

London Taught Course Centre

2025/26 Mock Paper Solutions

Graph Theory

Answers

- 1 (a) Given a tree T , we obtain an order $V(T) = \{v_1, \dots, v_n\}$ by removing a leaf (degree one vertex) v_n from T and repeating this in the resulting tree $T_{n-1} := T \setminus \{v_n\}$, and so on until T_1 is a single vertex v_1 .

Observe that the vertices of T^2 adjacent to v_i which come before v_i in this order are the unique v_j such that $v_i v_j \in E(T)$ and $j < i$, and the set $N(v_j) \cap \{v_1, \dots, v_{i-1}\}$.

In total at most $\Delta(T)$ neighbours in T^2 of v_i precede v_i in the ordering; so the greedy colouring algorithm, run on T^2 in this order, uses at most $\Delta(T) + 1$ colours. In the other direction, letting v be a vertex of T with $d(v) = \Delta(T)$, the set $\{v\} \cup N(v)$ is a set of $\Delta(T) + 1$ vertices which form a clique in T^2 , hence $\chi(T^2) \geq \Delta(T) + 1$.

- (b) This is obtained by modifying the tree argument from (a): namely, given a planar graph G , order $V(G)$ such that each vertex has at most five predecessors in the order. That this is possible follows from the bound $e(G') \leq 3v(G') - 6$ on the number of edges of a planar graph G' with 3 or more vertices, from which we see that the average degree is less than 6 so that there is a vertex of degree 5 or less. Colouring greedily in this order, when we come to colour v_i , each of its at most five predecessors has at most $\Delta(G) - 1$ neighbours other than v_i , which could all be before v_i in the order; in addition the at most $\Delta(G) - 5$ neighbours of v_i which come after it in the order have at most five predecessors each in the order, one of which is v_i but the rest of which could all come before v_i . No other vertex at distance one or two in G from v_i can precede v_i , so we conclude that in G^2 at most $5\Delta(G) + 4(\Delta(G) - 5)$ neighbours of v_i precede v_i in the order. The greedy algorithm thus uses at most $9\Delta(G) - 19$ colours, as desired.
- (c) This is an open problem. More accurately, there are planar graphs G with $\Delta(G) \leq 3$ such that $\chi(G^2) = 7$, for $4 \leq k \leq 7$ there are planar graphs G with $\Delta(G) \leq k$ such that $\chi(G^2) = k + 5$, and for $k \geq 8$ that there are planar graphs G with $\Delta(G) \leq k$ such that $\chi(G^2) = \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$, and it is conjectured (by Wegner) that for each value of k there are no planar graphs G with $\Delta(G) \leq k$ such that G^2 has larger chromatic number. A proof for the $k = 3$ case was announced in April 2016 by Hartke, Jahanbekam and Thomas (a previous 2006 announcement by Thomassen of the same seems to have suffered from a flaw in the proof); all other cases remain open at the time of writing. However it is known that the conjecture is at least close to true: Havet, van den Heuvel, McDiarmid and Reed showed in 2008 that for each $\gamma > 0$, if k is large enough then all planar graphs G with $\Delta(G) \leq k$ are such that G^2 can be properly vertex-coloured with $(\frac{3}{2} + \gamma)k$ colours.

Note that the cases $\Delta(G) = 1, 2$ are fairly easy to settle; the square of a matching ($\Delta(G) = 1$) is a matching and has chromatic number 2; the square of a path or cycle ($\Delta(G) = 2$) is always 4-colourable except for the square of C_5 , which is K_5 and needs five colours. In these cases the restriction to planar graphs is superfluous.

