

LTCC-2025-26: Stochastic processes

Choose 4 out of the 8 questions.

- Each question has equal weight.
- Please typeset your solutions.
- You can do the coding in R, Python, C++, or C64 Basic.
 - Do not use fancy packages. For example, if you are simulating a Markov chain, do not use a Markov chain package.
 - The idea is that the coding can help you understand a mathematical process and can also demonstrate your understanding; do not use AI to write the code!

Question 1

Let X be an irreducible and aperiodic Markov chain on a finite state space containing distinct states $s, t \in S$. Consider the average

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}[X_{k+2} = t, X_k = s].$$

- Let P be the transition matrix and let the stationary distribution be π . Identify the limit of A_n as $n \rightarrow \infty$.
- Using a renewal type argument, prove that A_n converges almost-surely to the limit.

Hint: you may use the fact that the limit does not depend on the distribution of X_0 .

Question 2

Illustrate Question 1, via simulations, using the following transition matrix

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/8 & 1/4 & 5/8 \end{bmatrix},$$

with $s = 1$ and $t = 3$.

Question 3

You are given Γ —a Poisson point process of intensity 3 on the interval $[0, \pi]$. Consider the function $g(t) = \sin(t) + 2$ and the following thinning procedure to produce another point process Γ' . Suppose the points of Γ are given by $\{t_1, \dots, t_n\}$. We consider independent $\text{Bern}(p_i)$ random variables, $X_1, \dots, X_n$, with

$$p_i = g(t_i)/3.$$

We let $\Gamma' \subseteq \Gamma$, where t_i is a point of Γ' if and only if $X_i = 1$.

- Let N be the total number of points of Γ' in the interval $[0, \pi]$. Prove that

$$\mathbb{E}N = \int_0^\pi g(t)dt = 2\pi + 2.$$

- Hint: if $g \equiv 2$ this would be easy, why?
- What is the distribution of N ? Explain.

Question 4

Illustrate the answer in Question 3 by running simulations.

Question 5

Let X be a Poisson random variable of mean μ . Let $k \geq 0$. Show (without doing any hard computations) that

$$\mathbb{P}(X \leq k) = \frac{1}{k!} \int_\mu^\infty e^{-t} t^k dt.$$

Question 6

Let $X_1, X_2, \dots$ be iid 3-sided dice roll with probabilities $p, q, r \in [0, 1]$ with $p + q + r = 1$; note that $r = 1 - (p + q)$. We let $S_n = (T_n^1, T_n^2)$, where

$$T_n^i = \sum_{k=1}^n \mathbf{1}[X_k = i].$$

Let $D = \{(u, v) : u + v \leq 1, 0 \leq u, 0 \leq v\}$. Let $f : D \rightarrow \mathbb{R}$ be continuous on the closed triangle D . Show that $f_n(p, q) := \mathbb{E}_{(p,q)} f(S_n/n)$ gives a sequence of polynomials (in two variables) that converge uniformly to f on D .

Question 7

Let X be a symmetric random variable, so that $-X$ has the same law as X . Also assume that $\mathbb{E}(X^2) < \infty$. Prove the Skorokhod embedding theorem in this case, with a randomized stopping time.

Question 8

Let P be an irreducible and aperiodic transition matrix on a finite state space. Let π be the stationary distribution. We let X and Y be independent chains both with transition matrix P , where X is started at some fixed state s , and Y is started at the stationary distribution. Let $T = \inf\{n \geq 0 : X_n = Y_n\}$. Consider the random variable $W = X_T$.

- Must W be distributed as π ?
- What about if X was also started at stationarity as well?