

MTP2024

Q1 (i) The function F is nondecreasing, right-continuous with left limits. For instance,

$$(-\infty, x) = \bigcup_n (-\infty, x - \frac{1}{n})$$

and $(-\infty, x - \frac{1}{n}) \subset (-\infty, x - \frac{1}{n+1})$
imply left continuity.

If $F(x-) < F(x)$, then $\mu(\{x\}) > 0$ and μ has an atom at x .

Thus F is continuous, hence takes all intermediate values

$$F(-\infty) = 0 < t < \mu(\mathbb{R}) = F(\infty).$$

(ii) (1). For $B \subset A$, $\mu(B) \leq \frac{1}{2} \mu(A)$ or $\mu(A \setminus B) \leq \frac{1}{2} \mu(A)$,

so we can find $B_1 \subset A$ s.t. $0 < \mu(B_1) \leq \frac{1}{2} \mu(A)$.

Since B_1 is not an atom, there exists $B_2 \subset B_1$ s.t.

$$0 < \mu(B_2) \leq \frac{1}{2} \mu(B_1) \leq \frac{1}{4} \mu(A).$$

Continuing so forth, construct $B_1 \supset B_2 \supset B_3 \supset \dots$

$$\text{with } \mu(B_k) \leq \frac{1}{2^k} \mu(A).$$

Now choose suitable k .

(2) By (1) we have $0 < \mu(A) \leq \varepsilon$ if $\mu(A) > 0$ and

there exists A_1 with $0 < \mu(A_1) \leq \mu(A) \leq \varepsilon$.

(2)

Note also that $A \subset B$ implies $\varphi(A) \leq \varphi(B)$.

If $\mu(\Omega \setminus A_1) = 0$ we are done, otherwise

$\varphi(\Omega \setminus A_1) > 0$ and by the definition of φ there exists $A_2 \subset \Omega \setminus A_1$ such that

$$\frac{1}{2} \varphi(\Omega \setminus A_1) \leq \mu(A_2) \leq \varphi(\Omega \setminus A_1) \leq \varepsilon.$$

Continuing so forth, either we construct a finite sequence $A_1, \dots, A_n$ with $\mu(\Omega \setminus \bigcup_{k=1}^n A_k) = 0$ or infinite sequence of disjoint $A_1, A_2, \dots$. In the latter case for $A_0 := \Omega \setminus \bigcup_{k=1}^{\infty} A_k$ we get

$$\varphi(A_0) \leq \varphi(\Omega \setminus \bigcup_{k=1}^n A_k) \leq 2\mu(A_{n+1}) \quad n=1, 2, \dots$$

But $\sum_n \mu(A_n) \leq \mu(\Omega) < \infty$ hence $\mu(A_n) \rightarrow 0$

and $\varphi(A_0) = 0$ implies $\mu(A_0) = 0$. Finally, choose

n to have $\sum_{k=n}^{\infty} \mu(A_k) < \varepsilon$; then

$$A_1, \dots, A_{n-1}, \left(\bigcup_{k=n}^{\infty} A_k \right) \text{ is } \subset A_0$$

is a partition of Ω with desired property.

(iii) Using (ii) it is possible to construct $A_1 \subset A_2 \subset \dots$ (3)

with $\mu(A_n) \rightarrow t$. Choose k large enough and find $B_1, \dots, B_m$ disjoint with $\mu(B_i) \leq \frac{1}{k}$ and $\bigcup_{i=1}^m B_i = \Omega$.

Take a subset $B_1, \dots, B_\ell$ such that

$$\mu(B_1 \cup \dots \cup B_\ell) \leq t < \mu(B_1 \cup \dots \cup B_\ell \cup B_{\ell+1})$$

then $t - \mu(B_1 \cup \dots \cup B_\ell) \leq \frac{1}{k}$.

Then partition $\Omega \setminus (B_1 \cup \dots \cup B_\ell)$ to find a better approximation to t , in terms of measure.

For $A_1 \subset A_2 \subset \dots$ sequence of such approximations take $A = \bigcup_n A_n$ to obtain $\mu(A) = t$.

Yes, it is enough to show existence of A with $\mu(A) = \frac{1}{2} \mu(\Omega)$. Binary expansion of t tells how iterates should be constructed.

Q2(i) $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(R_1, \dots, R_n)$, and

conditioning on $\mathcal{F}_n$ gives

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | \sigma(R_1, \dots, R_n)] = \mathbb{E}[X_{n+1} | R_n]$$

because $R_1, R_2, \dots$ is a Markov chain. Set $R_n = m$,

so $X_n = \frac{m}{nd+r+g}$, then computing transition probabilities for $m \rightarrow m+d$ and $m \rightarrow m$ gives

$$\begin{aligned} \mathbb{E}[X_{n+1} | R_n = m] &= \frac{m+d}{nd+d+r+g} \frac{m}{nd+r+g} \\ &\quad + \frac{m}{nd+d+r+g} \frac{nd+r+g-m}{nd+r+g} = \frac{m}{nd+r+g} \end{aligned}$$

$$\text{Thus } \mathbb{E}[X_{n+1} | R_n] = \frac{R_n}{nd+r+g}$$

$$\Leftrightarrow \mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | X_n] = X_n.$$

(ii) This follows by martingale convergence, by also noting that $0 \leq X_n \leq 1$ so (X_n) is uniformly integrable.

