcal{C}$ be a category. An object E is *initial* if for every object Y in $\mathcal{C}$ there is a unique morphism $E \to Y$ in $\mathcal{C}$. An object T is *terminal* if for every object Y in $\mathcal{C}$ there is a unique morphism $Y \to T$ in $\mathcal{C}$. A *zero object* is an object which is both initial and terminal. If O is a zero object in $\mathcal{C}$ and $f : X \to Y$ a morphism in $\mathcal{C}$ such that $f = h \circ g$, where $g : X \to O$ and $h : O \to Y$ are the unique morphisms, then f is called a *zero morphism* in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

The identity morphism of an initial or terminal object is its only endomorphism, and there is exactly one morphism between any two initial or terminal objects, and hence any such morphism is an isomorphism. Thus if a category has an initial or terminal or zero object, such an object is unique up to unique isomorphism. As a consequence, if $\mathcal{C}$ has a zero object, then for any two objects X, Y in $\mathcal{C}$ there is exactly one zero morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Composing the zero morphism with any other morphism yields again the zero morphism.

Examples A.25.

(1) The category $\mathbf{Vect}(k)$ of vector spaces over a field k has the zero space $\{0\}$ as zero object. Similarly, for A an algebra over a commutative ring, the category $\mathrm{Mod}(A)$ has the zero module $\{0\}$ as zero object.

(2) The category of groups **Grps** has the trivial group $\{1\}$ as zero object.

(3) The category of rings has $\mathbb{Z}$ has initial object: for any ring R there is a unique ring homomorphism $\mathbb{Z} \to R$ sending a positive integer n to $n \cdot 1_R = 1_R + 1_R + \cdots + 1_R$ (the sum of 1_R with itself n times); this is extended to Z by mapping 0 to 0_R and $-n$ to $-(n \cdot 1_R)$. Note though that Z is not a terminal object: for instance, there is no ring homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$.

(4) The space with a single element, denoted $\{*\}$, is terminal in the category of topological spaces **Top**, but not initial.

Definition A.26. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ with a zero object. A *kernel of f* is a pair consisting of an object in $\mathcal{C}$, denoted $\ker(f)$, and a morphism $i : \ker(f) \to X$, such that $f \circ i = 0$ and such that for any object Z and any morphism $g : Z \to X$ satisfying $f \circ g = 0$ there is a unique morphism $h : Z \to \ker(f)$ satisfying $g = i \circ h$. Dually, a *cokernel of f* is a pair consisting of an object in $\mathcal{C}$, denoted $\mathrm{coker}(f)$, and a morphism $p : Y \to \mathrm{coker}(f)$, such that $p \circ f = 0$ and such that for any object Z and any morphism $g : Y \to Z$ satisfying $g \circ f = 0$ there is a unique morphism $h : \mathrm{coker}(f) \to Z$ satisfying $g = h \circ p$.

The uniqueness properties in this definition imply that i is a monomorphism, p is an epimorphism, and the pairs $(\ker(f), i)$ and $(\mathrm{coker}(f), p)$, if they exist, are unique up to unique isomorphism. A kernel becomes a cokernel in the opposite category, and vice versa.

Definition A.27. Let $\mathcal{C}$ be a category, and let $\{X_j\}_{j\in I}$ be a familiy of objects in $\mathcal{C}$, where I is an indexing set. A *product of the family of objects* $\{X_j\}_{j\in I}$ is an object in $\mathcal{C}$, denoted $\prod_{j\in I} X_j$, together with a family of morphisms $\pi_i : \prod_{j\in I} X_j \to X_i$ for each $i \in I$, satisfying the following universal property: for any object Y in $\mathcal{C}$ and any family of morphisms $\varphi_i : Y \to X_i$, with $i \in I$, there is a unique morphism $\alpha : Y \to \prod_{j\in I} X_j$ satisfying $\varphi_i = \pi_i \circ \alpha$ for all $i \in I$.

