

Exam for the LTCC Course *Introduction to Optimal Transport*

March 2026

Exercise 1 (25 points).

- (a) Let $\mu = \mathcal{N}(m_0, \sigma_0^2)$ and $\nu = \mathcal{N}(m_1, \sigma_1^2)$ be Gaussian probability measures on $\mathbb{R}$ and consider the quadratic cost $c(x, y) = \frac{1}{2}|x - y|^2$. Explicitly compute the optimal transport map T . Compute the optimal transport cost.
- (b) Assume that μ, ν are probability measures on $\mathbb{R}^2$ such that

$$\mu = \mathcal{N}_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad \nu = \mathcal{N}_2 \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \right)$$

and consider the quadratic cost $c(x, y) = \frac{1}{2}|x - y|^2$ between them, where we recall that $|\cdot|$ is the Euclidean distance. For each of the following maps, say whether it is (or not) an optimal transport map between μ and ν and why.

- (i) $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$T_1(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x;$$

- (ii) $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$T_2(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x;$$

- (iii) $T_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$T_3(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Exercise 2 (20 points).

- (a) Given $a, b \in [0, 1]$ with $a + b = 1$ and $t < s$ consider $\mu := a\delta_t + b\delta_s$ as a probability measure on $\mathbb{R}$, and let ν be the uniform probability measure on the interval $[-1, 1]$. Compute $W_2(\mu, \nu)$ and find the values of a, b, t, s for which this distance is minimal.
- (b) Consider now a two-dimensional analogue of the previous situation: $\mu := a\delta_{(t,0)} + b\delta_{(s,0)}$ is a probability measure on $\mathbb{R}^2$, and let ν is the uniform probability measure on the square $[-1, 1]^2$. Again, compute $W_2(\mu, \nu)$ and find the values of a, b, t, s for which this distance is minimal.

Exercise 3 (20 points).

Let $\mathcal{M}_+(X)$ be the set of non-negative measures on X , and let $\mathcal{P}(Y)$ be the set of probability measures on Y . We consider the partial optimal transport problem between $\mu \in \mathcal{M}_+(X)$ with $1 < \mu(X) < +\infty$ and $\nu \in \mathcal{P}(Y)$, defined for some cost $c : X \times Y \rightarrow \mathbb{R}$ by

$$(\text{KPP}) = \inf_{\gamma \in \Gamma_-(\mu, \nu)} \int c d\gamma,$$

where $\Gamma_-(\mu, \nu) = \{\gamma \in \mathcal{M}_+(X \times Y) \text{ with marginals } \pi_X \text{ and } \pi_Y \mid \pi_X \# \gamma \leq \mu \text{ and } \pi_Y \# \gamma = \nu\}$.

(Note that for $\alpha, \beta \in \mathcal{M}_+(X)$, $\alpha \leq \beta$ means that for all functions u in the set $C^0(X)$ of continuous functions on X , we have $\int u d\alpha \leq \int u d\beta$.)

(a) Prove that $(\text{KPP}) \geq (\text{KPD})$ where

$$(\text{KPD}) = \sup_{\substack{(u, v) \in C^0(X) \times C^0(Y) \\ u \oplus v \leq c, v \leq 0}} \left(\int u d\mu + \int v d\nu \right)$$

(b) Recalling that for $v \in C^0(Y)$, $v^c(x) = \inf_{y \in Y} (c(x, y) - v(y))$, prove that

$$(\text{KPD}) = \sup_{v \in C^0(Y)} \left(\int \min(v^c, 0) d\mu + \int v d\nu \right).$$

(c) Let $\bar{Y} = Y \cup \{y_\infty\}$ where $y_\infty \notin Y$ is an arbitrary point, let $\bar{\nu} = \nu + \alpha \delta_{y_\infty}$ where $\alpha = \mu(X) - 1$, and let $\bar{c} : X \times \bar{Y} \rightarrow \mathbb{R}$ defined by $\bar{c}|_{X \times Y} = c$ and $\bar{c}(x, y_\infty) = 0$ for all $x \in X$.

Using an explicit bijection between $\Gamma(\mu, \bar{\nu})$ and $\Gamma_-(\mu, \nu)$, prove that $(\text{KPP}) = (\overline{\text{KP}})$, where $(\overline{\text{KP}})$ is the “standard” optimal transport problem

$$(\overline{\text{KP}}) = \min_{\bar{\gamma} \in \Gamma(\mu, \bar{\nu})} \int_{X \times \bar{Y}} \bar{c} d\bar{\gamma}.$$

Exercise 4 (20 points).

Consider the Coulomb cost $C(x, y) = 1/|x - y|$, $x, y \in \mathbb{R}$. Assume that μ is the uniform probability measure on $[0, 1]$. Find the optimal transport map from μ to μ .

[**Hint:** You may want to use in your proof the functions $u(x) = 2 - 4|x - \frac{1}{2}|$ and $v(y) = 2 - 4|y - \frac{1}{2}|$ as a guess for the Kantorovich potentials.]

Exercise 5 (15 points).

Assume that μ and ν have pdfs f and g respectively, both with support on $[0, 1]$. Show that

$$W_2^2(\mu, \nu) \leq \int_0^1 |f(x) - g(x)| dx.$$

[**Hint:** You may want to use in your solution the Kantorovich-Rubinstein Theorem.]