Topology, geometry and dynamics of higher-order networks

An introduction to simplicial complexes
Lesson V

LTCC module 2024

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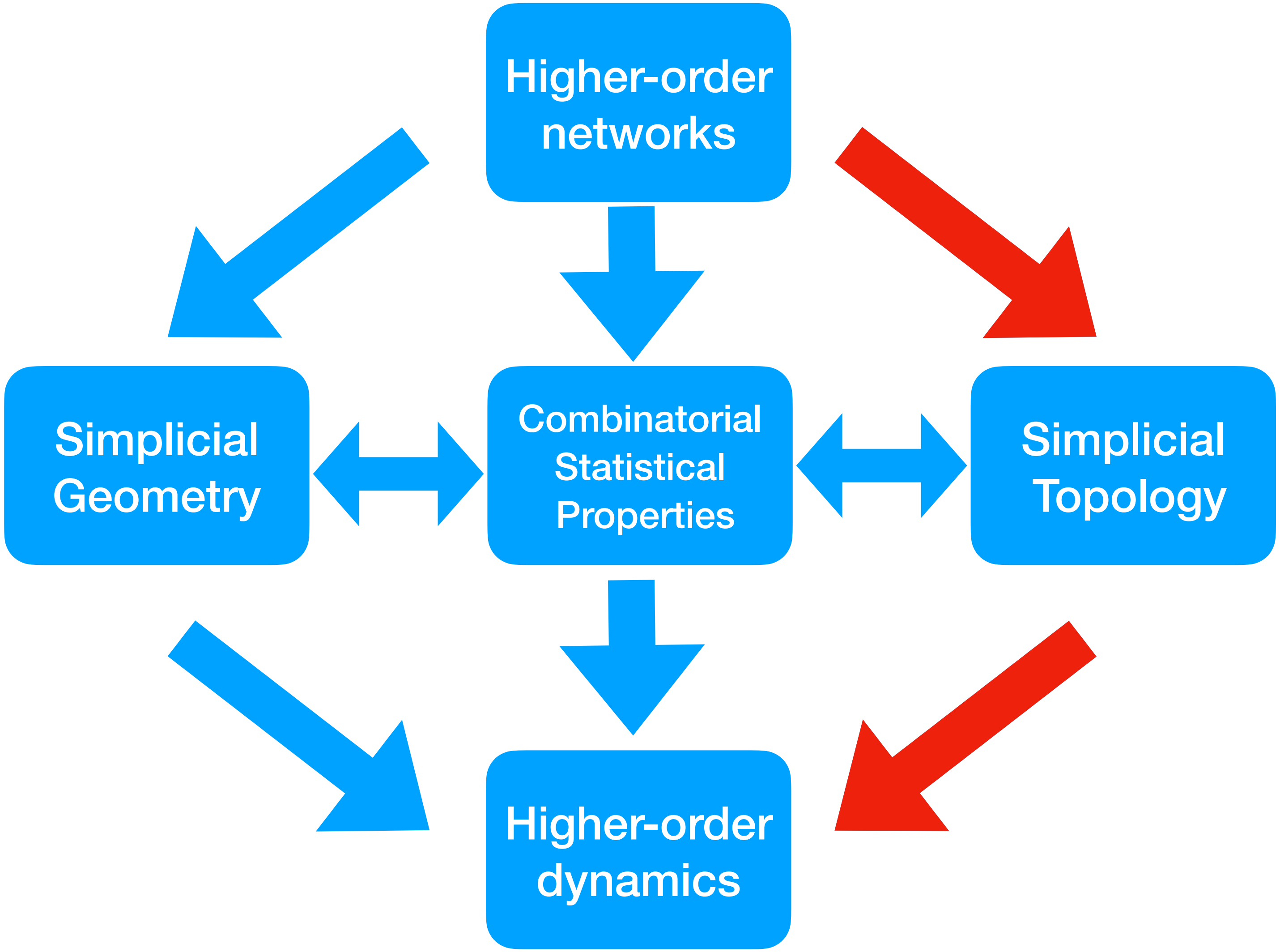
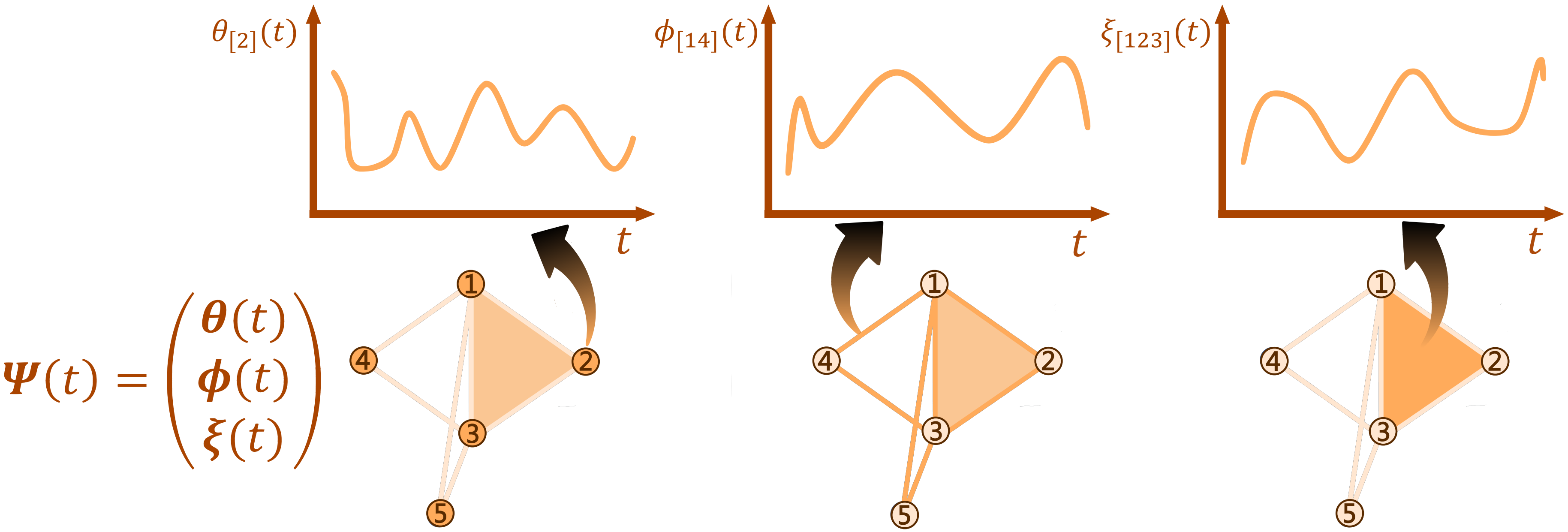
Higher-order structure and dynamics

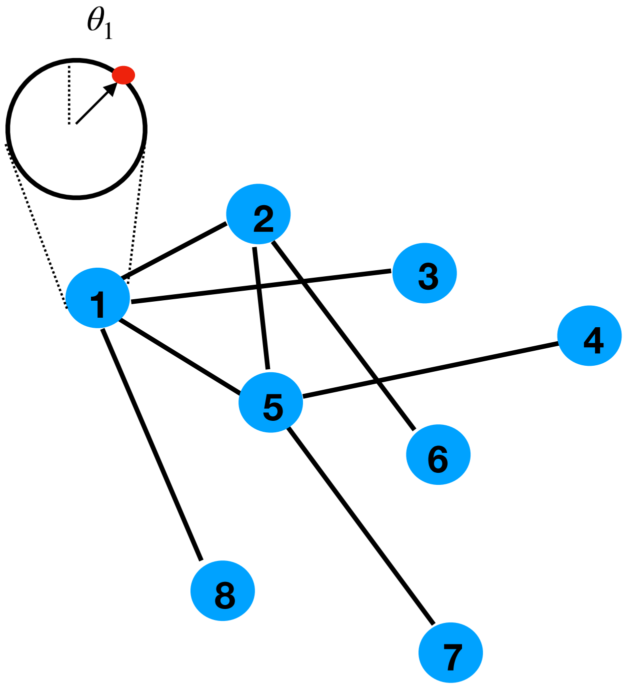


Topological signals

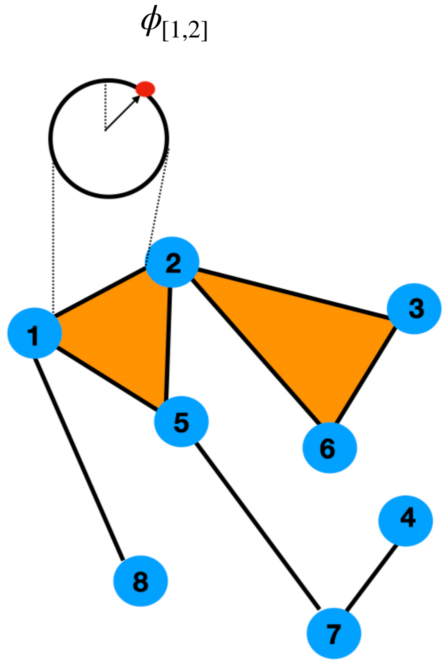
Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals



Topological Syncrhonization



Node-based dynamics



Higher-order Topological Dynamics

A. P. Millan, J.J. Torres and G. Bianconi PRL (2020)
T. Carletti, L. Giambagli, G. Bianconi PRL (2023)

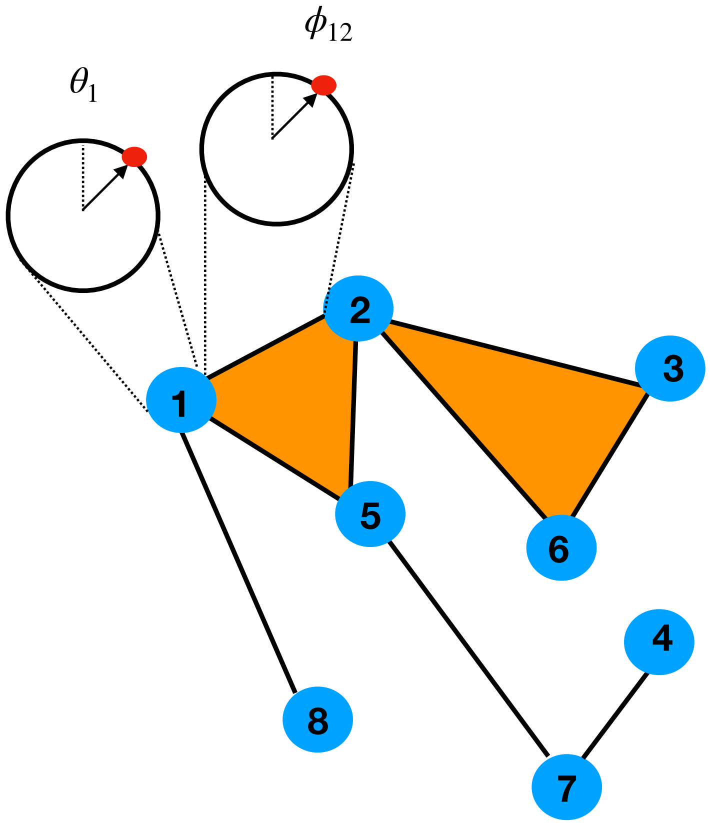
Coupling topological signals of different dimension

How can we couple

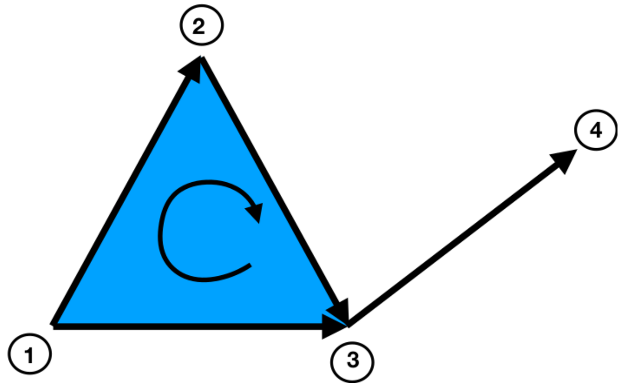
topological signal

of different dimension

locally and topologically?