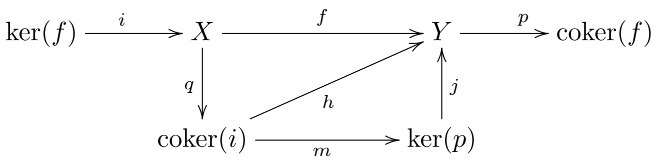
The uniqueness of α implies that the product, if it exists at all, is uniquely determined up to unique isomorphism. By reversing the direction of morphisms, one obtains coproducts or direct sums.

Definition A.28. Let $\mathcal{C}$ be a category, and let $\{X_j\}_{j\in I}$ be a familiy of objects in $\mathcal{C}$, where I is an indexing set. A *coproduct or direct sum of the family of objects* $\{X_j\}_{j\in I}$ is an object in $\mathcal{C}$, denoted $\coprod_{j\in I} X_j$, together with a family of morphisms $\iota_i : X_i \to \coprod_{j\in I} X_j$ for each $i \in I$, satisfying the following universal property: for any object Y in $\mathcal{C}$ and any family of morphisms $\varphi_i : X_i \to Y$, with $i \in I$, there is a unique morphism $\alpha : \coprod_{j\in I} X_j \to Y$ satisfying $\varphi_i = \alpha \circ \iota_i$ for all $i \in I$.

Definition A.29. A category $\mathcal{C}$ with a zero object is called *additive* if the morphism classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are abelian groups, such that the composition of morphisms is biadditive, and such that coproducts of finite families of objects exist. A category $\mathcal{C}$ with a zero object is called *k-linear* if the morphism classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are k-vector spaces, such that the composition of morphisms is bilinear, and such that coproducts of finite families of objects exist.

Remark A.30. In an additive or k-linear category we also have products of finite families, and products and coproducts of finite families of objects are isomorphic. To see this, let I be a finite indexing set and let $\{X_i\}_{i\in I}$ be a finite family of objects in an additive category $\mathcal{C}$. In order to simplify notation, we write $\coprod$ instead of $\coprod_{j\in I}$. We need to construct morphisms $\coprod X_j \to X_i$ for any $i \in I$ satisfying the universal property as in the definition of the product of the X_i. Let $i \in I$. For $j \in I$, denote by $\varphi : X_i \to X_j$ the morphism Id_{X_i} if $i = j$, and the zero morphism if $i \neq j$. The universal property of the coproduct yields a unique morphism $\pi_i : X_i \to \coprod X_j$ with the property $\pi_i \circ \iota_i = \mathrm{Id}_{X_i}$ and $\pi_j \circ \iota_i = 0$, where $i, j \in I$, $i \neq j$. To see that $\coprod X_j$, together with the morphisms $\pi_i : \coprod X_j \to X_i$, is a product, we consider a family of morphisms $\psi_i : Y \to X_i$, for $i \in I$, where Y is some object in $\mathcal{C}$. Then $\alpha = \sum_{j\in I} \iota_j \circ \psi_j$ is a morphism from $Y \to \coprod X_j$; this is well-defined since I is finite. Thus $\pi_i \circ \alpha = \sum_{j\in I} \pi_i \circ \iota_j \circ \psi_j = \psi_i$ for all $i \in I$. To see the uniqueness of α with this property, note first that the endomorphism $\gamma = \sum_{j\in I} \iota_j \circ \pi_j$ of $\coprod X_j$ satisfies $\gamma \circ \iota_i = \iota_i$ for all $i \in I$. But the identity morphism of $\coprod X_j$ is the unique endomorphism with this propery, where we use the universal property of coproducts. Thus γ is equal to the identity on $\coprod X_j$. Therefore, if $\beta : Y \to \coprod X_j$ is any other morphism satisfying $\pi_i \circ \beta = \psi_i$ for all $i \in I$, then $\beta = \sum_{j\in I} \iota_j \circ \pi_j \circ \beta = \sum_{j\in J} \iota_j \circ \psi_j = \alpha$, which shows the uniqueness of α. This proves that $\coprod X_j$, together with the family of morphisms $\pi_i : \coprod X_j \to X_i$, with $i \in I$, is indeed product of the family $\{X_i\}_{i\in I}$.

