

Measure Theory:

The λ^* stands for Lebesgue outer measure and λ for the Lebesgue measure defined on Lebesgue measurable sets.

1. Let $f_0 \geq f_1 \geq \dots \geq 0$ be a sequence of real valued measurable functions defined on $\mathbf{R}$ such that $\int_I f_i(x)dx < \infty$ for every finite interval I of $\mathbf{R}$ and for every fixed $x \in \mathbf{R}$ $\lim_{n \rightarrow \infty} f_n(x) = 0$. Show that $f = \sum_{n=0}^{\infty} (-1)^n f_n$ is a Lebesgue measurable function, $-\infty < \int_I f(x)dx < \infty$, and $\int_I f(x)dx = \sum_{i=0}^{\infty} (-1)^i \int_I f_i(x)dx$ for every finite interval I .
2. Let X be a (probability) measure space with sigma-algebra $\mathcal{A}$, m a countably additive measure defined on $\mathcal{A}$ and $m(X) = 1$. Assume additionally that there exists no point x in X such that $m(\{x\}) > 0$. True or false: there is a subset B of X such that $m(B) = \frac{1}{2}$. Justify your answer.
3. Let A be a subset of the real numbers in $[0, 1)$. For any decimal expression of $r \in [0, 1)$ as $.r_1 r_2 \dots$ with $r = \sum_{i=1}^{\infty} r_i 10^{-i}$ and for every i $r_i \in \{0, 1, \dots, 9\}$ we assume that r is never represented with $r_k = 9$ for all $k \geq l$ for some l , $r_{l-1} \neq 9$, as then r is already represented by $.r_1 \dots r_{l-1} + 1, 0, \dots$ with $r_k = 0$ for all $k \geq l$. Assume that from any r in A by switching a single digit from an odd digit to an even digit or from an even digit to an odd digit the result is a number outside of A . Likewise assume that from any r outside of A by switching a single digit from an odd digit to an even digit or from an even digit to an odd digit the result is a number in A . Show that A cannot be a Lebesgue measurable subset of measure $\frac{1}{2}$.
- 4 Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a Lebesgue measurable function. Show that $f(x) = 0$ almost everywhere if and only if for all Lebesgue measurable sets E it follows that $\int_E f(x)dx = 0$.
5. Let $X := \mathbf{Z}^2$ be the set $\{(n, m) \mid n, m \in \mathbf{Z}\}$. For every $x \in X$ and $n, m \in \mathbf{Z}$ let $A_x^{n,m}$ be the subset of X defined by $A_x^{n,m} := \{x + jn(1, 0) + km(0, 1) \mid j, k \in \mathbf{Z}\}$.
 (a) If $\mathcal{A}$ is the collection of all sets of the form $A_x^{n,m}$ what is the smallest

algebra $\mathcal{A}'$ containing $\mathcal{A}$?

(b) What is the smallest sigma-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$?

(c) Define a finitely additive probability measure on $\mathcal{A}'$.

(d) Can one define a countably additive measure on $\sigma(\mathcal{A})$ that extends the finitely additive measure on $\mathcal{A}'$ that you found in (c) to a countably additive measure?

6. Let A be a Lebesgue measurable set and let $[a, b]$ and $[c, d]$ be intervals such that $\lambda([a, b] \setminus A) < \epsilon$, $\lambda([c, d] \setminus A) < \delta$ and $\epsilon + \delta < \min(b - a, d - c)$. Show that all of $(a + c + \epsilon + \delta, b + d - \epsilon - \delta)$ belongs to $A + A$ (where $A + A := \{a_1 + a_2 \mid a_1, a_2 \in A\}$).

7. Let A and B be subsets such that $A \cap B = \emptyset$, $\mathbf{R} \setminus (A \cup B)$ is a set of Lebesgue measure zero, and $A - B \subseteq A$ and $B - A \subseteq B$. Show that it is not possible for both A and B to be Lebesgue measurable if all four subsets $A \cap (-\infty, 0)$, $A \cap [0, \infty)$, $B \cap (-\infty, 0)$, and $B \cap [0, \infty)$ have positive measures.