

You may refer without proof to results from the course (theorems, examples, etc.).

Q1

- (i) Suppose $X \stackrel{d}{=} \mathcal{N}(0, 1)$. Does the joint distribution of random variables X and X^2 possess a density (relative to the Lebesgue measure on the plane)?
- (ii) In the setting of (i), determine $\mathbb{E}[X^2 | X]$ and $\mathbb{E}[X | X^2]$.
- (iii) Suppose positive random variables $X_1, X_2, \dots, X_n$ are i.i.d. with finite mean. Determine $\mathbb{E}[X_1 | X_1 + \dots + X_n]$.

Q2 In this question U is a random variable with uniform distribution on $[0, 1]$.

- (i) Let $F(x), x \in \mathbb{R}$, be a strictly increasing, continuous distribution function, and denote F^{-1} the function inverse to F , i.e. satisfying $F(F^{-1}(p)) = p$ for $0 < p < 1$. Show that the random variable $X = F^{-1}(U)$ has distribution function F .
- (ii) Let $F_1(x), F_2(x), x \in \mathbb{R}$, be strictly increasing continuous distribution functions satisfying $F_1(x) < F_2(x)$. Construct a probability space and random variables X_1, X_2 defined on this space to satisfy the following three conditions:
 1. X_1 has distribution function F_1 ,
 2. X_2 has distribution function F_2 ,
 3. the inequality $X_1 > X_2$ holds almost surely.
- (iii) For X_1, X_2 as in part (ii), can X_1, X_2 be independent? Answer this question but with condition $X_1 > X_2$ omitted.

Q3 Let $x_1, x_2, \dots$ be a numerical sequence with $x_n \in [0, 1]$ for $n \in \mathbb{N}$, and let U be a random variable with uniform distribution on $[0, 1]$. Consider

$$N = \#\{n \in \mathbb{N} : x_n > U\}$$

and note that

$$N = \sum_{n=1}^{\infty} X_n,$$

where $X_n = 1(U < x_n)$ (indicator random variable).

- (i) Find a necessary and sufficient condition on the sequence to ensure that $\mathbb{P}[N < \infty] = 1$.
- (ii) Find a necessary and sufficient condition on the sequence to ensure that $\mathbb{E}[N] < \infty$.
- (iii) Let $x_n = q^n, n \in \mathbb{N}$, for some $q \in (0, 1)$. Calculate $\text{Var}[N]$.
- (iv) Give an example of sequence $x_1, x_2, \dots$ such that $\mathbb{P}(N < \infty) = 1$ and $\mathbb{E}N = \infty$. Explain why $N < \infty$ a.s. does not contradict the Borel-Cantelli lemma.

Q4 Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$, and let τ and ρ be stopping times adapted to this filtration.

- (i) Show that $\tau \wedge \rho$ and $\tau \vee \rho$ are stopping times ($\wedge = \min, \vee = \max$).
- (ii) Let $\mathcal{F}_\tau$ be the set of events $A \in \mathcal{F}$ with the property

$$A \cap \{\tau = n\} \in \mathcal{F}_n, \text{ for } n \in \mathbb{N}.$$

Show that $\mathcal{F}_\tau$ is a σ -algebra.

- (iii) Let $B_1, B_2, \dots$ independent Bernoulli variables with probability $0 < p < 1$ for $B_n = 1$, and let $\tau = \min\{n : B_{n-2} = B_{n-1} = B_n = 1\}$. Assuming that $\mathcal{F}_n = \sigma(B_1, \dots, B_n)$ give two nontrivial examples of such events $A \in \mathcal{F}_\tau$ (see (ii) for the definition of $\mathcal{F}_\tau$).
- (iv) A version of Doob's optional sampling theorem asserts that if $X_1, X_2, \dots$ is a martingale and $\tau \leq \rho \leq N$ for some $N \in \mathbb{N}$ then $\mathbb{E}[X_\rho | \mathcal{F}_\tau] = X_\tau$. Dropping the assumption $\tau \leq \rho$ use the result to prove that $\mathbb{E}[X_\rho | \mathcal{F}_\tau] = X_{\rho \wedge \tau}$.

Q5 Let $(B(t), t \geq 0)$ be a Brownian motion (BM). For $t > 0, x > 0$ consider

$$M(t) := \max_{s \in [0, t]} B(s), \text{ and } \tau_x = \min\{t : B(t) = x\}.$$

- (i) Find out how the random variable $M(t)$ relates to $M(1)$, and write down the density function of $M(t)$.
- (ii) Find out how the random variable τ_x relates to τ_1 , and write down the density function of τ_x .
- (iii) The same question as in (ii), but now with $x < 0$.