Module categories are additive, but they have more structure: all morphisms have kernels and cokernels, and there are isomorphism theorems relating kernels and images. Consider a k-algebra A and a homomorphism of A-mdoules $\varphi : U \to V$. Then $U/\ker(\varphi)$ is obtained by first taking the kernel $\ker(\varphi)$ and then taking the cokernel of the inclusion $\ker(\varphi) \subseteq U$. The image $\mathrm{Im}(\varphi)$ is obtained by first taking the cokernel $V \to \mathrm{coker}(\varphi) = V/\mathrm{Im}(\varphi)$, and then $\mathrm{Im}(\varphi)$ is the kernel of the map $V \to \mathrm{coker}(\varphi)$. The isomorphism theorem $U/\ker(\varphi) \cong \mathrm{Im}(\varphi)$ amounts therefore to stating that taking kernels and cokernels 'commute' in a canonical way. These considerations can be extended to additive categories. If $\mathcal{C}$ is an additive category, then for any morphism $f : X \to Y$ in $\mathcal{C}$ which has a kernel $i : \ker(f) \to X$ and a cokernel $p : Y \to \operatorname{coker}(f)$ there is a canonical morphism $\operatorname{coker}(i) \to \ker(p)$. This morphism is constructed as follows. Taking the cokernel of i yields an epimorphism $q : X \to \operatorname{coker}(i)$, and taking the kernel of p yields a monomorphism $j : \ker(p) \to Y$. Since $f \circ i = 0$, the definition of $\operatorname{coker}(i)$ yields a uniqe morphism $h : \operatorname{coker}(i) \to Y$ such that $h \circ q = f$. Then $0 = p \circ f = p \circ h \circ q$. Since q is an epimorphism, this implies that $p \circ h = 0$. Then the definition of $\ker(p)$ yields a unique morphism $m : \operatorname{coker}(i) \to \ker(p)$ satisfying $j \circ m = h$.



Definition A.31. An additive category $\mathcal{C}$ is called an *abelian category* if for every morphism $f : X \to Y$ there exists a kernel $i : \ker(f) \to X$ and a cokernel $p : Y \to \operatorname{coker}(f)$, and if the canonical morphism $\operatorname{coker}(i) \to \ker(p)$ is an isomorphism.

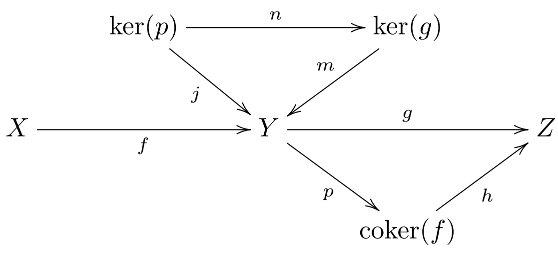
Every module category of a ring is an abelian category. Other examples of abelian categories include categories of sheaves on topological spaces. The Freyd-Mitchell embedding theorem states that every small abelian category is equivalent to a full subcategory of a module category of some ring A. (For the precise definition of equivalent categories see A.37 below.) The notion of exactness can be generalised as follows. A sequence of two composable A-homomorphisms in the category of A-modules

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

is *exact* if $\operatorname{Im}(\varphi) = \ker(\psi)$. With the technique from above, describing $\operatorname{Im}(\varphi)$ as the kernel of a cokernel of φ, consider a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in an abelian category $\mathcal{C}$, such that $g \circ f = 0$. Let $p : Y \to \operatorname{coker}(f)$ be a cokernel of f. Since $g \circ f = 0$, there is a unique morphism $h : \operatorname{coker}(f) \to Z$ such that $h \circ p = g$. Let $j : \ker(p) \to Y$ be a kernel of p. Thus $p \circ j = 0$, hence $g \circ j = h \circ p \circ j = 0$. Let $m : \ker(g) \to Y$ be a kernel of g. Thus there is a unique morphism $n : \ker(p) \to \ker(q)$ satisfying $j = m \circ n$. We say that the above sequence is *exact* if n is an isomorphism in $\mathcal{C}$.



The philosophy of considering any mathematical object together with its structure preserving maps applies to categories as well. Functors are 'morphisms' between categories.

