CHAPTER 1

Projective resolutions

1. R-Modules

In this section we will quickly review the basic definitions of modules over a ring, projective resolutions and the definition of $\mathrm{Ext}^n(M, N)$. In general we denote a ring by R and assume that R has a unit.

Let R be a ring. A **left R-module** is an abelian group $(M, +)$ together with a multiplication

$$\begin{array}{ccc} R \times M & \to & M \\ (r, m) & \mapsto & rm \end{array}$$

satisfying the following axioms:

(M1) $r(m + n) = rm + rn$ for all $r \in R$ and $m, n \in M$
(M2) $(r + s)m = rm + sm$ for all $r, s \in R$ and $m \in M$
(M3) $(rs)m = r(sm)$ for all $r, s \in R$ and $m \in M$
(M4) $1_R m = m$ for all $m \in M$.

We usually write M_R - or M if it is clear which ring is meant. Right R-modules are defined analogously. If R is commutative a left R-module can be made into a right R-module by defining the multiplication by $(m, r) \mapsto rm$.

Let M and N be R-modules. A map $\alpha : M \to N$ is called R-linear or an R-module homomorphism if

- $\alpha(m + m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$
- $\alpha(rm) = r\alpha(m)$ for all $m \in M, r \in R$.

Let M and N be R-modules. We denote by $\mathrm{Hom}_R(M, N)$ the set of all R-linear maps $\alpha : M \to N$.

Remark. $\mathrm{Hom}_R(M, N)$ is an abelian group with addition defined pointwise. Furthermore $End_R(M) = \mathrm{Hom}_R(M, M)$ is a ring where multiplication is defined by composition of maps.

Naturality means that for every R-module homomorphism $\alpha : M \to N$ the following diagram commutes,

$$\begin{array}{ccc} \mathrm{Hom}_R(R, M) & \xrightarrow{\phi_M} & M \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_R(R, N) & \xrightarrow{\phi_N} & N \end{array}$$

where $\alpha_*(f) = \alpha \circ f$ and $\alpha \circ \phi_M = \phi_N \circ \alpha_*$.

A sequence

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

($i \in \mathbb{Z}$) of linear maps is called **exact at** M_i if $im(\alpha_{i+1}) = ker\alpha_i$.
The sequence is called exact if it is exact at every $M_i (i \in \mathbb{Z})$.

EXERCISE 1. Show that:

(1) $0 \longrightarrow L \xrightarrow{\alpha} M$ is exact if and only if α is a monomorphism.

(2) $M \xrightarrow{\beta} N \longrightarrow 0$ is exact if and only if β is an epimorphism.

(3) $0 \longrightarrow L \xrightarrow{\alpha} M \longrightarrow 0$ is exact iff α is an isomomorphism.

Remark. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

In particular, α is a monomorphism, β is an epimorphism and $im(\alpha) = ker(\beta)$. Hence $N \cong M/\alpha(L)$. Conversely, if $N \cong M/L$, then there is a short exact sequence

$$L \hookrightarrow M \twoheadrightarrow N.$$

Let us get back to the groups $\mathrm{Hom}_R(M, N)$: Let $\alpha \in \mathrm{Hom}_R(M, N)$ and let $\xi : N \to X$ be an R-module homomorphism. We then define

$$\xi_* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, X)$$

by $\xi_*(\alpha) = \xi \circ \alpha$. In other words, $\mathrm{Hom}_R(M, -)$ is a covariant functor. Now let $\psi : Y \to M$ be an R-module homomorphism. We define

$$\psi^* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(Y, N)$$

by $\psi^*(\alpha) = \alpha \circ \psi$. We say $\mathrm{Hom}_R(-, N)$ is a contravariant functor.

THEOREM 1.1. *Let X and Y be R-modules and let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following sequences are exact:

(1) $0 \longrightarrow \mathrm{Hom}_R(Y, L) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Y, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Y, N)$

(2) $0 \longrightarrow \mathrm{Hom}_R(N, X) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, X) \xrightarrow{\alpha^*} \mathrm{Hom}_R(L, X)$.

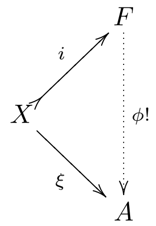
Proof: We leave (2) as exercise and do (1) in class. □

We say $\mathrm{Hom}_R(-, X)$ and $\mathrm{Hom}_R(Y, -)$ are left exact functors. Neither β_* nor α^* have to be surjective. We'll come back to conditions on X and Y for Hom to be an exact functor.

