

LTCC: Representation Theory of Finite Groups

Final Exam

Throughout this exam, let G denote a finite group.

1. Show that if V is an irreducible $\mathbb{C}[G]$ -module, then there exists $\lambda \in \mathbb{C}$ such that

$$\sum_{g \in G} g \cdot v = \lambda v \quad \text{for all } v \in V.$$

2. Give examples, with brief justification, of each of the following:

- (a) a noncyclic group with 8 distinct (i.e. non-equivalent) irreducible representations
- (b) a degree 2 faithful representation of C_3 over $\mathbb{C}$

3. Find the character table of the group $G = \langle a, b \mid a^5 = b^4 = e, bab^{-1} = a^2 \rangle$.

[Please include details of the methods used, e.g. finding conjugacy classes, normal subgroups, etc.]

4. The character table of S_4 appears in Lecture 5. The character table of $D_4 = \langle r, f \mid r^4 = f^2 = e, fr = r^{-1}f \rangle$ is given below.

$g_i :$ $ \text{Cl}(g_i) $	e	r	r^2	f	rf
	1	2	1	2	2
ψ_1	1	1	1	1	1
ψ_2	1	1	1	-1	-1
ψ_3	1	-1	1	1	-1
ψ_4	1	-1	1	-1	1
ψ_5	2	0	-2	0	0

- (a) We may consider D_4 to be a subgroup of S_4 by letting $r = (1\ 2\ 3\ 4)$ and $f = (1\ 2)(3\ 4)$. Decompose the induced representation $\psi_5 \uparrow_{D_4}^{S_4}$ into irreducible characters of S_4 .

- (b) Let S be the square in $\mathbb{R}^4$ with corners $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Define a representation of D_4 on $\mathbb{R}^4$ by linearly extending the usual action of D_4 on a square from S to $\mathbb{R}^4$. Find the character χ of this representation, and decompose χ into irreducible pieces.
5. In this problem we will examine the symmetric group S_5 . Let V be the defining *permutation module* of S_5 , i.e. $V = \langle v_1, \dots, v_5 \rangle$ and S_5 acts on V by $\sigma v_i = v_{\sigma(i)}$ for $\sigma \in S_5$. Let U be the 1-dimensional *trivial module* and let W be the *standard module* of S_5 , i.e. W is the complement of U in V .
- (a) Find a representative for each conjugacy class of S_5 , and find the values of the character ψ of W on each class.
- (b) Given a character χ , we can define $\chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$ and $\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$. [Note that these are the characters of $\text{Alt}(W)$ and $\text{Sym}(W)$, studied in Q6 of the Week 4 Exercises.] Compute ψ_A and ψ_S for the character ψ found in the previous part.
- (c) Show that the values of characters of S_n are always real.
- (d) Show that for any character χ with all real values, the trivial character is a constituent of χ^2 .
- (e) Show that if π is the character of V , then π^2 contains π , and in particular, π_S contains π .