Definition A.32. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *functor* or *covariant functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$* is a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps, abusively all denoted by the same letter $\mathcal{F}$, from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(a) For all objects X in $\mathrm{Ob}(\mathcal{C})$ we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(b) For all objects X, Y, Z in $\mathrm{Ob}(\mathcal{C})$ and morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$ we have

$$\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\psi) \circ \mathcal{F}(\varphi) \ .$$

Similarly, a *contravariant functor from $\mathcal{C}$ to $\mathcal{D}$* is map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$ for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(c) For all objects X in $\mathrm{Ob}(\mathcal{C})$ we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(d) For all objects X, Y, Z in $\mathrm{Ob}(\mathcal{C})$ and morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$ we have

$$\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi) \ .$$

Equivalently, a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Functors can be composed in the obvious way, by composing the maps on objects and on morphisms. Composing a covariant functor with a contravariant functor (in either order) yields a contravariant functor. Composing two contravariant functors yields a covariant functor. On every catagory $\mathcal{C}$ there is the identity functor $\mathrm{Id}_{\mathcal{C}}$ which is the identity map on $\mathrm{Ob}(\mathcal{C})$ and the family of identity maps on the morphism sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, X, $Y \in \mathrm{Ob}(\mathcal{C})$. Since the object classes of categories need not be sets, we cannot consider the category having all categories as objects and functors as morphisms. We can though consider the category **Cat** having as objects small categories and as morphisms all functors between small categories; that is, for two small categories $\mathcal{C}$, $\mathcal{D}$, we denote by $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ the set of functors from $\mathcal{C}$ to $\mathcal{D}$.

Examples A.33.

(1) There is a class of trivial functors, called *forgetful functors*, obtained from ignoring a part of the structure of a mathematical object. For instance, we have a forgetful functor $\mathbf{Alg}(k) \to \mathbf{Vect}(k)$ which sends a k-algebra to its underlying k-vector space (that is, we ignore the multiplication in the algebra). Every k-vector space is in particular an abelian group, so this yields a forgetful functor $\mathbf{Vect}(k) \to \mathbf{Ab}$ sending a vector space to the underlying abelian group (that is, we ignore the scalar muliplication). Every abelian group is in particular a set, so we get a forgetful functor $\mathbf{Ab} \to \mathbf{Sets}$.

(2) There is a functor from **Grps** to $\mathbf{Alg}(k)$ sending a group G to the group algebra kG and sending a group homomorphism $\varphi : G \to H$ to the algebra homomorphism $kG \to kH$ obtained by extending φ linearly. There is also a functor $\mathbf{Alg}(k) \to \mathbf{Grps}$ sending a k-algebra A to the group of invertible elements $A^\times$. To see that this is functorial, one verifies that an algebra homomorphism $\alpha : A \to B$ sends $A^\times$ to $B^\times$, hence induces a group homomorphism $A^\times \to B^\times$.

(3) There is a class of functors called *representable functors.* Let $\mathcal{C}$ be a category such that for any two objects X, X' the class $\mathrm{Hom}_{\mathcal{C}}(X, X')$ is a set. Fix an an object X in $\mathcal{C}$. We define a functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ from $\mathcal{C}$ to the category of sets as follows. For any object Y in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ sends Y to the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. For any morphism $f : Y \to Z$ in $\mathcal{C}$ the functor

$\mathrm{Hom}_{\mathcal{C}}(X, -)$ sends f to the map, denoted $\mathrm{Hom}_{\mathcal{C}}(X, f)$ which is induced by composition with f; that is, which sends $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $f \circ h \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$. One easily sees that this is a functor. This construction applied to $\mathcal{C}^{\mathrm{op}}$ yields also a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$, sending Y to $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ and sending f to the map denoted $\mathrm{Hom}_{\mathcal{C}}(f, X)$ induced by precomposition with f; that is, $\mathrm{Hom}_{\mathcal{C}}(f, X)$ sends $h \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$ to $h \circ f \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$. Functors of this form are called *representable.* If we consider both X and Y as variables, then $\mathrm{Hom}_{\mathcal{C}}(-, -)$ is what we call a *bifunctor* . Depending on what additional structures the category $\mathcal{C}$ has, the representable functors may have as target category not just the category of sets but categories with more structure. For instance, if A is a k-algebra and U an A-module, then the representable functor $\mathrm{Hom}_A(U, -)$ and its contravariant analogue $\mathrm{Hom}_A(-, U)$ are functors from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$.