Boundary Operators



Boundary operators

$$\mathbf{B}_{[1]} = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \qquad \mathbf{B}_{[2]} = \begin{array}{c} \\ [1,2] \\ [1,3] \\ [2,3] \\ [3,4] \end{array} \begin{array}{c} [1,2,3] \\ 1 \\ -1 \\ 1 \\ 0 \end{array}.$$

$\mathbf{B}_{[1]}$ **Discrete divergence**

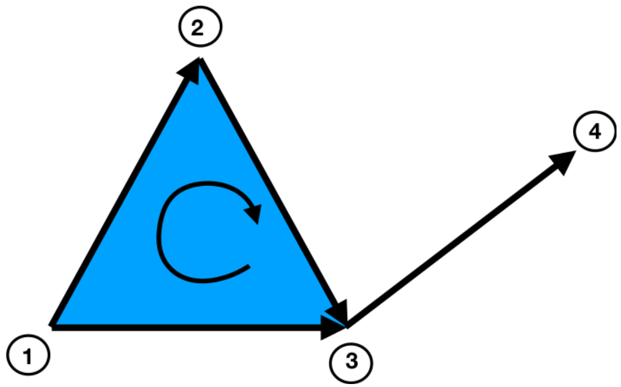
$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[2]}^{\top}$ **Discrete Curl**

The boundary of the boundary is null

$$\mathbf{B}_{[m-1]}\mathbf{B}_{[m]} = \mathbf{0}, \quad \mathbf{B}_{[m]}^{\top}\mathbf{B}_{[m-1]}^{\top} = \mathbf{0}$$

Simplicial complexes and Hodge Laplacians



Hodge Laplacians

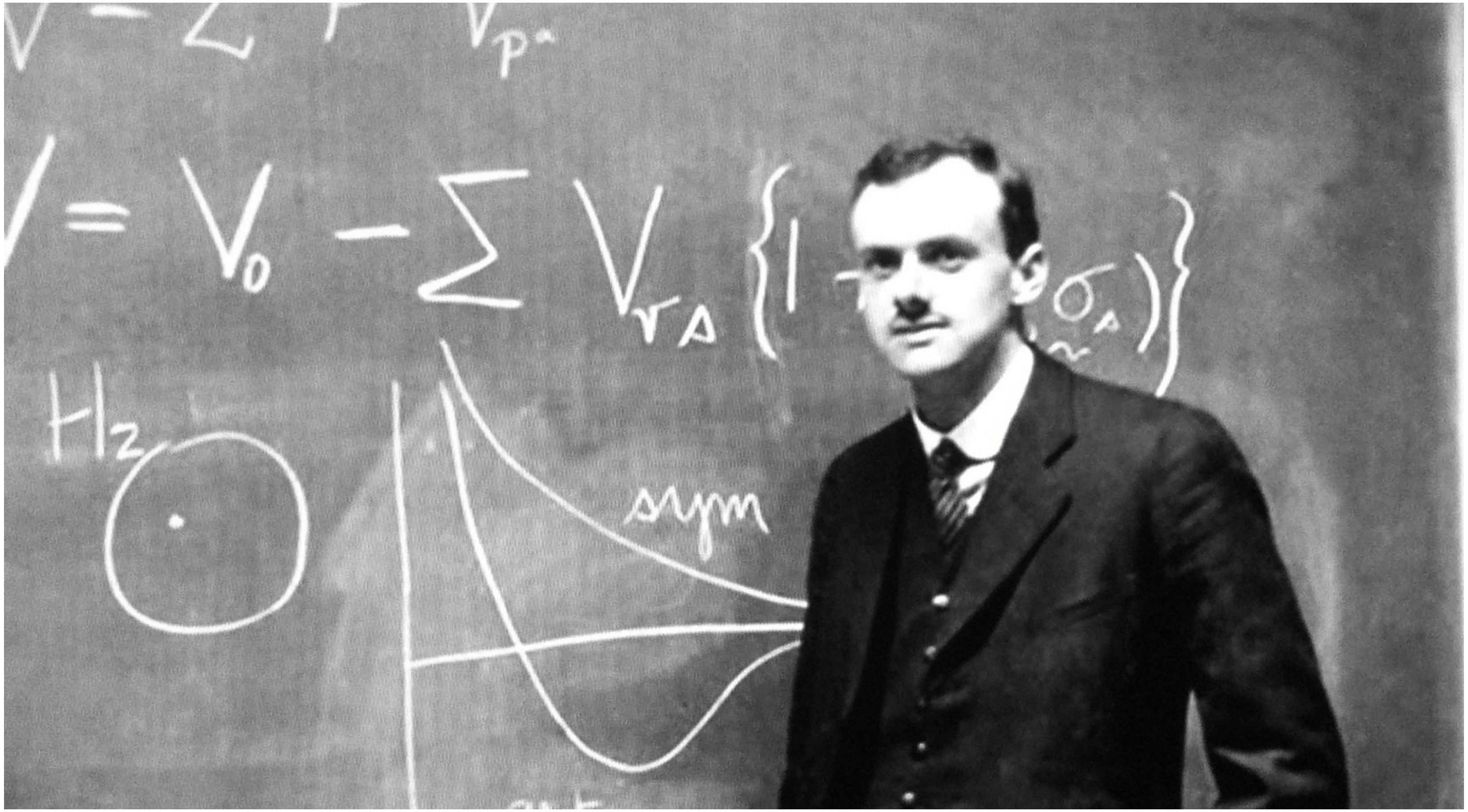
The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices

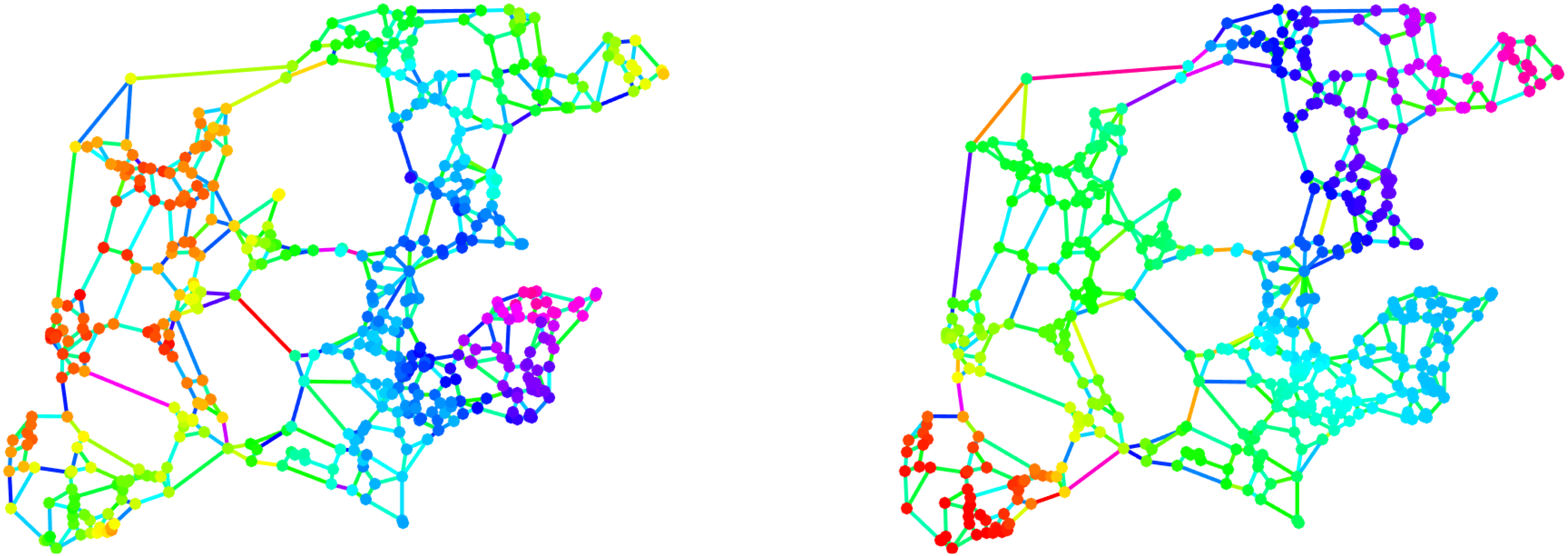
For a 2-dimensional simplicial complex we have

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^\top \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^\top\mathbf{B}_{[1]} + \mathbf{B}_{[2]}\mathbf{B}_{[2]}^\top \qquad \mathbf{L}_{[2]} = \mathbf{B}_{[2]}^\top\mathbf{B}_{[2]}$$

Dirac legacy



Dirac operator on graphs



Lesson V:
The Dirac operator
on graphs and simplicial complexes

- **Dirac operator on graphs**
  - **Eigenvalues, Eigenvectors Chirality**
  - **Weighted and Normalised Dirac operator**
  - **Topological Dirac equation**
  - **Insights into the mathematical interpretation of gamma matrices**

- **Dirac operator on simplicial complexes**

- **Dirac operator in dynamical systems and signal processing**
  - **Dirac Synchronisation and Global Dirac Synchronization**
  - **Dirac Turing patterns**
  - **Dirac signal processing**

The Dirac operator on graphs

Topological spinor

The topological spinor is defined on both nodes and edges of a graph $G = (V, E)$

as $\mathbf{\Psi} = \chi \bigoplus \psi \in C^0 \bigoplus C^1$ or equivalently

$$\mathbf{\Psi} = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\psi} \end{pmatrix}$$

with

- $\boldsymbol{\chi}$ defined on nodes, i.e. $\chi \in C^0$
- $\boldsymbol{\psi}$ defined on edges, i.e. $\psi \in C^1$

Exterior derivative and its dual

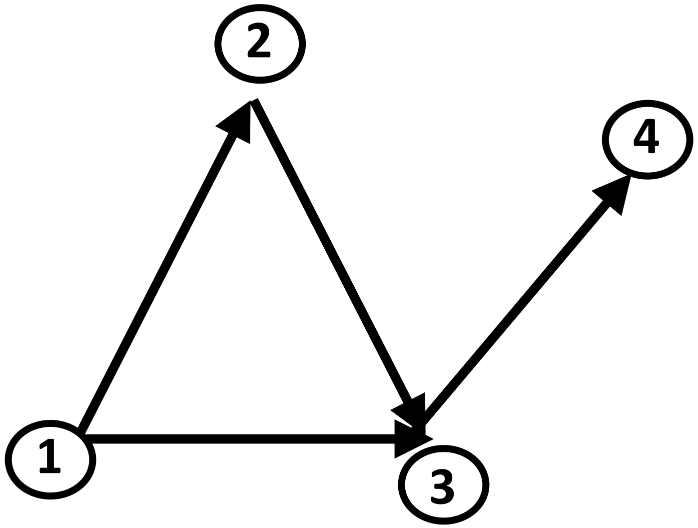
- The exterior derivative $d : C^0 \to C^1$ is defined as

$$(d\chi)_{e=[i,j]} = \chi_j - \chi_i \quad \textbf{gradient}$$

- It adjoint operator $d^* : C^1 \to C^0$ is defined as

$$(d^*\psi)_i = \sum_{e \in E_i^+} \psi_e - \sum_{e \in E_i^-} \psi_e \quad \textbf{divergence}$$

Boundary matrix



Boundary matrix

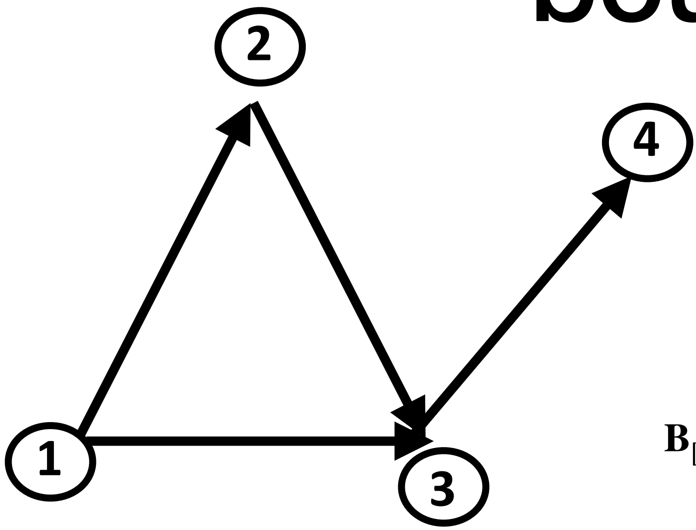
$\mathbf{B}_{[1]}$ is a $N_0 \times N_1$ matrix of elements

$$\mathbf{B}_{[1]}(r,\ell) = \begin{cases} 1 \textbf{ if } \ell = [s,r] \\ -1 \textbf{ if } \ell = [r,s] \\ 0 \textbf{ otherwise} \end{cases}$$

$\mathbf{B}_{[1]}^{\top}$	**Discrete gradient**
$\mathbf{B}_{[1]}$	**Discrete divergence**

The discrete gradient can be represented by the coboundary matrix $\bar{\mathbf{B}}_{[1]} = \mathbf{B}_{[1]}^{\top}$

Boundary operator and co-boundary matrix



Boundary and co-boundary matrices

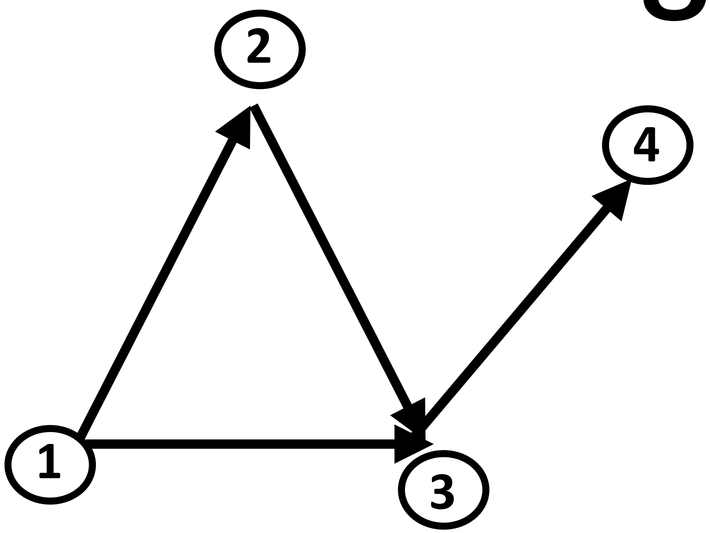
$$
\mathbf{B}_{[1]} = \begin{array}{c} \\ {[1]} \\ {[2]} \\ {[3]} \\ {[4]} \end{array} \begin{array}{cccc} [1,2] & [1,3] & [2,3] & [3,4] \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}, \quad \mathbf{B}_{[1]}^{\top} = \begin{array}{c} \\ {[1,2]} \\ {[1,3]} \\ {[2,3]} \\ {[3,4]} \end{array} \begin{array}{cccc} [1] & [2] & [3] & [4] \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array},
$$

