

LTCC Homological algebra examination 2025/26.

1. Let A be an algebra over some commutative ring. Let X be a chain complex of A -modules. Suppose that there are only finitely many integers n such that $H_n(X) \neq \{0\}$.

(a) Show that there exists a bounded below complex of A -modules Y and a quasi-isomorphism $f : Y \rightarrow X$.

(b) Show that there is a bounded above complex of A -modules Z and a quasi-isomorphism $g : X \rightarrow Z$.

2. Let A be an algebra over some commutative ring. Let X be a chain complex of A -modules. For any integer n set $C(X)_n = X_{n-1} \oplus X_n$ and denote by $\Delta : C(X)_n \rightarrow C(X)_{n-1}$ the A -homomorphism given by

$$\Delta_n(x, y) = (-\delta_{n-1}(x), x + \delta_n(y))$$

(a) Show that $(C(X)_n, \Delta_n)_{n \in \mathbb{Z}}$ is a contractible chain complex.

(b) Show that there is an exact sequence of chain complexes

$$0 \longrightarrow X \longrightarrow C(X) \longrightarrow X[1] \longrightarrow 0$$

3. For any integer $n \geq 0$ calculate the abelian groups $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ and $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}, \mathbb{Q})$.

4. Let A be an algebra over some commutative ring k . Let $d : A \rightarrow A$ be a k -linear derivation. Let e be an idempotent in the centre $Z(A)$ of A . Show that $d(e) = 0$. Show, by giving an example, that the hypothesis on e being central is necessary.

5. Suppose that k is a field. Calculate the dimension of $HH^1(k[x]/(x^4))$. (*Hint:* calculate all derivations on $k[x]/(x^4)$.)

6. Let $(E_r^{p,q})_{r \geq a}$ be a first quadrant spectral sequence of abelian groups (that is $E_r^{p,q}$ is zero if at least one of p or q is negative). Show that for any integer $n \geq a$ and any $(p, q) \in \mathbb{Z}^2$ such that $p + q = n$ we have $E_{n+2}^{p,q} = E_{\infty}^{p,q}$.