(4) Let A, B be k-algebras, and let M be an A-B-bimodule. There is a functor $M \otimes_B -$ from $\mathrm{Mod}(B)$ to $\mathrm{Mod}(A)$ sending a B-module V to the A-module $M \otimes_B V$ and a B-homomorphism $\psi : V \to V'$ to the A-homomorphism $\mathrm{Id}_M \otimes \psi : M \otimes_B V \to M \otimes_B V'$. There is a similar functor $- \otimes_A M$ from $\mathrm{Mod}(A^{\mathrm{op}})$ to $\mathrm{Mod}(B^{\mathrm{op}})$. There is a functor $\mathrm{Hom}_A(M, -)$ from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(B)$, sending an A-module U to $\mathrm{Hom}_A(M, U)$, viewed as a B-module via $(b \cdot \varphi)(m) = \varphi(mb)$, where $\varphi \in \mathrm{Hom}_A(M, U)$, $m \in M$, $b \in B$. There is a similar functor $\mathrm{Hom}_{B^{\mathrm{op}}}(M, -)$ from $\mathrm{Mod}(B^{\mathrm{op}})$ to $\mathrm{Mod}(A^{\mathrm{op}})$.

Pushing our philosophy of considering mathematical objects with their structural maps even further, we view now functors as objects and define morphisms between functors as follows.

Definition A.34. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $\mathcal{F}$, $\mathcal{F}'$ be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation from $\mathcal{F}$ to $\mathcal{F}'$* is a family $\varphi = (\varphi(X))_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms $\varphi(X) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}'(X))$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$ we have $\mathcal{F}'(f) \circ \varphi(X) = \varphi(Y) \circ \mathcal{F}(f)$; that is, we have a commutative diagram of morphisms in the category $\mathcal{D}$ of the form

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\varphi(X)} & \mathcal{F}'(X) \\ {\scriptstyle \mathcal{F}(f)} \downarrow & & \downarrow {\scriptstyle \mathcal{F}'(f)} \\ \mathcal{F}(Y) & \xrightarrow[\varphi(Y)]{} & \mathcal{F}'(Y) \end{array}$$

By considering contravariant functors from $\mathcal{C}$ to $\mathcal{D}$ as convariant functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ we get an abvious notion of natural transformation between contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

Every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ gives rise to the *identity transformation* $\mathrm{Id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ consisting of the family of identity morphisms $\mathrm{Id}_{\mathcal{F}(X)}$, $X \in \mathrm{Ob}(\mathcal{C})$. Natural transformations can be composed: if $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ are functors from $\mathcal{C}$ to $\mathcal{D}$ and $\varphi : \mathcal{F} \to \mathcal{F}'$, $\psi : \mathcal{F}' \to \mathcal{F}''$ are natural transformations, then the family $\psi \circ \varphi$ of morphisms $\psi(X) \circ \varphi(X) : \mathcal{F}(X) \to \mathcal{F}''(X)$ is a natural transformation from $\mathcal{F}$ to $\mathcal{F}''$, and this composition of natural transformations is associative. As in the case of the category of categories there are set theoretic issues if we consider the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations as morphisms. If we assume that $\mathcal{C}$ is small, then the functors from $\mathcal{C}$ to an arbitrary category $\mathcal{D}$, together with natural transformations as morphisms, form a category. There is an obvious extension of the natural transformation to bifunctors.

Examples A.35.

(1) Let $\mathcal{C}$ be a category, X, X' objects, and let $\varphi : X \to X'$ be a morphism in $\mathcal{C}$. Then φ induces a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(X', -)$ to $\mathrm{Hom}_{\mathcal{C}}(X, -)$, given by the family of maps $\mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ sending $\tau \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$ to $\tau \circ \varphi$, and φ induces a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(-, X)$ to $\mathrm{Hom}_{\mathcal{C}}(-, X')$ sending $\tau \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ to $\varphi \circ \tau$, for all objects Y in $\mathcal{C}$.