$\mathbf{B}_{[1]}^{\top}$ **Discrete gradient**

$\mathbf{B}_{[1]}$ **Discrete divergence**

The discrete gradient can be represented by a coboundary matrix $\bar{\mathbf{B}}_{[1]} = \mathbf{B}_{[1]}^{\top}$

Hodge Laplacians



Betti numbers of a connected network

$\beta_0 = 1$ one connected component

$\beta_1 = N_1 - (N_0 - 1)$ number of independent cycles

dim ker$(\mathbf{L}_{[m]}) = \beta_m$

Hodge Laplacians

The Hodge Laplacians describe diffusion

from m-simplices to m-simplices through (m-1) and (m+1) simplices:

for a graph we have

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top} \qquad \mathbf{L}_{[1]} = \mathbf{B}_{[1]}^{\top}\mathbf{B}_{[1]}$$

Exterior derivation and its adjoint on a graph

The exterior derivative and its adjoint

$$d = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix} \qquad d^{*} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

act on a topological spinor

$$\mathbf{\Psi} = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\psi} \end{pmatrix}$$

Basic definition of the Dirac operator on graphs

The Dirac operator in its simplest form

is the self-adjoint operator $D : C^0 \bigoplus C^1 \to C^0 \bigoplus C^1$ defined as

$$\boxed{D = d + d^*}$$

satisfying

$$D(\chi \oplus \psi) = (d + d^*)(\chi \oplus \psi) = (d^*\psi) \oplus (d\chi)$$

Dirac operator on a network

Exterior divergence

$$d = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{[1]}^\top & \mathbf{0} \end{pmatrix} \qquad d^* = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Dirac operator is a self-adjoint operator

$$D = d + d^*$$

Dirac operator on graph

Dirac operator on a graph

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^\top & \mathbf{0} \end{pmatrix}$$

Action of the Dirac operator on the topological spinor

$$\mathbf{D}\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{[1]}\psi \\ \mathbf{B}_{[1]}^\top \chi \end{pmatrix}$$

The Dirac as the square-root of the Laplacian

The Dirac operator
can be interpreted as the
"square-root" of the Laplacian

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} \\ \mathbf{B}_{[1]}^\top & 0 \end{pmatrix}, \qquad \mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}$$

The non-zero eigenvalues of the Dirac operator
are the square root of the non-zero eigenvalues of the graph Laplacian.

The spectrum of the Dirac operator

Since $\mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]} \end{pmatrix}$ and $\mathbf{L}_{[0]}, \mathbf{L}_{[1]}$ are isospectral, it follows that:

Spectrum: For every positive eigenvalue μ of $\mathbf{L}_{[0]}$ there is one positive and one negative eigenvalue λ of the Dirac operator $\mathbf{D}$ with

$$\lambda = \pm\sqrt{\mu}$$

Chirality

Let us define $\gamma_0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$

obeying the anti commutator relation $\{\mathbf{D}, \gamma_0\} = \mathbf{0}$

- **Chirality**:If $\boldsymbol{\Psi} = (\chi, \psi)^\top$ is an eigenvector of the Dirac operator with eigenvalue λ, i.e. if $\mathbf{D}\boldsymbol{\Psi} = \lambda\boldsymbol{\Psi}$ then $\gamma_0\boldsymbol{\Psi} = (\chi, -\psi)^\top$ is an eigenvector of $\mathbf{D}$ with eigenvalue $-\lambda$

- Indeed from the anti-commutator relation it follows that $\mathbf{D}\gamma_0\boldsymbol{\Psi} = -\gamma_0\mathbf{D}\boldsymbol{\Psi} = -\lambda\gamma_0\boldsymbol{\Psi}$

Eigenvectors of the Dirac operator

- It follows that the matrix of eigenvectors of the Dirac operator can be expressed as

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{U}^{[1]} & \mathbf{U}^{[1]} & \mathbf{U}_0^{harm} & \mathbf{0} \\ \mathbf{V}^{[1]} & -\mathbf{V}^{[1]} & \mathbf{0} & \mathbf{U}_1^{harm} \end{pmatrix}$$

- where $\mathbf{U}^{[1]}, \mathbf{V}^{[1]}$ Indicates the left and right singular vectors of the boundary operator and $\mathbf{U}_0^{harm}, \mathbf{U}_1^{harm}$ are the matrices of the harmonic eigenvectors of $\mathbf{L}_{[0]}, \mathbf{L}_{[1]}$ respectively.

Index of the Dirac operator

The index of the Dirac operator D is given

by the Euler number χ_E of the graph

$$\text{ind}\, D = \dim\ker d - \dim\ker d^* = \chi_E$$

Indeed

$$\text{ind}\, D = \chi_E = N_0 - N_1$$

Introducing an algebra

Dirac operator on a network can be enriched by an algebra

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b\mathbf{B}_{[1]} \\ b^{\star}\mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

with $b \in \mathbb{C}, \quad |b| = 1$

Weighted and Normalised Dirac operator on graphs

Weighted Dirac operator on a network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \bar{\mathbf{B}}^*_{[1]} \\ \bar{\mathbf{B}}_{[1]} & \mathbf{0} \end{pmatrix}$$

with $$\bar{\mathbf{B}}_{[1]} = \mathbf{B}^\top_{[1]} \quad \bar{\mathbf{B}}^*_{[1]} = \mathbf{G}_{[0]}\mathbf{B}_{[1]}\mathbf{G}^{-1}_{[1]}$$

$$\tilde{\mathbf{D}}^2 = \mathscr{L} = \begin{pmatrix} \tilde{\mathbf{L}}_{[0]} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{L}}_{[1]} \end{pmatrix}$$

with

$$\tilde{\mathbf{L}}_{[0]} = \bar{\mathbf{B}}^*_{[1]}\bar{\mathbf{B}}_{[1]}, \tilde{\mathbf{L}}_{[1]} = \bar{\mathbf{B}}_{[1]}\bar{\mathbf{B}}^*_{[1]}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Normalised Dirac operator

If the matrix $\mathbf{G}_{[1]}^{-1}$,$\mathbf{G}_{[0]}^{-1}$ are the diagonal matrices with elements

$$\mathbf{G}_{[1]}^{-1}(\ell,\ell) = w_\ell/2$$

$$\mathbf{G}_{[0]}^{-1}(r,r) = \sum_{\ell \in E_r} w_\ell$$

The weighted Dirac operator is also called normalised Dirac operator and has eigenvalues bounded in absolute value by one $|\lambda| \leq 1$

F. Baccini, F. Geraci and G. Bianconi (2022)

Normalised Dirac operator of unweighted networks

If the weights of all the links are one, i.e. $w_\ell = 1$ we have

That the matrices $\mathbf{G}_{[1]}^{-1}$,$\mathbf{G}_{[0]}^{-1}$ are the diagonal matrices with elements

$$\mathbf{G}_{[1]}^{-1}(\ell,\ell) = 1/2 \quad \mathbf{G}_{[0]}^{-1}(r,r) = k_r$$

The normalised Dirac operator with spectrum satisfying $|\lambda| \leq 1$

is given by

$$\tilde{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \mathbf{K}^{-1}\mathbf{B}_{[1]}/2 \\ \mathbf{B}_{[1]}^\top & \mathbf{0} \end{pmatrix}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Symmetric Weighted Dirac operator on a network

$$\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{0} & \bar{\mathbf{B}}^*_{[1]} \\ \bar{\mathbf{B}}_{[1]} & \mathbf{0} \end{pmatrix}$$

with

$$\bar{\mathbf{B}}^*_{[1]} = \mathbf{G}^{1/2}_{[0]} \bar{\mathbf{B}}_{[1]} \mathbf{G}^{-1/2}_{[1]} \quad \bar{\mathbf{B}}_{[1]} = \mathbf{G}^{-1/2}_{[1]} \bar{\mathbf{B}}^\top_{[1]} \mathbf{G}^{1/2}_{[0]}$$

$$\hat{\mathbf{D}}^2 = \mathscr{L} = \begin{pmatrix} \hat{\mathbf{L}}_{[0]} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}}_{[1]} \end{pmatrix}$$

with

$$\hat{\mathbf{L}}_{[0]} = \bar{\mathbf{B}}^*_{[1]} \bar{\mathbf{B}}_{[1]}, \hat{\mathbf{L}}_{[1]} = \bar{\mathbf{B}}_{[1]} \bar{\mathbf{B}}^*_{[1]}$$

F. Baccini, F. Geraci and G. Bianconi (2022)

Topological Dirac equation

G. Bianconi,
Topological Dirac equation on networks and simplicial complexes
JPhys Complexity (2021)

Topological spinor

On a network we consider the topological spinor

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\psi} \end{pmatrix}$$

Characterising the dynamical state of the topological signals of the network, being a vector with a block structure formed by a 0-cochain and a 1-cochain

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_N \end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_{\ell_1} \\ \psi_{\ell_2} \\ \vdots \\ \psi_{\ell_L} \end{pmatrix}.$$

Topological Dirac equation

The topological Dirac equation is then given by

$$\mathrm{i}\partial_{\mathrm{t}}\tilde{\mathbf{\Psi}} = \mathscr{H}\tilde{\mathbf{\Psi}}$$

with Hamiltonian

$$\mathscr{H} = \mathbf{D} + m\boldsymbol{\beta}$$

Where $\boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$ with anti-commutator $\{\mathbf{D}, \boldsymbol{\beta}\} = \mathbf{0}$

Topological Dirac equation: Eingenstates

The topological Dirac equation is then given by

$$\mathrm{i}\partial_{\mathrm{t}}\tilde{\mathbf{\Psi}} = \mathscr{H}\tilde{\mathbf{\Psi}}$$

Has solution $\tilde{\mathbf{\Psi}} = e^{-\mathrm{iEt}}\mathbf{\Psi}$ with $\mathbf{\Psi}$ independent of time

if and only if

$$E\mathbf{\Psi} = \mathscr{H}\mathbf{\Psi}$$

Proof: Substituting $\tilde{\mathbf{\Psi}} = e^{-\mathrm{iEt}}\mathbf{\Psi}$ into $\mathrm{i}\partial_{\mathrm{t}}\tilde{\mathbf{\Psi}} = \mathscr{H}\tilde{\mathbf{\Psi}}$ we get

$\mathrm{i}\partial_{\mathrm{t}}\mathrm{e}^{-\mathrm{iEt}}\mathbf{\Psi} = \mathrm{E}\mathrm{e}^{-\mathrm{iEt}}\mathbf{\Psi} = \mathscr{H}\mathrm{e}^{-\mathrm{iEt}}\mathbf{\Psi}$ thus this implies $E\mathbf{\Psi} = \mathscr{H}\mathbf{\Psi}$

Energy Eigenstates

The energy eigenstates satisfy $E\boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$

which leads to

$$\begin{aligned} E\boldsymbol{\chi} &= \mathbf{B}^{\dagger}\boldsymbol{\psi} + m\boldsymbol{\chi}, \\ E\boldsymbol{\psi} &= \mathbf{B}\boldsymbol{\chi} - m\boldsymbol{\psi} \end{aligned}$$

$\boldsymbol{\chi}, \boldsymbol{\psi}$ are respectively the singular vectors of $\mathbf{B}$

with eigenvalue λ and the energy E

is given by $E = \pm\sqrt{|\lambda|^2 + m^2}$

Sketch of the derivation

The eigenvalue problem $E\boldsymbol{\Psi} = \mathscr{H}\boldsymbol{\Psi}$ is equivalent to

$$E\boldsymbol{\chi} = b\mathbf{B}\boldsymbol{\psi} + m\boldsymbol{\chi},$$
$$E\boldsymbol{\psi} = b^{\star}\mathbf{B}^{\top}\boldsymbol{\chi} - m\boldsymbol{\psi}$$

Let us re-order obtaining

$$(E-m)\boldsymbol{\chi} = b\mathbf{B}\boldsymbol{\psi},$$
$$(E+m)\boldsymbol{\psi} = b^{\star}\mathbf{B}^{\top}\boldsymbol{\chi}$$

Therefore

$$(E-m)(E+m)\boldsymbol{\chi} = \mathbf{B}\mathbf{B}^{\top}\boldsymbol{\chi} = \mathbf{L}_{[0]}\boldsymbol{\chi},$$
$$(E+m)(E-m)\boldsymbol{\psi} = \mathbf{B}^{\top}\mathbf{B}\boldsymbol{\psi} = \mathbf{L}_{[1]}^{down}\boldsymbol{\psi}$$

$\longrightarrow$ This implies $E^2 = m^2 + |\lambda|^2$

Matter-Antimatter asymmetry and homology

For $E^2 > m^2$ there is symmetry between positive energy eigenstates and negative energy eigenstates.