(iii) $R_{n+1} = 1$ if $R_n = 1$ and green ball drawn at step $n+1$, so (5)

$$\mathbb{P}(R_{n+1} = 1) = \frac{1}{n+1} \cdot \frac{n+1}{n+2} = \frac{1}{n+2},$$

where we assumed by induction that R_n is uniform.

Similarly, for $1 < m \leq n+1$

$$\mathbb{P}(R_{n+1} = m) = \frac{1}{n+1} \cdot \frac{m-1}{n+2} + \frac{1}{n+1} \cdot \frac{n+2-m}{n+2} = \frac{1}{n+2}.$$

This proves induction step.

(iv) Appropriate filtration is $\mathcal{F}_n = \sigma(R_n, R_{n+1}, \dots)$,

so
$$\mathbb{E}\left[\frac{R_{n+1}-1}{n} \mid \mathcal{F}_{n+1}\right] = \mathbb{E}\left[\frac{R_n-1}{n} \mid R_{n+1}\right] \text{ by}$$

Markov property. The backward transition probability is computed as (using uniform distribution)

$$\mathbb{P}(R_n = m-1 \mid R_{n+1} = m) = \frac{\mathbb{P}(R_n = m-1, R_{n+1} = m)}{\mathbb{P}(R_{n+1} = m)} =$$

$$= \frac{\mathbb{P}(R_{n+1} = m \mid R_n = m-1) \mathbb{P}(R_n = m-1)}{\mathbb{P}(R_{n+1} = m)} =$$

$$\frac{\frac{m-1}{n+2} \cdot \frac{1}{n+1}}{\frac{1}{n+2}} = \frac{m-1}{n+1} \quad \text{hence}$$

$$\mathbb{P}(R_n = m \mid R_{n+1} = m) = \frac{n+2-m}{n+1}.$$

(6)

The conditional expectation becomes

$$E\left[\frac{R_n - 1}{n} \mid R_{n+1} = m\right] = \frac{m-1}{n+1} \frac{m-2}{n} + \frac{h+2-m}{n+1} \frac{m-1}{n}$$

$$= \frac{m-1}{n+1}$$

That is $E\left[\frac{R_n - 1}{n} \mid \mathcal{Y}_{n+1}\right] = E\left[\frac{R_n - 1}{n} \mid R_{n+1}\right] =$

$$\frac{R_{n+1} - 1}{n+1}$$

$\Rightarrow \left(\frac{R_n - 1}{n}, n = 1, 2, \dots\right)$ is a reversed martingale.

Q3 (i) Suppose $s < u < t$, then by the Markov property

$$E[B(t) \mid B(s)=x, B(u)=y] = E[B(t) \mid B(u)=y] = y$$

because the BM is a martingale. Case $t < s < u$ is similar.

Suppose $t < s < u$, then by independence of increments

$$E[B(t) \mid B(s)=x, B(u)=y] = E[B(t) \mid B(s)=x] =$$

$$= E\left[\left(B(t) - \frac{t}{s} B(s)\right) + \frac{t}{s} B(s) \mid B(s)=x\right] =$$

$$= E\left[B(t) - \frac{t}{s} B(s)\right] + \frac{t}{s} E[B(s) \mid B(s)=x] = \frac{t}{s} x$$

(7)

where we used that $B(t) - \frac{t}{s} B(s)$ and $B(s)$ for $t < s$ are independent. Indeed, they are jointly normal and

$$\text{Cov}(B(t) - \frac{t}{s} B(s), B(s)) = t - \frac{t}{s} \cdot s = 0.$$

Case $t < u < s$ is similar.

The answer in the case $s < t < u$ is

$$E[B(t) | B(s)=x, B(u)=y] = \frac{t-s}{u-s} x + \frac{u-t}{u-s} y.$$

This is reduced to the case $E[B(s) | B(t)]$, $s < t$ using that for fixed $c > 0$

$W(t) = B(t+c) - B(c)$ is a BM independent of the path $(B(p), p \leq c)$.

(ii) (a) Let $M(t) = \max \{B(s), s \leq t\}$.

Then $\{\tau_x \leq t\} = \{M(t) \geq x\}$ and (see Lecture 5)

$$P[M(t) \geq x] = P[M(1) \geq \frac{x}{\sqrt{t}}] = P[|B(1)| \geq \frac{x}{\sqrt{t}}]$$

$$= \sqrt{\frac{2}{\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy.$$

Differentiating in t gives the density of τ_x :

$$f(t) = \frac{x}{t^{3/2} \sqrt{2\pi}} e^{-\frac{x^2}{2t}}, \quad t > 0.$$

$$\text{Then } E\tau_x = \int_0^{\infty} t f(t) dt = \infty.$$

(b) By the strong Markov property

$W(t) = B(t + \tau_x) - B(\tau_x)$ is a BM independent of τ_x . Therefore the time for W to reach level y has same distribution as τ_y . The claim follows since W reaches y when B reaches $x+y$.

(c) This is shown by induction, similarly to (b).

(iii) Given $\tau_x = t$ the distribution of $W(\tau_x)$ is $N(0, t)$.

Therefore by total probability, the density of $W(\tau_x)$

$$g(y) = \int_0^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \frac{x}{t^{3/2} \sqrt{2\pi}} e^{-\frac{x^2}{2t}} dt =$$

$$\text{substitute } z = \frac{1}{2}t(x^2 + y^2)$$

$$= \int_0^{\infty} \frac{x}{\pi(x^2 + y^2)} e^{-z} dz = \frac{x}{\pi(x^2 + y^2)}$$

Which is the density function (in variable y) of the Cauchy distribution with parameter x .