

London Taught Course Centre

2026 Examination

Graph Theory

Instructions to candidates

This open-book exam has 3 questions. Some parts are harder than others. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed and you may use results you find there (remember though that you should show own work for each question), but you should reference your sources.

Question 1

Let $\mathcal{G}$ be the class of graphs G with the following property. We can identify a set of edges E^* in G such that the graph $G - E^*$ obtained from G by removing all edges in E^* is planar, and such that for any pair of distinct edges $e, f \in E^*$ and any pair $u \in e, v \in f$ of vertices the distance of u and v in G is at least 2.

- (a) Provide and prove a tight upper bound on the edge chromatic number of graphs in $\mathcal{G}$ on n vertices.
 - (b) Provide and prove a tight upper bound on the chromatic number of graphs in $\mathcal{G}$ on n vertices.
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Question 2

Given an integer k , a C_k -factor of size s (where s is a multiple of k) is a vertex disjoint union of s/k cycles C_k of length k .

For U and V disjoint with n vertices each, let (U, V) be an ε -regular pair with density at least d . Given k and $d > 0$, what is the largest size of a C_k -factor you can find in (U, V) when ε is sufficiently small?

Question 3

Consider the following decision problem FIXED-3-COL. The input for this problem is a graph $G = (V, E)$ and a subset of vertices $S \subseteq V$ with a fixed colouring. More precisely, each vertex in S is already given a fixed colour from the colour palette $\{1, 2, 3\}$. We assume further that the following property is satisfied. Each vertex of G either has a fixed colour or has a neighbour with a fixed colour. The decision problem FIXED-3-COL then is to decide if the so far uncoloured vertices of G can also be coloured with colours from the palette $\{1, 2, 3\}$ so that we obtain a proper (vertex) colouring of G . Show that FIXED-3-COL is in P.