However the symmetry between positive energy states and negative energy states breaks down for
$$|E| = m$$

The states at energy states at $E = m$
are localised on nodes and they have a degeneracy given by the Betti number β_0

The energy states $E = -m$
are localised on links and they have a degeneracy given by the Betti number β_1

Eigenvectors of the Dirac equation

The eigenvectors associate to non-zero eigenvalues λ of the Dirac operator are

$$\boldsymbol{\phi}_\lambda^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_\lambda \\ \mathbf{v}_\lambda\end{pmatrix} \quad \boldsymbol{\phi}_\lambda^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_\lambda \\ -\mathbf{v}_\lambda\end{pmatrix}$$

where $\mathbf{u}_\lambda, \mathbf{v}_\lambda$ are the right and left singular vector of $\mathbf{B}_{[1]}$ corresponding to singular value λ and $\mathscr{C}$ indicates the normalisation constants.

The eigenvectors associated to $E = \pm\sqrt{m^2+\lambda^2}, \quad \lambda \neq 0$ of the topological Dirac equation are instead

$$\boldsymbol{\phi}_\lambda^{[+]} = \mathscr{C}\begin{pmatrix}\mathbf{u}_\lambda \\ \frac{b^*\lambda^*}{|E|+m}\mathbf{v}_\lambda\end{pmatrix} \quad \boldsymbol{\phi}_\lambda^{[+]} = \mathscr{C}\begin{pmatrix}\frac{b\lambda}{|E|+m}\mathbf{u}_\lambda \\ -\mathbf{v}_\lambda\end{pmatrix}$$

Therefore the overall normalisation of the nodes signal changes with respect

to the normalisation of the edge signal.

Eigenvectors of the Dirac equation

The harmonic eigenvectors associated to the zero eigenvalue $\lambda = 0$ of the Dirac operator are

$$\boldsymbol{\phi}_0^N = \begin{pmatrix} \mathbf{u}^{\mathrm{h}} \\ \mathbf{0} \end{pmatrix} \quad \boldsymbol{\phi}_0^E = \begin{pmatrix} \mathbf{0} \\ \mathbf{v}^{\mathrm{h}} \end{pmatrix}$$

where $\mathbf{u}^{\mathrm{h}}, \mathbf{v}^{\mathrm{h}}$ are the harmonic eigenvector of $\mathbf{L}_{[0]}$ and $\mathbf{L}_{[1]}$ respectively

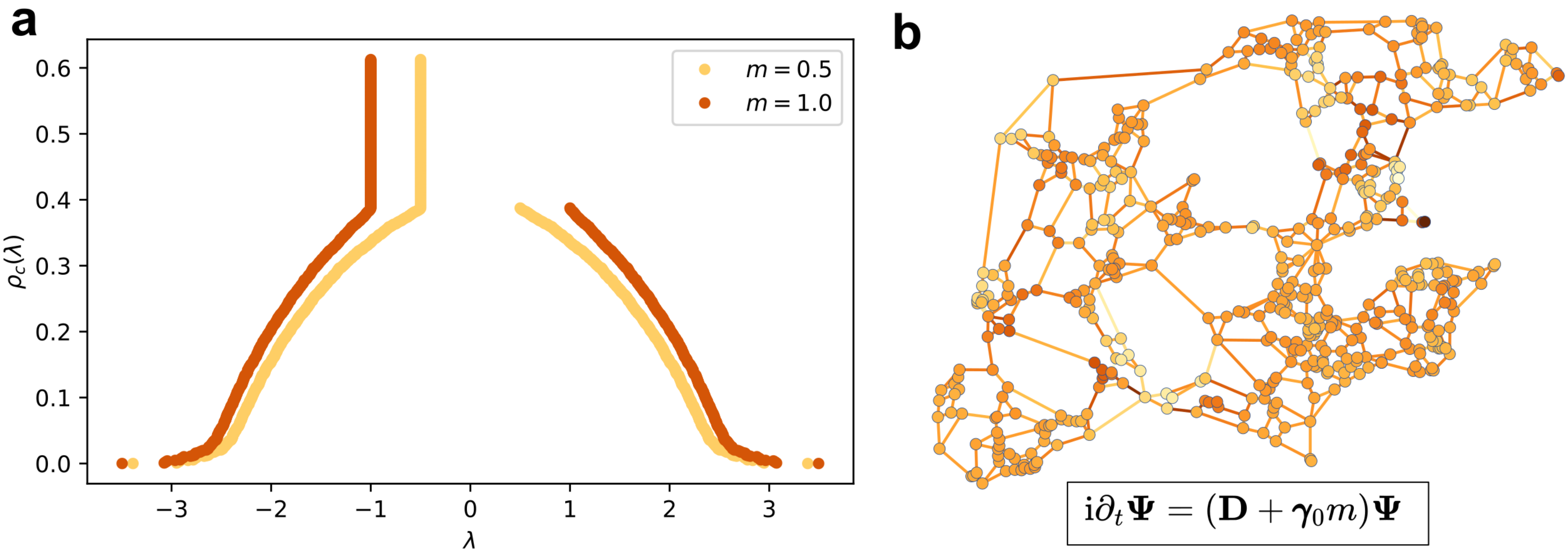
The eigenvectors associated to energies $E = \pm m$ of the topological Dirac equation are instead

$$\boldsymbol{\phi}_0^+ = \begin{pmatrix} \mathbf{u}^{\mathrm{h}} \\ \mathbf{0} \end{pmatrix} \quad \boldsymbol{\phi}_0^- = \begin{pmatrix} \mathbf{0} \\ \mathbf{v}^{\mathrm{h}} \end{pmatrix}$$

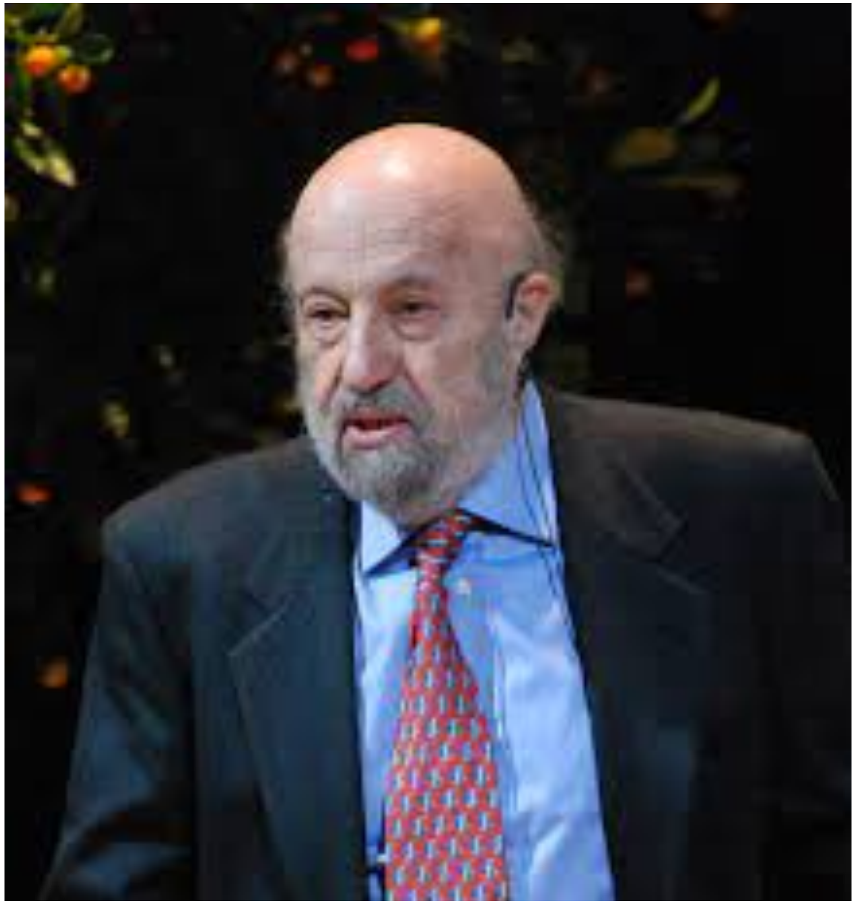
With eigenstates of energy $E = m$ having degeneracy β_0 and the

eigenstates of energy $E = -m$ having degeneracy β_1

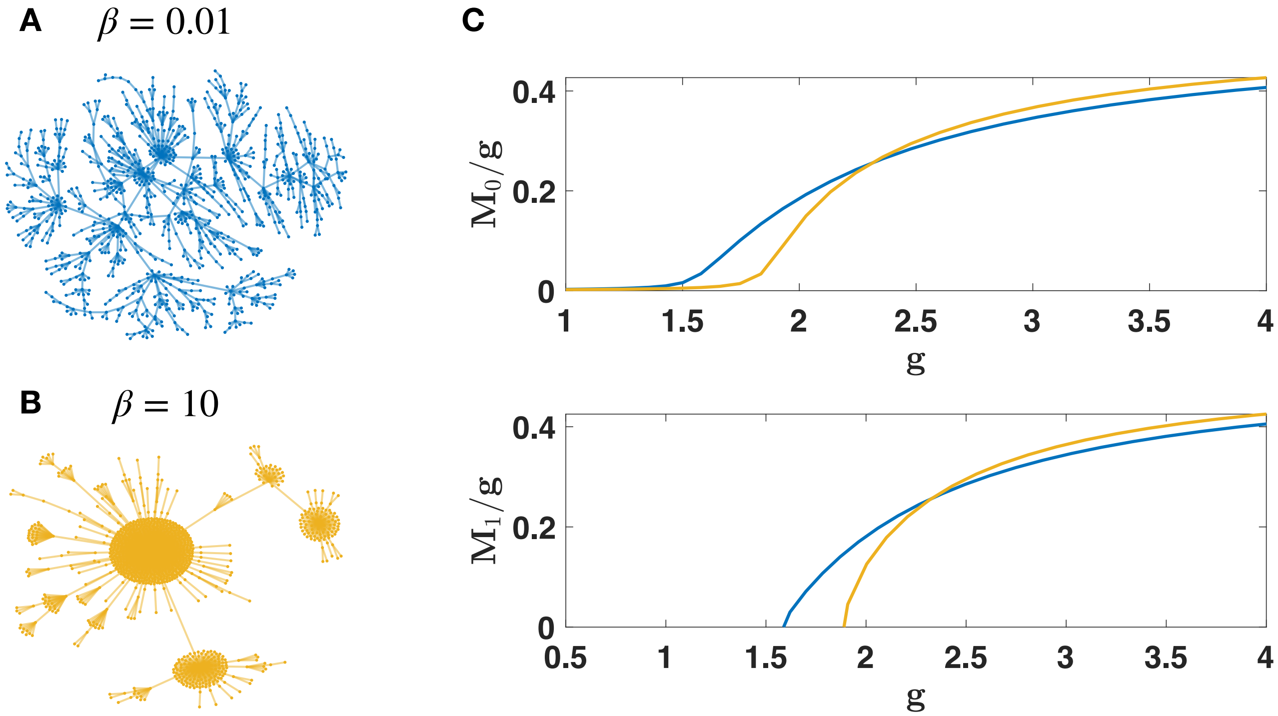
Dirac equation spectrum and eigenstate



Nambu-Jona Lasinio legacy



Mass of simple and higher-order networks



Gap equation

$$M_0 = \frac{g}{\mathcal{N}} \sum_{i=1} \beta_{2i} + \frac{g}{\mathcal{N}} \sum_{\lambda>0} \frac{M_0}{\sqrt{M_0^2 + \lambda^2}}$$

$$M_1 = \frac{g}{\mathcal{N}} \sum_{i=1} \beta_{2i+1} + \frac{g}{\mathcal{N}} \sum_{\lambda>0} \frac{M_1}{\sqrt{M_1^2 + \lambda^2}}$$

The mass of simple or higher-order networks depends on their geometry and topology

G. Bianconi The mass of simple and higher-order networks JPhysA (2023)

Combining the Dirac operator with algebra
Topological Dirac equation on 3 dimensional lattice

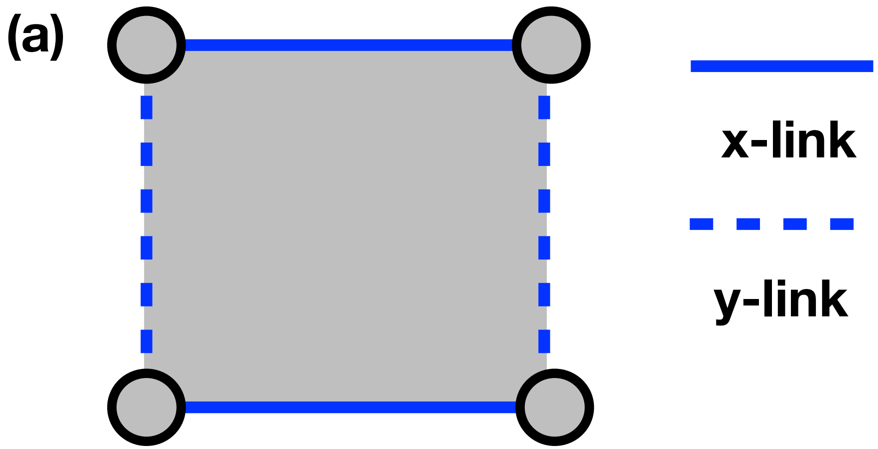
[Non-examinable]

