

LTCC — Special Functions and Applications

Exam 2025–26 — Both questions are compulsory

1. The Tzitzéica equation,

$$u_{tt} - u_{xx} = e^u - e^{-2u}, \quad (1)$$

has applications in nonlinear optics, quantum field theory and differential geometry.

- (a) Show that equation (1) has a travelling wave solution of the form $u(x, t) = v(z)$, where $z = x - ct$, if and only if $w(z) = e^{v(z)}$ satisfies

$$\frac{c^2 - 1}{2} \left(\frac{dw}{dz} \right)^2 = w^3 + kw^2 + \frac{1}{2}, \quad (2)$$

where k is a constant. From now on we will assume that $|c| < 1$.

- (b) Show that there are real constants α and β such that under the transformation $w(z) = \alpha W(Z)$ and $Z = \beta z$, equation (2) becomes

$$\left(\frac{dW}{dZ} \right)^2 = P(W), \quad (3)$$

where $P(W) = W^3 - kW^2 - \frac{1}{2}$.

- (c) Show that if $k < -3/2$, then $P(W)$ has three real roots, two negative and one positive. Show that if $k > -3/2$, then there is only one real root. (Hint: consider the critical points of P .)
- (d) Assume that $k < -3/2$ and label the roots of $P(W)$ as $e_1 > 0 > e_2 > e_3$. (There is no need to find these roots!). Show that the general form of the travelling wave solution to equation (1) in this case is

$$u(x, t) = \ln \left\{ -e_3 - (e_2 - e_3) \operatorname{sn}^2 \left(\sqrt{\frac{e_1 - e_3}{2(1 - c^2)}} (z - z_0); \kappa \right) \right\}, \quad \kappa = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}},$$

where z_0 is a constant. (Hint: Let $W(Z) = e_3 + \gamma Y(Z)^2$, for some constant γ .)

2. (a) Recall that the (generalised) hypergeometric series is given by

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

Show that

$$\begin{aligned} & {}_{p+1}F_{q+1} \left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_{q+1} \end{matrix}; z \right) \\ &= \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1}) \Gamma(b_{q+1} - a_{p+1})} \int_0^1 t^{a_{p+1}-1} (1-t)^{b_{q+1}-a_{p+1}-1} {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; tz \right) dt. \end{aligned}$$

- (b) Explain why there are constants α and β such that

$${}_2F_1 \left(\begin{matrix} a, b \\ a + b - c + 1 \end{matrix}; 1 - z \right) = \alpha {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) + \beta z^{1-c} {}_2F_1 \left(\begin{matrix} a - c + 1, b - c + 1 \\ 2 - c \end{matrix}; z \right).$$

- (c) Assume that $\operatorname{Re}(c) < 1$. Show that

$$\alpha = \frac{\Gamma(a + b - c + 1) \Gamma(1 - c)}{\Gamma(a - c + 1) \Gamma(b - c + 1)}$$

and find β . You are not required to find the simplest form for β .