Projective modules are basically the bread and butter of homological algebra, so let's define them. But first, let's do free modules:

Let F be an R-module and X be a subset of F. We say F is **free on** X if for every R-module A and every map $\xi : X \to A$ there exists a unique R-module homomorphism $\phi : F \to A$ such that $\phi(x) = \xi(x)$ for all $x \in X$.

In other words F is free if there's a unique R-module homorphism ϕ making the following diagram commute:



A very hard look at this diagram now gives us the following lemma.

Proposition 1.2. *Let P be an R-module. Then the following statements are equivalent:*

1. $\mathrm{Hom}_R(P, -)$ *is an exact functor*
2. P *is a direct summand of a free module.*
3. *Every epimorphism $M \twoheadrightarrow P$ splits.*
4. *For every epimorphism $\pi : A \twoheadrightarrow B$ of R-modules and every R-module map $\alpha; P \to B$ there is an R-module homomorphism $\phi : P \to A$ such that $\pi \circ \phi = \alpha$.*

Every R-module satisfying the conditions of Proposition 1.2 is called a **projective R-module.**

Definition 1.3. Let M be an R-module. A projective resolution of M is an exact sequence

$$\cdots \longrightarrow P_{i+1} \xrightarrow{d_i} P_i \xrightarrow{d_{i+1}} \cdots \xrightarrow[1]{d} P_1 \xrightarrow[0]{d} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where every P_i, $i \geq 0, i \in \mathbb{Z}$, is a projective module.

We also use the short notation

$$\mathbf{P}_* \twoheadrightarrow M.$$

Given an R-module N, we apply $\mathrm{Hom}_R(-, N)$ to the projective resolution above to get a complex

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \cdots .$$

We define:

$$\mathrm{Ext}^n_R(M, N) = ker(\mathrm{Hom}_R(P_n, N) \to \mathrm{Hom}_R(P_{n+1}, N))/im(\mathrm{Hom}_R(P_{n-1}, N) \to \mathrm{Hom}_R(P_n, N)).$$

We use the convention that $P_i = 0$ for all $i < 0$.

Theorem 1.4. $\mathrm{Ext}^n_R(M, N)$ *is independent of the choice of projective resolution of M.*

Exercise 2. Prove that $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$.

Definition 1.5. Let M be an R-module. We say M has finite projective dimension over R, $\mathrm{pd}_R M < \infty$, if M admits a projective resolution $\mathbf{P}_* \twoheadrightarrow M$ of finite length. In particular, there exists an $n \geq 0$ such that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resulution of n. The smallest such n is called the projective dimension of M.

Proposition 1.6. *Let M be an R-module. Then the following statments are equivalent:*

(1) $\mathrm{pd}_R M \leq n$.
(2) $\mathrm{Ext}^i_R(M,-) = 0$ *for all* $i > n$
(3) $\mathrm{Ext}^{n+1}_R(M,-) = 0$
(4) *Let* $0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *be an exact sequence with* P_i *projective for all* $0 \leq i \leq n-1$. *Then* K_{n-1} *is projective.*

Exercise 3. Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Prove the following:

(1) $\mathrm{pd}M' \leq sup\{\mathrm{pd}M, \mathrm{pd}M'' + 1\}$.
(2) $\mathrm{pd}M \leq sup\{\mathrm{pd}M'', \mathrm{pd}M'\}$.
(3) $\mathrm{pd}M'' \leq sup\{\mathrm{pd}M, \mathrm{pd}M' - 1\}$.

(This is an exercise in applying Theorem 1.7)

Exercise 4. Let M be an R-module such that $\mathrm{pd}M = n$. Then there exists a free R-module F such that

$$\mathrm{Ext}^n(M,F) \neq 0.$$

Theorem 1.7. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. And let N be an arbitrary R-module. Then there are long exact sequences in cohomology*

(1)

$$\cdots \to \mathrm{Ext}^n(N,M'') \to \mathrm{Ext}^n(N,M) \to \mathrm{Ext}^n(N,M') \to \mathrm{Ext}^{n+1}(N,M'') \to \cdots$$

(2)

$$\cdots \to \mathrm{Ext}^n(M',N) \to \mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(M'',N) \to \mathrm{Ext}^{n+1}(M',N) \to \cdots$$