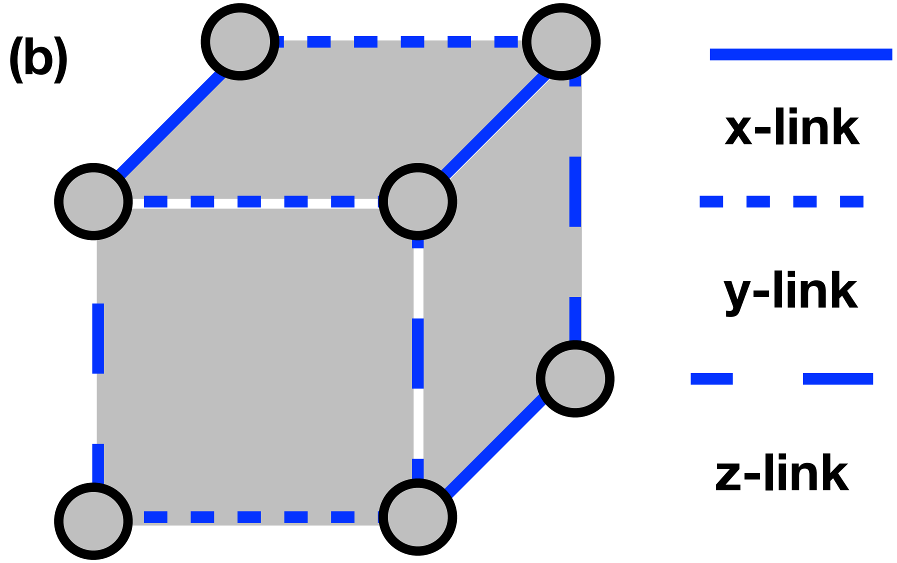
G. Bianconi,
Topological Dirac equation on networks and simplicial complexes
JPhys Complexity (2021)

Directional Dirac operator on lattices

On a lattice links have different directions

The Directional Dirac operator induces a
phase rotation of the topological signal depending on the direction of the links





Introducing an algebra

Dirac operator on a network can be enriched by an algebra

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & b\mathbf{B}_{[1]} \\ b^{\star}\mathbf{B}_{[1]}^{\top} & \mathbf{0} \end{pmatrix}$$

with $b \in \mathbb{C}, \quad |b| = 1$

Topological spinor for 3-dimensional lattice

In order to treat every type of link differently

by inducing different "rotations"of the topological spinor,

in 3-d we need to consider the spinor $\boldsymbol{\Psi}$ formed by two 0-cochains and two 1-cochains, i.e.

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Xi} \\ \hat{\boldsymbol{\Psi}} \end{pmatrix},$$

with

$$\boldsymbol{\Xi} = \begin{pmatrix} \chi^{(1)} \\ \chi^{(2)} \end{pmatrix}, \hat{\boldsymbol{\Psi}} = \begin{pmatrix} \boldsymbol{\psi}^{(1)} \\ \boldsymbol{\psi}^{(2)} \end{pmatrix}$$

Directional Boundary operators and Hodge Laplacians on the 3-dimensional lattice

We consider directional boundary operators only acting between nodes and w-type links

$$[\mathbf{B}_{(w)}]_{r\ell} = \begin{cases} 1 \text{ if } \ell = [s, r] \text{ and } \ell \text{ is a type } w\text{−link} \\ -1 \text{ if } \ell = [r, s] \text{ and } \ell \text{ is a type } w\text{−link} \\ 0 \text{ otherwise} \end{cases}$$

The directional Hodge Laplacians are given by

$$\mathbf{L}_{[0]}^{(w)} = \mathbf{B}_{(w)}\mathbf{B}_{(w)}^{\top} \quad \mathbf{L}_{[1]}^{(w)} = \mathbf{B}_{(w)}^{\top}\mathbf{B}_{(w)}$$

The graph Laplacian is given by

$$\mathbf{L}_{[0]} = \mathbf{L}_{[0]}^{(x)} + \mathbf{L}_{[0]}^{(y)} + \mathbf{L}_{[0]}^{(z)}$$

Directional Dirac operators on 3-dimensional lattice

In 3d the Directional Dirac operators are defined as

$$\mathbf{D}_{(w)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{w})} \\ \mathscr{B}^{\dagger}_{(\mathbf{w})} & \mathbf{0} \end{pmatrix}$$

with

$$\mathscr{B}_{(x)} = \boldsymbol{\sigma}_1 \otimes \mathbf{B}_{(x)}, \qquad \mathscr{B}_{(y)} = \boldsymbol{\sigma}_2 \otimes \mathbf{B}_{(y)}, \qquad \mathscr{B}_{(z)} = \boldsymbol{\sigma}_3 \otimes \mathbf{B}_{(z)},$$

where we make use of the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Spatial directional Dirac operators

The spatial directional Dirac operators

$$\mathbf{D}_{(w)} = \begin{pmatrix} \mathbf{0} & \mathscr{B}_{(\mathbf{w})} \\ \mathscr{B}^{\dagger}_{(\mathbf{w})} & \mathbf{0} \end{pmatrix}$$

are Hermitian

and their square is given by the directional Laplacians

$$[\mathbf{D}_{(w)}]^2 = \mathscr{L}_{(w)} = \begin{pmatrix} \mathbf{I}_2 \otimes \mathbf{L}^{(w)}_{[0]} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \otimes \mathbf{L}^{(w)}_{[1]} \end{pmatrix}$$

The Dirac operator on simplicial complexes

The exterior derivative on simplicial complexes

The exterior derivative acts directly on the topological spinor in dimension n we have

Exterior derivative operator

$$d = \oplus_{m=0}^{n-1} \delta_m$$

Topological signal "spinor"

$$s \in \oplus_{m=1}^{n} C^m$$

Thus for n=2, in matrix form we obtain

$$d = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{B}_{[1]}^\top & 0 & 0 \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix}, \qquad \mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

$\mathbf{s}_0$ Node signal

$\mathbf{s}_1$ Link signal

$\mathbf{s}_2$ Triangle signal

The Dirac operator on simplicial complexes

**The Dirac operator allows
to study interacting topological signals of different dimensions
coexisting in the same network topology**

$$D = d + d^*$$

Assuming a L_2 norm between cochains we obtain

Dirac operator

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix},$$

Topological signal “spinor”

$$\mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix}$$

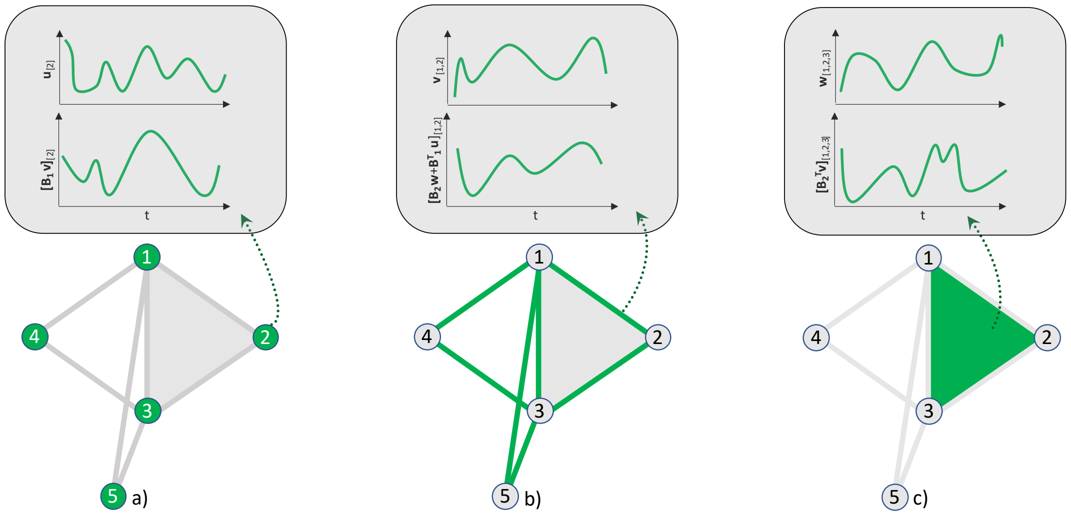
$\mathbf{s}_0$ **Node signal**

$\mathbf{s}_1$ **Link signal**

$\mathbf{s}_2$ **Triangle signal**

The action of the Dirac operator

The Dirac operator allows cross-talking between signals of different dimension



$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix}, \text{ acts on } \mathbf{s} = \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}\mathbf{s} = \begin{pmatrix} \mathbf{B}_{[1]}\mathbf{s}_1 \\ \mathbf{B}_{[1]}^\top \mathbf{s}_0 + \mathbf{B}_{[2]}\mathbf{s}_2 \\ \mathbf{B}_{[2]}^\top \mathbf{s}_1 \end{pmatrix}$$

Dirac decomposition

$$D = \sum_{m=1}^{n} D_{[m]}$$

where $D_{[m]} = \delta_{m-1} + \delta^{\star}_{m-1}$

only couples $(m-1)$-dimensional simplices

to m-dimensional simplices

Dirac decomposition

$$D_{[m]}D_{[m']} = D_{[m']}D_{[m]} = 0 \quad \forall m \neq m'$$

Dirac decomposition for n=2

$$\mathbf{D} = \mathbf{D}_{[1]} + \mathbf{D}_{[2]}$$

Here

$\mathbf{D}_{[1]}$ only couples node and link signals and

$\mathbf{D}_{[2]}$only couples link and triangle signals

$$\mathbf{D}_{[1]} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^{\top} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{D}_{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^{\top} & 0 \end{pmatrix}$$

$$\mathbf{D}_{[1]}^2 = \mathscr{L}_{[1]} = \begin{pmatrix} \mathbf{L}_{[0]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]}^{down} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{D}_{[2]}^2 = \mathscr{L}_{[2]} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{[1]}^{up} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{[2]}^{down} \end{pmatrix}$$

Dirac decomposition

Since the boundary of the boundary is null we obtain

$$\mathbf{D}_{[1]}\mathbf{D}_{[2]} = \mathbf{D}_{[2]}\mathbf{D}_{[1]} = \mathbf{0}$$

which implies

$$\boxed{\begin{gathered}\ker(\mathbf{D}_{[1]}) \supseteq \operatorname{im}(\mathbf{D}_{[2]}) \\ \ker(\mathbf{D}_{[2]}) \supseteq \operatorname{im}(\mathbf{D}_{[1]})\end{gathered}}$$

Dirac decomposition

Every topological signal can be decomposed in a unique way thanks to the Dirac decomposition

$$\mathbb{R}^{N_0+N_1+N_2} = \text{im}(\mathbf{D}_{[1]}) \oplus \text{ker}(\mathbf{D}) \oplus \text{im}(\mathbf{D}_{[2]})$$

therefore every signals defined on nodes, links and triangles can be decomposed in a unique way as

$$\mathbf{s} = \mathbf{s}^{[1]} + \mathbf{s}^{[2]} + \mathbf{s}^{harm}$$

With

$$\mathbf{s}^{[1]} = \mathbf{D}_{[1]}\mathbf{D}_{[1]}^{+}\mathbf{s}$$

$$\mathbf{s}^{[2]} = \mathbf{D}_{[2]}\mathbf{D}_{[2]}^{+}\mathbf{s}$$

Eigenvalues of the Dirac operator

Due to the Dirac decomposition
the eigenvalues of the Dirac operator $\mathbf{D}$
are the direct sum
of the non-zero eigenvalues
of $\mathbf{D}_{[1]}$ and of $\mathbf{D}_{[2]}$
plus the zero eigenvalue
with degeneracy $\beta_0 + \beta_1 + \beta_2$

Eigenvectors of the Dirac operator

Due to the Dirac decomposition

the eigenvectors of the Dirac operator $\mathbf{D}$

are the eigenvectors

corresponding to non-zero eigenvalues

of $\mathbf{D}_{[1]}$ or of $\mathbf{D}_{[2]}$

r the harmonic eigenvectors of $\mathbf{D}$

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}^{[1]} & \mathbf{\Phi}^{[2]} & \mathbf{\Phi}^{harm} \end{pmatrix}$$