(2) Let A, B be k-algebras and let M, M' be A-B-bimodules. Any bimodule homomorphism $\alpha : M \to M'$ induces a natural transformation from $M \otimes_B -$ to $M' \otimes_B -$ given by the family of maps $\alpha \otimes \mathrm{Id}_V : M \otimes_B V \to M' \otimes_B V$ for all B-modules V. Similarly, any such α induces a natural transformation from $\mathrm{Hom}_A(M', -)$ to $\mathrm{Hom}_A(M, -)$, as in the previous example.

Definition A.36. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Two functors $\mathcal{F}$, $\mathcal{F}'$ from $\mathcal{C}$ to $\mathcal{D}$ are called *isomorphic* if there are natural transformations $\varphi : \mathcal{F} \to \mathcal{F}'$ and $\psi : \mathcal{F}' \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{Id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathcal{F}'}$.

If $\varphi : \mathcal{F} \to \mathcal{F}'$ is a natural transformation such that all morphisms $\varphi(X) : \mathcal{F}(X) \to \mathcal{F}'(X)$ are isomorphisms, then the family of morphisms $\psi(X) = \varphi(X)^{-1}$ is a natural transformation from $\mathcal{F}'$ to $\mathcal{F}$ satisfying $\psi \circ \varphi = \mathrm{Id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathcal{F}'}$.

Definition A.37. Two categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if there are functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{F}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$, and the functors $\mathcal{F}$, $\mathcal{G}$ arising in this way are called *equivalences of categories.*

Thus an equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ need not induce a bijection between $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$, but it induces a bijection between the isomorphism classes in $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$.

Definition A.38. Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be covariant functors. We say that $\mathcal{G}$ is *left adjoint to* $\mathcal{F}$ and that $\mathcal{F}$ is *right adjoint to* $\mathcal{G}$, if there is an isomorphism of bifunctors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$. If $\mathcal{G}$ is left and right adjoint to $\mathcal{F}$ we say that $\mathcal{F}$ and $\mathcal{G}$ are *biadjoint.*

An isomorphism of bifunctors as in Definition A.38 is a family of isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), U) \cong \mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(U)) ,$$

with U an object in $\mathcal{C}$ and V an object in $\mathcal{D}$, such that for fixed U we get an isomorphism of contravariant functors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), U) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(U))$, and for fixed V we get an isomorphism of covariant functors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), -) \cong \mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(-))$. Such an isomorphism of bifunctors, if it exists, need not be unique. If $\mathcal{C}$, $\mathcal{D}$ are k-linear categories for some commutative ring k, we will always require such an isomorphism of bifunctors to be k-linear. Given an adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$, evaluating Φ at an object V in $\mathcal{D}$ and $\mathcal{G}(V)$ yields an isomorphism $\mathrm{Hom}_{\mathcal{D}}(V, \mathcal{F}(\mathcal{G}(V))) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(V))$. We denote by $f(V) : V \to \mathcal{F}(\mathcal{G}(V))$ the morphism corresponding to $\mathrm{Id}_{\mathcal{G}(V)}$ through this isomorphism; that is, $f(V) = \Phi(V, \mathcal{G}(V))(\mathrm{Id}_{\mathcal{G}(V)})$. One checks that the family of morphisms $f(V)$ defined in this way is a natural transformation

$$f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F} \circ \mathcal{G}$$

called the *unit* of the adjunction isomorphism Φ, where $\mathrm{Id}_{\mathcal{D}}$ denotes the identity functor on $\mathcal{D}$ (sending every object and every morphism in $\mathcal{D}$ to itself). Similarly, evaluating Φ at an object

U in $\mathcal{C}$ and at $\mathcal{F}(U)$ yields an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U)), U) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U), \mathcal{F}(U))$. We denote by $g(U) : \mathcal{G}(\mathcal{F}(U)) \to U$ the morphism corresponding to $\mathrm{Id}_{\mathcal{F}(U)}$ through the isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U)), U) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U), \mathcal{F}(U))$; that is, $g(U) = \Phi(\mathcal{F}(U), U)^{-1}(\mathrm{Id}_{\mathcal{F}(U)})$. Again, this is a natural transformation

$$g : \mathcal{G} \circ \mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$$

called the *counit* of the adjunction isomorphism Φ.