Exercise 5. **[Dimension shifting]** Let $K \hookrightarrow P \twoheadrightarrow M$ be the beginning of a projective resolution of M and let N be an R-module. Then for all $n \geq 1$,

$$\mathrm{Ext}^n(K,N) \cong \mathrm{Ext}^{n+1}(M,N).$$

Proof: Apply Theorem 1.7 and the fact that Ext vanishes on projectives. □

2. The Group Ring

Throughout we denote a group by G. Let $\mathbb{Z}G$ denote the free $\mathbb{Z}$-module with basis the elements of G. In particular, every $x \in \mathbb{Z}G$ can be written in a unique way as

$$x = \sum_{g\in G} n_g g$$

where $n_g \in \mathbb{Z}$and almost all $n_g = 0$. Define a multiplication on $\mathbb{Z}G$ as follows:

$$xy = (\sum_{g\in G} n_g g)(\sum_{h\in G} n_h h) = \sum_{g,h\in G} n_g n_h (gh).$$

this makes $\mathbb{Z}G$ into a ring, the **integral group ring.**

EXAMPLE 1.8. (1) Let $G = \langle x \rangle$ be infinite cyclic. Then $\mathbb{Z}G$ has $\mathbb{Z}$-basis $\{x^i \mid i \in \mathbb{Z}\}$ and can be identified with the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials $\sum_{i \in \mathbb{Z}} a_i x^i$, where almost all $a_i = 0$.

(2) Let G be cyclic order n and t be a generator for G. $\{1, t, t^2, ..., t^{n-1}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}G$ and $t^n - 1 = 0$ hence

$$\mathbb{Z}G \cong \mathbb{Z}[T]/T^n - 1.$$

DEFINITION 1.9. Let M be an abelian group and let G act on M

$$\begin{array}{ccc} G \times M & \to & M \\ (g, m) & \mapsto & gm \end{array}$$

such that for all $m, n \in M$ and $g, h \in G$:

- $1_G m = m$
- $(gh)m = g(hm)$
- $g(m + n) = gm + gn$

we say that M is a G-module.

A G-module can be made in a $\mathbb{Z}G$-module by "linearly extending" the action, i.e. $xm = (\sum_{g \in G} n_g g)m = \sum_{g \in G} n_g(gm)$. Furthermore, G is a subgroup of the multiplicative group $\mathbb{Z}G^*$ and hence there's the follwing universal property:
Let R be a ring and $f : G \to R^*$ be a group homomorphism. Then f can be extended uniquely to a ring homomorphism $\mathbb{Z}G \to R$. Hence

$$Hom_{rings}(\mathbb{Z}G, R) \cong Hom_{groups}(G, R^*)$$

and a G-module is nothing but a $\mathbb{Z}G$-module.

EXAMPLE 1.10. Every abelian group A is a trivial G-module with the action defined by $ag = a$ for all $a \in A, g \in G$. Hence for $x = \sum_{g \in G} n_g g$ it follows that $xa = \sum_{g \in G} n_g a$.

For every group G there is a ring homomorphism

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$$

defined by $\varepsilon(g) = 1$. for all $g \in G$. Hence for $x = \sum_{g \in G} n_g g$, $\varepsilon(x) = \sum_{g \in G} n_g$. The kernel of ε is called the **augmentation ideal** and is denoted by $\mathfrak{g}$ or IG.

LEMMA 1.11. *$\mathfrak{g}$ is a free $\mathbb{Z}$-module with basis*

$$X = \{g - 1 \mid 1 \neq g \in G\}.$$

ε is a G-module homomorphism and $\mathfrak{g}$ is a G-module.

LEMMA 1.12. (1) *Let S be generating set for G. Then $\mathfrak{g}$ is generated as a G-module by*

$$S - 1 = \{s - 1 \mid s \in S\}.$$

(2) *Let S be a set of elements of G such that $S - 1$ generates $\mathfrak{g}$ as a G-module. Then S generates the group G.*

Proof: We do (1) in class and leave (2) as an exercise. □

Now let Ω be a G-set and consider the free abelian group $\mathbb{Z}\Omega$ on Ω. The operation of G on Ω can be extended to a $\mathbb{Z}$-linear operation of G on $\mathbb{Z}\Omega$. Hence $\mathbb{Z}\Omega$ is a G-module, the so called **Permutation module**.