With $\mathbf{\Phi}^{[1]}$ localised on nodes and links and

$\mathbf{\Phi}^{[2]}$ localised on links and triangles

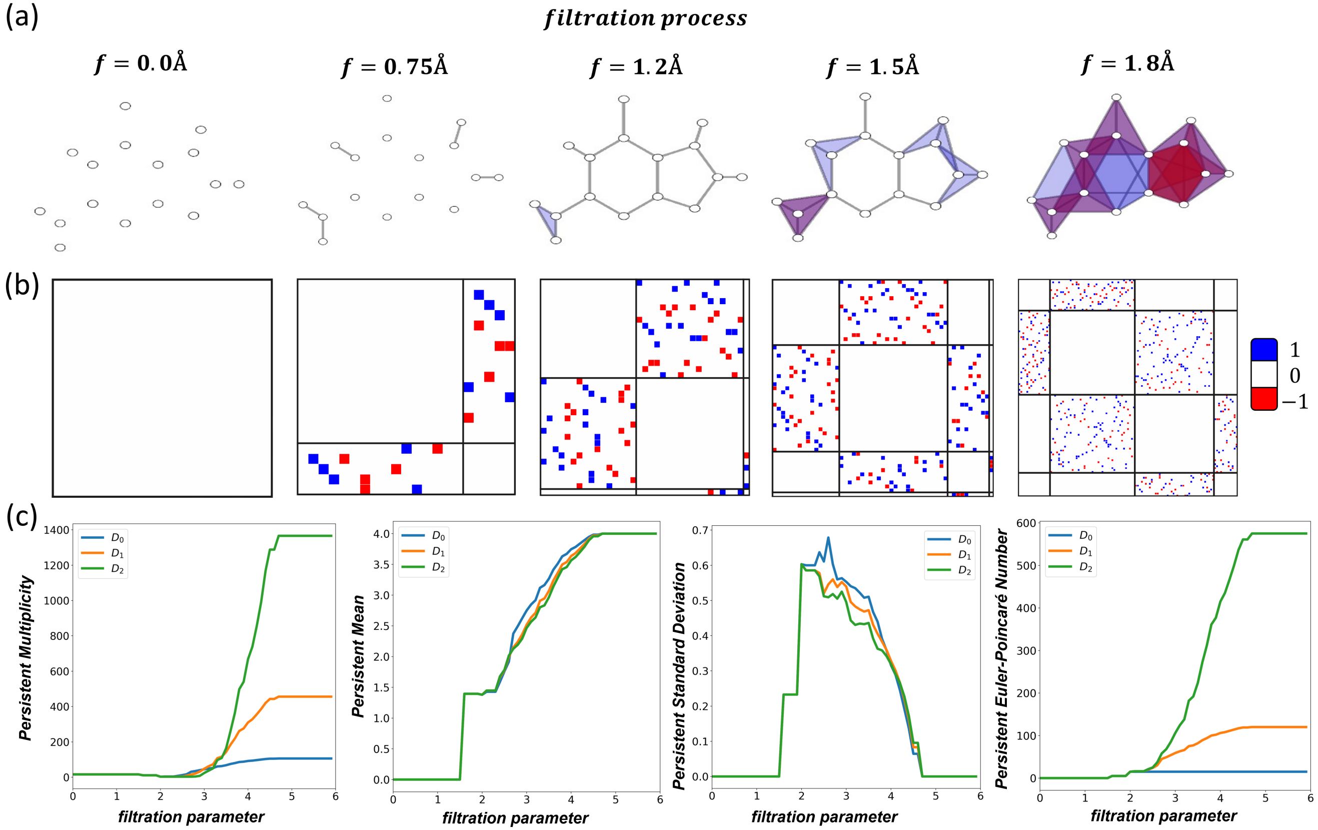
Eigenvectors or the Dirac operator

In summary the eigenvectors of the Dirac operator

defined on a simplicial complex of dimension 2 have the structure

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{U}^{[1]} & \mathbf{U}^{[1]} & \mathbf{0} & \mathbf{0} & \mathbf{U}_0^{harm} & \mathbf{0} & \mathbf{0} \\ \mathbf{V}^{[1]} & -\mathbf{V}^{[1]} & \mathbf{U}^{[2]} & \mathbf{U}^{[2]} & \mathbf{0} & \mathbf{U}_1^{harm} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}^{[2]} & -\mathbf{V}^{[2]} & \mathbf{0} & \mathbf{0} & \mathbf{U}_2^{harm} \end{pmatrix}$$

Topological Data analysis
Persistent Dirac for molecular representations



JJ Wee, G. Bianconi, K. Xia (2023)

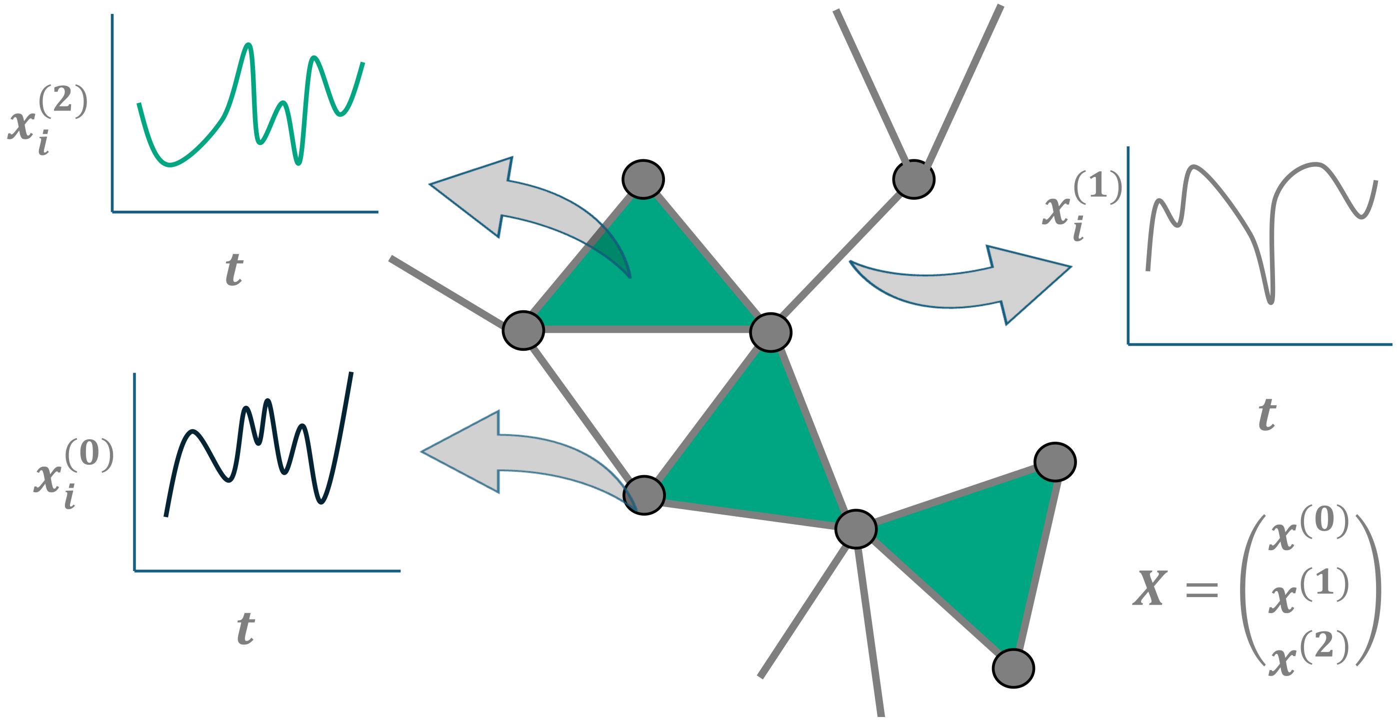
Dirac operator

and

Dynamical systems

[Non examinable]

Coupling topological signals of different dimension



Dirac Synchronization on graphs

Dirac Synchronization allows to couple locally and topologically signals defined on nodes and links.

Given $\mathbf{\Phi} = (\boldsymbol{\theta}, \boldsymbol{\phi})^\top$ Dirac synchronisation obeys

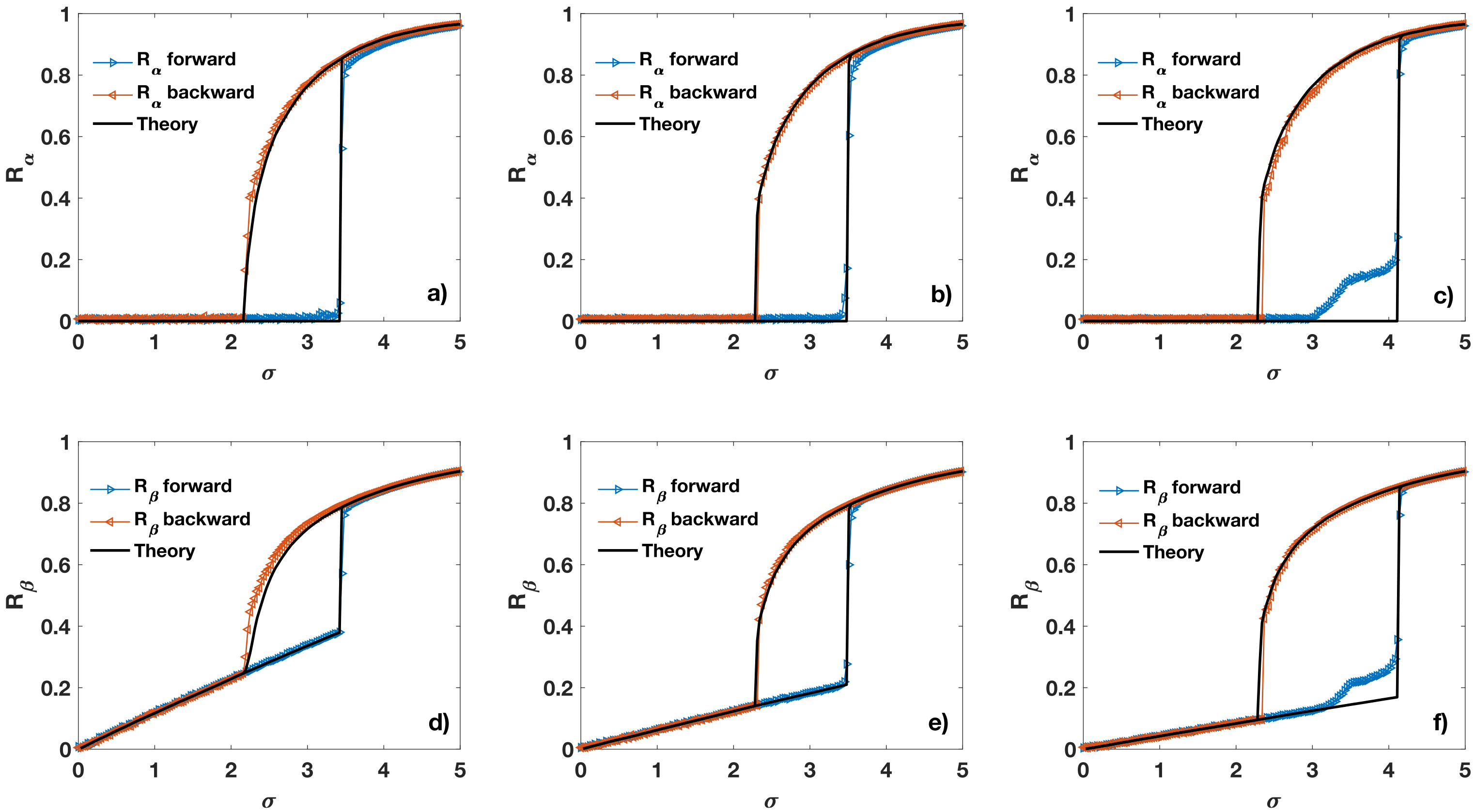
$$\dot{\mathbf{\Phi}} = \mathbf{\Omega} - \sigma\tilde{\mathbf{D}}\sin((\tilde{\mathbf{D}} - z\boldsymbol{\gamma}\tilde{\mathbf{D}}^2)\mathbf{\Phi})$$

- **Node and links signals are not independent.** The order parameters depend on linear combinations of nodes and link signals