An adjunction isomorphism is uniquely determined by its unit and counit. To state this properly we need the following notation. Given two functors $\mathcal{F}$, $\mathcal{F}' : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi : \mathcal{F} \to \mathcal{F}'$, we denote for any functor $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ by $\mathcal{G}\varphi : \mathcal{G}\circ\mathcal{F} \to \mathcal{G}\circ\mathcal{F}'$ the natural transformation given by $(\mathcal{G}\varphi)(U) = \mathcal{G}(\varphi(U)) : \mathcal{G}(\mathcal{F}(U)) \to \mathcal{G}(\mathcal{F}'(U))$ for any object U in $\mathcal{C}$. Similarly, for any functor $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ we denote by $\varphi\mathcal{H} : \mathcal{F} \circ \mathcal{H} \to \mathcal{F}' \circ \mathcal{H}$ the natural transformation given by $\varphi(\mathcal{H}(W)) : \mathcal{F}(\mathcal{H}(W)) \to \mathcal{F}'(\mathcal{H}(W))$ for any object W in $\mathcal{E}$. We denote by $\mathrm{Id}_{\mathcal{F}}$ the identity natural transformation on $\mathcal{F}$, given by the family of identity morphisms $\mathrm{Id}_{\mathcal{F}(U)}$, with U running over the objects of $\mathcal{C}$.

Theorem A.39. *Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be covariant functors.*

(i) Suppose there is an adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$. The unit f and counit g of Φ satisfy $(\mathcal{F}g) \circ (f\mathcal{F}) = \mathrm{Id}_{\mathcal{F}}$ and $(g\mathcal{G}) \circ (\mathcal{G}f) = \mathrm{Id}_{\mathcal{G}}$.

(ii) Let $f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F}\circ\mathcal{G}$ and $g : \mathcal{G}\circ\mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$ be two natural transformations satisfying $(\mathcal{F}g)\circ(f\mathcal{F}) = \mathrm{Id}_{\mathcal{F}}$ and $(g\mathcal{G}) \circ (\mathcal{G}f) = \mathrm{Id}_{\mathcal{G}}$. There is a unique adjunction isomorphism $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$ such that f is the unit of Φ and g is the counit of Φ.

(iii) Let $\Phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-), -) \cong \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F}(-))$ be an adjunction isomorphism with unit f and counit g. Then $\Phi(V, U)(\varphi) = \mathcal{F}(\varphi) \circ f(V)$ for any object U in $\mathcal{C}$, any object V in $\mathcal{D}$ and any morphism $\varphi : \mathcal{G}(V) \to U$ in $\mathcal{C}$, and $\Phi(V, U)^{-1}(\psi) = g(U) \circ \mathcal{G}(\psi)$ for any morphism $\psi : V \to \mathcal{F}(U)$ in $\mathcal{D}$. In particular, we have $\varphi = g(U) \circ \mathcal{G}(\mathcal{F}(\varphi) \circ f(V))$ and $\psi = \mathcal{F}(g(U) \circ \mathcal{G}(\psi)) \circ f(V)$.

See e. g. [6, Chapter 2, Section 3] for a proof and more details. The following adjunction is known as the Tensor-Hom adjunction.

Theorem A.40. *Let A, B be k-algebras and let M be an A-B-bimodule. For any A-module U and any B-module V we have natural inverse isomorphisms of k-modules*

$$\begin{cases} \mathrm{Hom}_A(M \otimes_B V, U) & \cong & \mathrm{Hom}_B(V, \mathrm{Hom}_A(M, U)) \\ \varphi & \to & (v \mapsto (m \mapsto \varphi(m \otimes v))) \\ (m \otimes v \mapsto \psi(v)(m)) & \longleftarrow & \psi \end{cases}$$

In particular, the functor $M \otimes_B - : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$ is left adjoint to the functor $\mathrm{Hom}_A(M, -) : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$.

The proof of Theorem A.40 is a straightforward verification.

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