For the specific (non-optimal but still an open problem) bound I asked for, you can also do $\Delta(G) = 3$ fairly quickly. If G is planar and has maximum degree 3, we want to show $\chi(G^2) \leq \frac{3}{2} \cdot 3 + 5$, in other words $\chi(G^2) \leq 9$. We can assume G is connected (otherwise we can colour components of G^2 independently). Now G^2 has maximum degree at most 9, because any given $v \in G$ has at most three neighbours each of which has at most two neighbours other than v . So by Brooks' Theorem, we are done unless G^2 is K_{10} . But the only way we can get $G^2 = K_{10}$ is if each vertex of G^2 has degree

9; in other words, for each vertex $v \in G$, we have $d(v) = 3$, the neighbours of v form an independent set, and there are six second neighbours of v (the neighbours of the neighbours of v which aren't v), which means that G is a 3-regular graph on 10 vertices which doesn't contain a triangle or a C_4 . There is only one such graph. To see this, fix a vertex v , its neighbours x, y, z and their respective neighbours $x', x'', y', y'', z', z''$. The only edges we don't already know in G are those between $x', x'', y', y'', z', z''$ which have to form a 2-regular graph. There are only two 2-regular graphs on six vertices, namely C_6 and two disjoint K_3 . But G doesn't contain a triangle, so we have to have C_6 . And because G doesn't contain a triangle or C_4 , the pair x', x'' have to be opposite on the C_6 , and the same for y', y'' and z', z'' . This fixes the graph up to isomorphism: it is the Petersen graph, which contains K_5 as a minor (check it!) and therefore is not planar.

- 2** (a) This is a standard algorithmic task and solutions can easily be found online.
- (b) Given $\gamma > 0$, choose $d = \gamma/10$ and $\varepsilon = d/10$. Now given G , take an ε -regular partition $V_0, \dots, V_t$ of $V(G)$ with $t + 1$ parts, where $\varepsilon^{-1} \leq t \leq K(\varepsilon)$ parts, which we are told is possible in polynomial time. Draw a graph H on $[t]$ by putting an edge ij whenever (V_i, V_j) is a pair in G of density at least d . We can construct this graph in time $O(n^2)$, as we simply have to count edges. We examine all 3-colourings of $[t]$. We answer ‘Yes’ if there is a 3-colouring of $[t]$ such that at most εt^2 edges are not properly coloured, and otherwise ‘No’. This last step takes a constant (independent of n) time and hence the algorithm in total runs in polynomial time in n .

If G is 3-colourable, fix a proper 3-colouring c of $V(G)$. We derive a colouring c' of $[t]$ by colouring i with a majority colour used on V_i (breaking ties arbitrarily). Observe that if ij is an edge of H such that $c'(i) = c'(j)$, then (V_i, V_j) is not ε -regular in G , because otherwise by ε -regularity there would be an edge from the vertices in V_i of colour $c'(i)$ to the vertices in V_j of the same colour. Since an ε -regular partition contains at most εt^2 pairs which are not ε -regular, there are at most εt^2 edges of H which are not properly coloured by c' ; so our algorithm indeed returns ‘Yes’.

If our algorithm returns ‘Yes’, fix a colouring c' of $[t]$ such that at most εt^2 edges are not properly coloured. We construct a set S of edges of G as follows: we put into S each edge intersecting V_0 , each edge in a part V_i , and each edge between a pair (V_i, V_j) such that $c'(i) = c'(j)$. Now let c on $V(G)$ be defined by $c(v) = 1$ if $v \in V_0$, and $c(v) = c'(i)$ if $v \in V_i$. By definition of S this is a proper 3-colouring of $G - S$; on the other hand, since $|V_0| \leq \varepsilon n$ there are at most εn^2 edges intersecting V_0 , for each i since $|V_i| \leq n/t$ there are at most n^2/t^2 edges in V_i , and hence at most $n^2/t \leq \varepsilon n^2$ edges within parts. Since there are at most εt^2 pairs ij which are edges of H but not properly coloured, at most $\varepsilon t^2(n/t)^2 = \varepsilon n^2$ edges are in pairs (V_i, V_j) such that ij is an edge of H not properly coloured, and by definition less than $d(n/t)^2 \binom{t}{2} \leq dn^2$ edges are in pairs (V_i, V_j) such that ij is not an edge of H . This covers all the edges which can be in S , so $|S| \leq (3\varepsilon + d)n^2 < \gamma n^2$. In particular, if our algorithm returns ‘Yes’ then there is a set S of at most γn^2 edges of G such that $G - S$ is 3-colourable.