- **The synchronization transition is discontinuous**

L. Calmon, J. Restrepo, J.J. Torres and G. Bianconi (2022) L. Calmon, S. Khrisnagopal, G. Bianconi (2023)

Dirac synchronization is explosive on a fully connected network

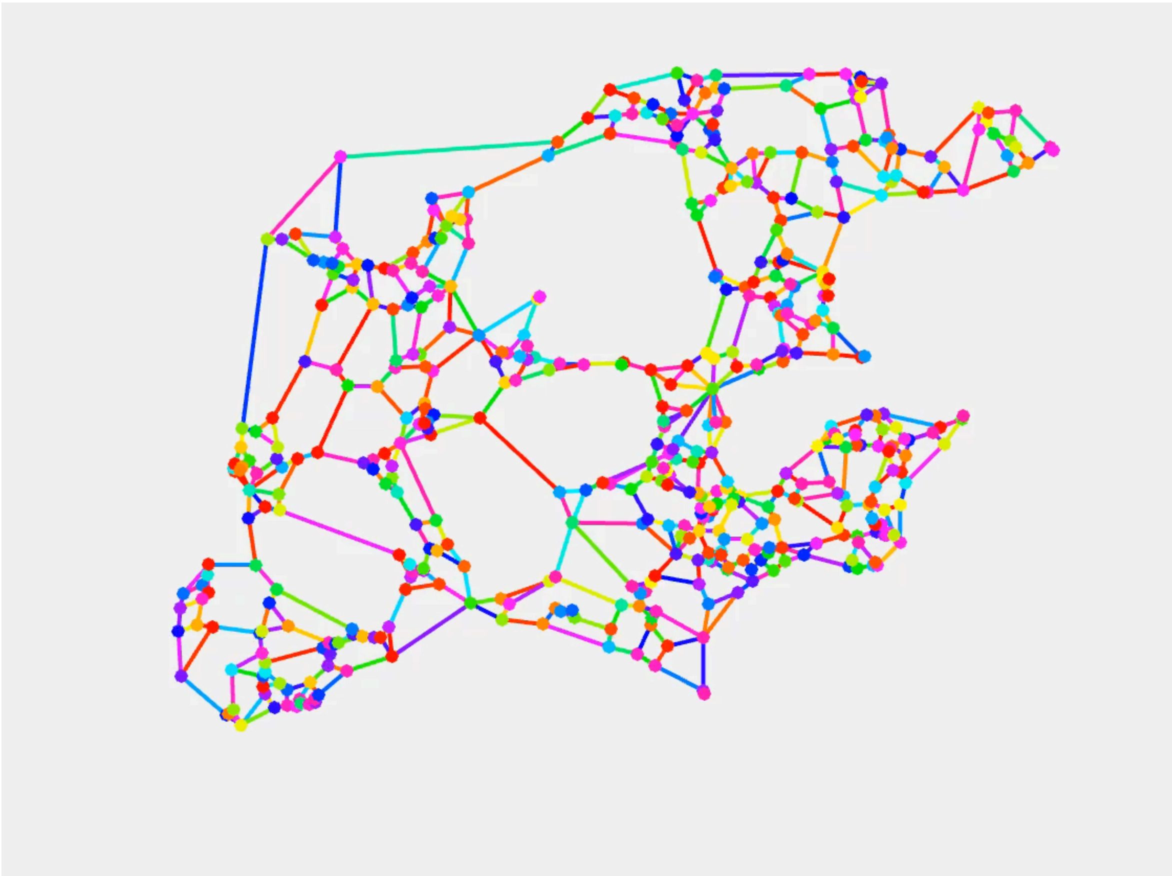


In the Dirac Synchronization the free dynamics of the synchronized state is localised on the links around 1-dimensional holes (since we are in a network)

$$\frac{d\langle \mathbf{u}_{harm}, \boldsymbol{\phi} \rangle}{dt} = \langle \mathbf{u}_{harm}, \hat{\boldsymbol{\omega}} \rangle$$

The free dynamics is localised on harmonic components

Dirac synchronisation on the fungi network



L. Calmon, J. Restrepo, J.J. Torres and G. Bianconi (2022) L. Calmon,S. Khrisnagopal, G. Bianconi (2023)

Global Dirac Synchronization

Consider a 2-dimensional cell complex whose dynamical state is encoded in the topological spinor

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{pmatrix}$$

Consider the uncoupled dynamics of identical oscillators placed on nodes, edges and 2-cells

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) + \cancel{\mathbf{D}}\mathbf{H}(\mathbf{X})$$

where $\mathbf{F}(\mathbf{X}) = \begin{pmatrix} \mathbf{f}(\mathbf{x}^{(0)}) \\ \mathbf{f}(\mathbf{x}^{(1)}) \\ \mathbf{f}(\mathbf{x}^{(2)}) \end{pmatrix}, \quad \mathbf{H}(\mathbf{X}) = \begin{pmatrix} \mathbf{h}(\mathbf{x}^{(0)}) \\ \mathbf{h}(\mathbf{x}^{(1)}) \\ \mathbf{h}(\mathbf{x}^{(2)}) \end{pmatrix},$

Dirac operator

In Dirac Global Synchronization the coupling
between identical oscillators
is captured by the Dirac operator

$$\boxed{\mathbf{D}\!\!\!/ = \gamma_{[1]}\mathbf{D}_{[1]} + \gamma_{[2]}\mathbf{D}_{[2]}}$$

where
$\mathbf{D}_{[1]}$ couples nodes with edges and
$\mathbf{D}_{[2]}$ couples edges with two-cells
while the matrices $\boldsymbol{\gamma}_{[n]}$ encode for the coupling constants

Dirac operator and gamma matrices

$$\boxed{\not{D} = \gamma_{[1]}\mathbf{D}_{[1]} + \gamma_{[2]}\mathbf{D}_{[2]}}$$

where

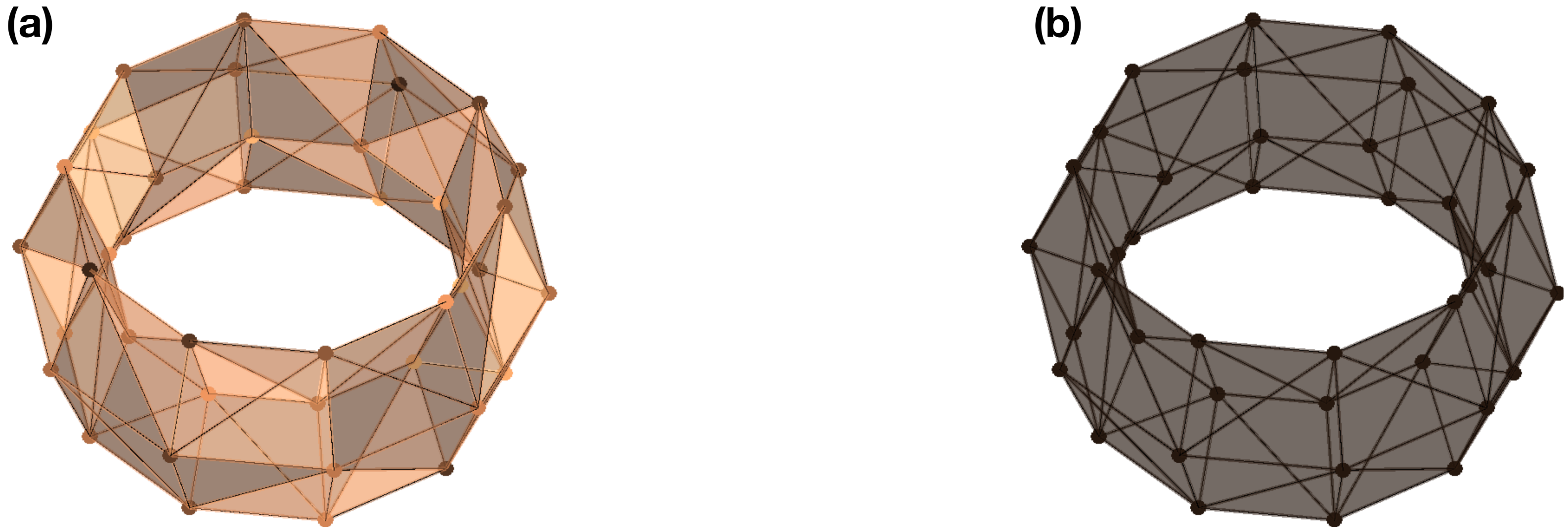
$$\mathbf{D}_{[1]} = \begin{pmatrix} 0 & \mathbf{I}_d \otimes \mathbf{B}_{[1]} & 0 \\ \mathbf{I}_d \otimes \mathbf{B}_{[1]}^\top & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{D}_{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_d \otimes \mathbf{B}_{[2]} \\ 0 & \mathbf{I}_d \otimes \mathbf{B}_{[2]}^\top & 0 \end{pmatrix}$$

And

$$\gamma_{[1]} = \begin{pmatrix} \gamma_0^{(1)} \otimes \mathbf{I}_{N_0} & 0 & 0 \\ 0 & \gamma_1^{(1)} \otimes \mathbf{I}_{N_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_1^{(2)} \otimes \mathbf{I}_{N_1} & 0 \\ 0 & 0 & \gamma_2^{(2)} \otimes \mathbf{I}_{N_2} \end{pmatrix}$$

Global Dirac Synchronization

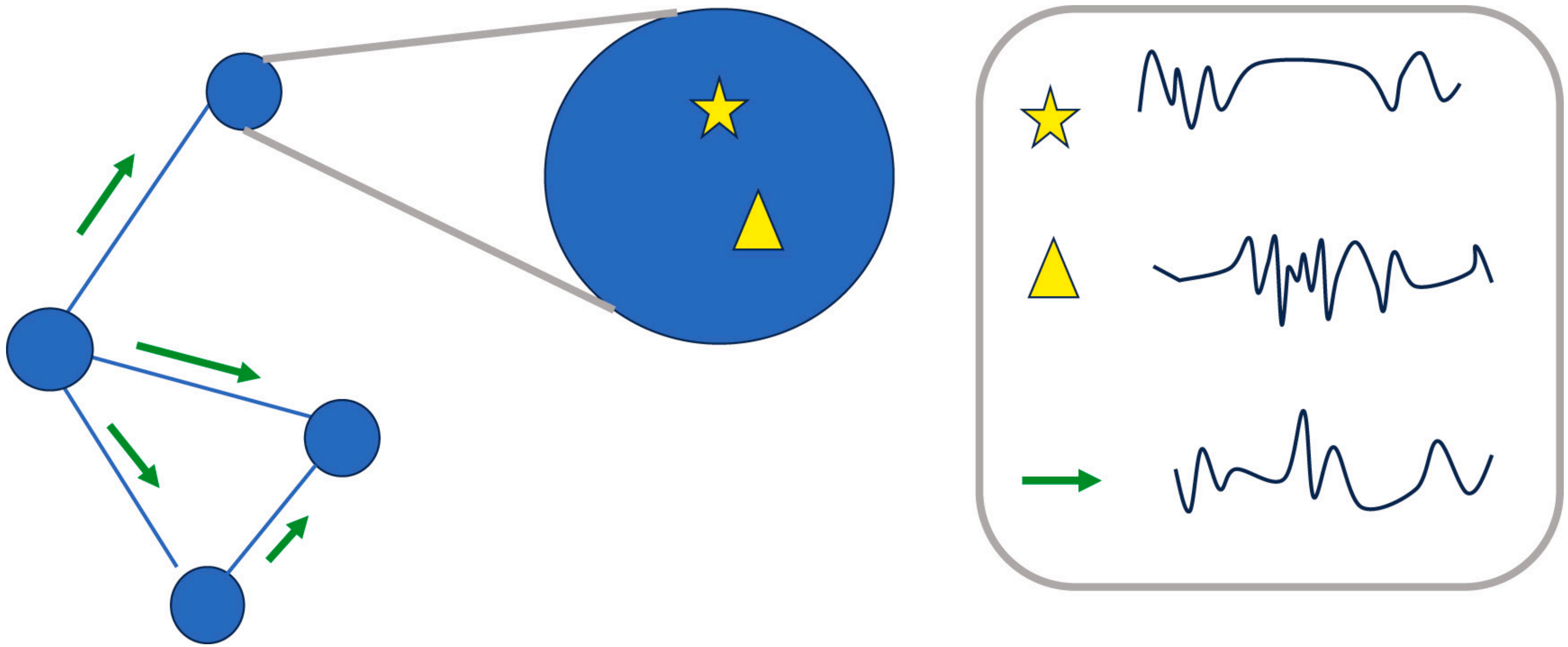
Global Dirac Synchronization is achieved on a square lattice tessellation of a torus and on a triangular tessellation of a weighted torus



Transient state

Globally synchronised state

Dynamical Turing patterns comprising three topological signals



Muolo, Carletti Bianconi Chaos Solitons Fractals (2024)

The three way Dirac operator

Treating three topological signal requires the three-way Dirac operator coupled with a non-trivial gamma matrix

$$\partial = \begin{pmatrix} \mathbf{0} & \mathbf{I} \otimes \mathbf{B} \\ \mathbf{I}_2 \otimes \mathbf{B}^\top & \mathbf{0} \end{pmatrix} \quad \gamma = \begin{pmatrix} \boldsymbol{\alpha} \otimes \mathbf{I}_{N_0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta} \otimes \mathbf{I}_{N_1} \end{pmatrix} \longrightarrow \partial = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\top & \mathbf{0} \end{pmatrix} \quad \gamma = \begin{pmatrix} \alpha_u \mathbf{I}_{N_0} & \mathbf{0} & \mathbf{0} \\ \alpha_v \mathbf{I}_{N_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta_u \mathbf{I}_{N_1} & \beta_u \mathbf{I}_{N_1} \end{pmatrix}$$

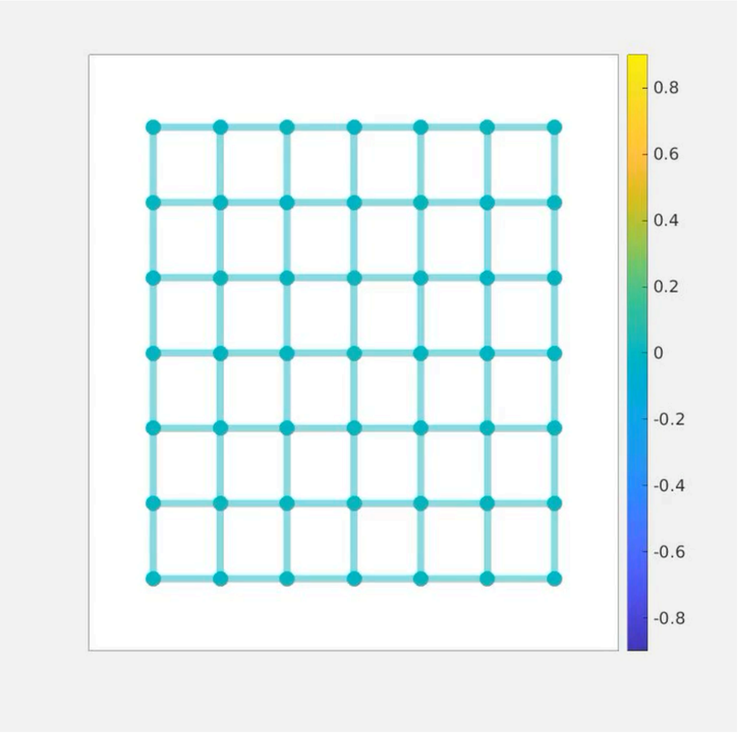
$$\not{D} = \gamma \partial = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \alpha_u \mathbf{B} \\ \mathbf{0} & \mathbf{0} & \alpha_v \mathbf{B} \\ \beta_u \mathbf{B}^\top & \beta_u \mathbf{B}^\top & \mathbf{0} \end{pmatrix}$$

Muolo, Carletti Bianconi Chaos Solitons Fractals (2024)

Dirac patterns are different!

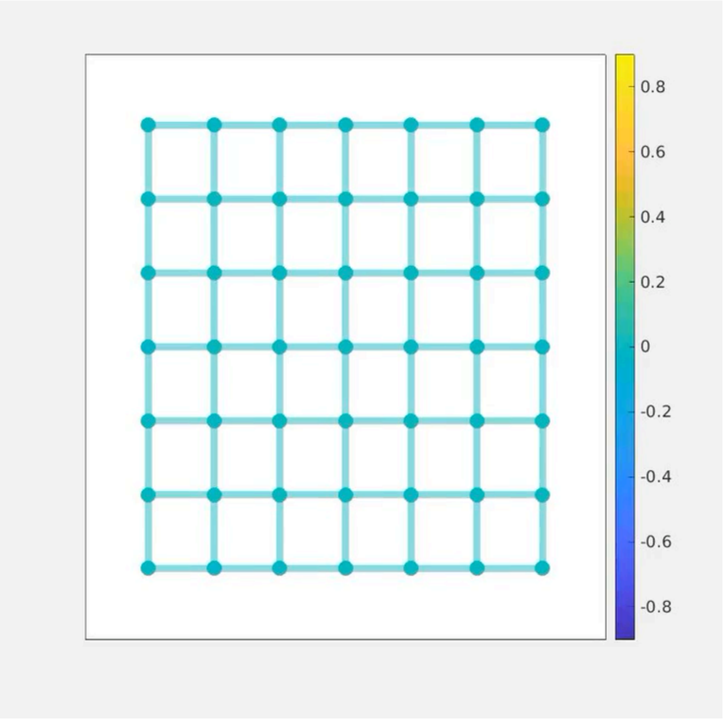
Turing patterns

$$\dot{\boldsymbol{\phi}} = \mathbf{F}(\boldsymbol{\phi}) - \sigma \mathscr{L} \boldsymbol{\phi}$$



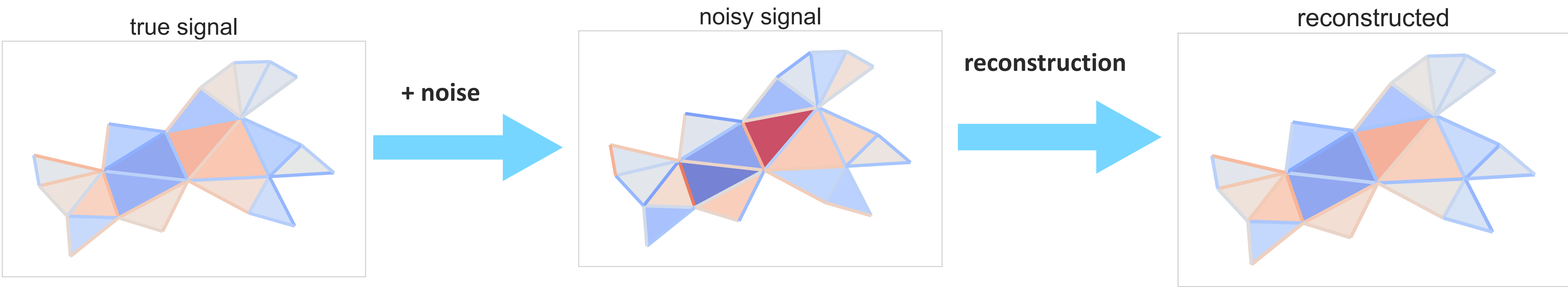
Dirac patterns

$$\dot{\boldsymbol{\phi}} = \mathbf{F}(\boldsymbol{\phi}) - \sigma \not{\mathcal{D}} \boldsymbol{\phi}$$



Muolo, Carletti Bianconi Chaos Solitons Fractals (2024)

Dirac Signal Processing



The Dirac operator allows us to filter out nodes and links signals **jointly**

L. Calmon, M. Schaub and G. Bianconi
Dirac signal processing of topological signals
(2023)

Processing with the Dirac operator

Given a **noisy topological signal defined on simplices of different dimension**

$\tilde{\mathbf{s}} = \mathbf{s} + \boldsymbol{\epsilon}$ with $\boldsymbol{\epsilon}$ noise

The reconstructed signal is $\hat{\mathbf{s}}$

Found by Minimising the Loss $\mathscr{L}$

that jointly filters the signal with the Dirac operator:

$$\mathscr{L} = \|\tilde{\mathbf{s}} - \hat{\mathbf{s}}\|_2^2 + \gamma \hat{\mathbf{s}}^T (\mathbf{D} - E\mathbf{I})^2 \hat{\mathbf{s}}$$

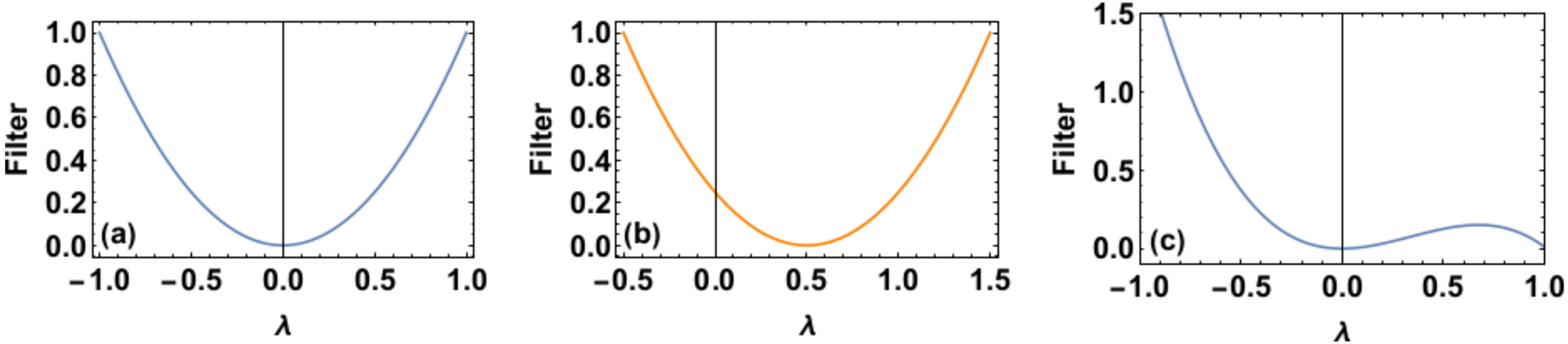
$m = 0$ Hodge Laplacian kernel

$m \neq 0$ Dirac kernel coupling signals of different dimension

Processing with the Dirac operator

Possible choices for the regularisation term

$$\mathscr{L} = \|\tilde{\mathbf{s}} - \hat{\mathbf{s}}\|_2^2 + \gamma \hat{\mathbf{s}}^T \mathscr{R} \hat{\mathbf{s}}$$



$\hat{\mathbf{s}}^T \mathscr{R} \hat{\mathbf{s}} = \hat{\mathbf{s}}^T \mathbf{D}^2 \hat{\mathbf{s}}$

$\hat{\mathbf{s}}^T \mathscr{R} \hat{\mathbf{s}} = \hat{\mathbf{s}}^T \left(\mathbf{D} - E\mathbf{I}\right)^2 \hat{\mathbf{s}}$

$\hat{\mathbf{s}}^T \mathscr{R} \hat{\mathbf{s}} = \hat{\mathbf{s}}^T (\mathbf{D}^2 - z\mathbf{D}^3) \hat{\mathbf{s}}$

Interpretation of the parameter m

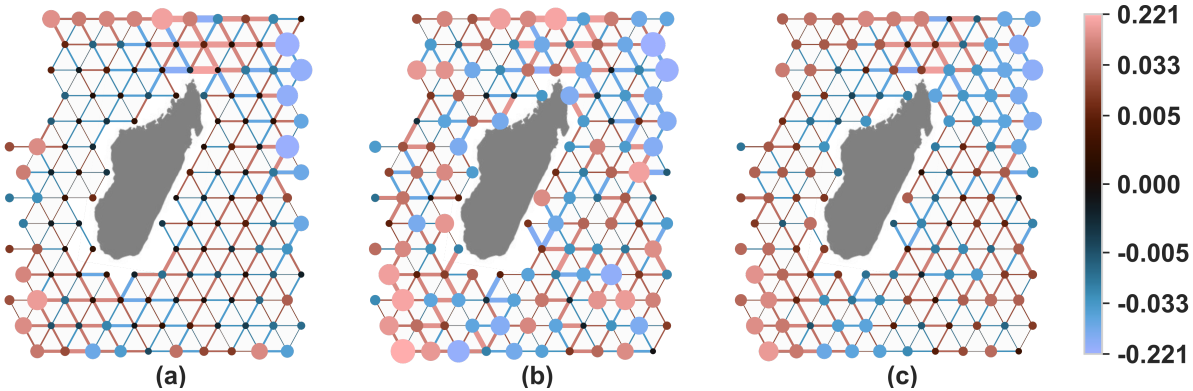
The parameter E can be interpreted as

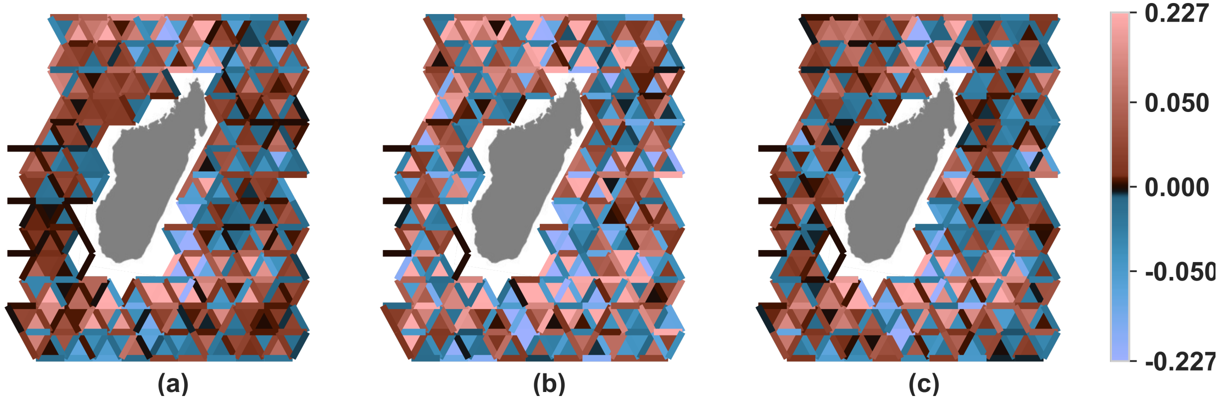
$$\boxed{E = \frac{\mathbf{s}^\top \mathbf{D} \mathbf{s}}{\mathbf{s}^\top \mathbf{s}}}$$

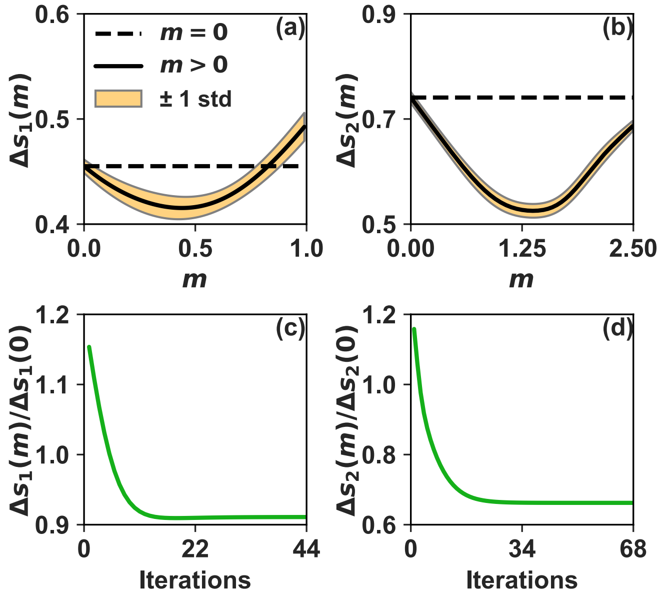
Which allow us to interpret the regularisation as a minimization of the mean square error of the signal around E

The parameter E can be learned from data

Dirac signal processing on buoys data







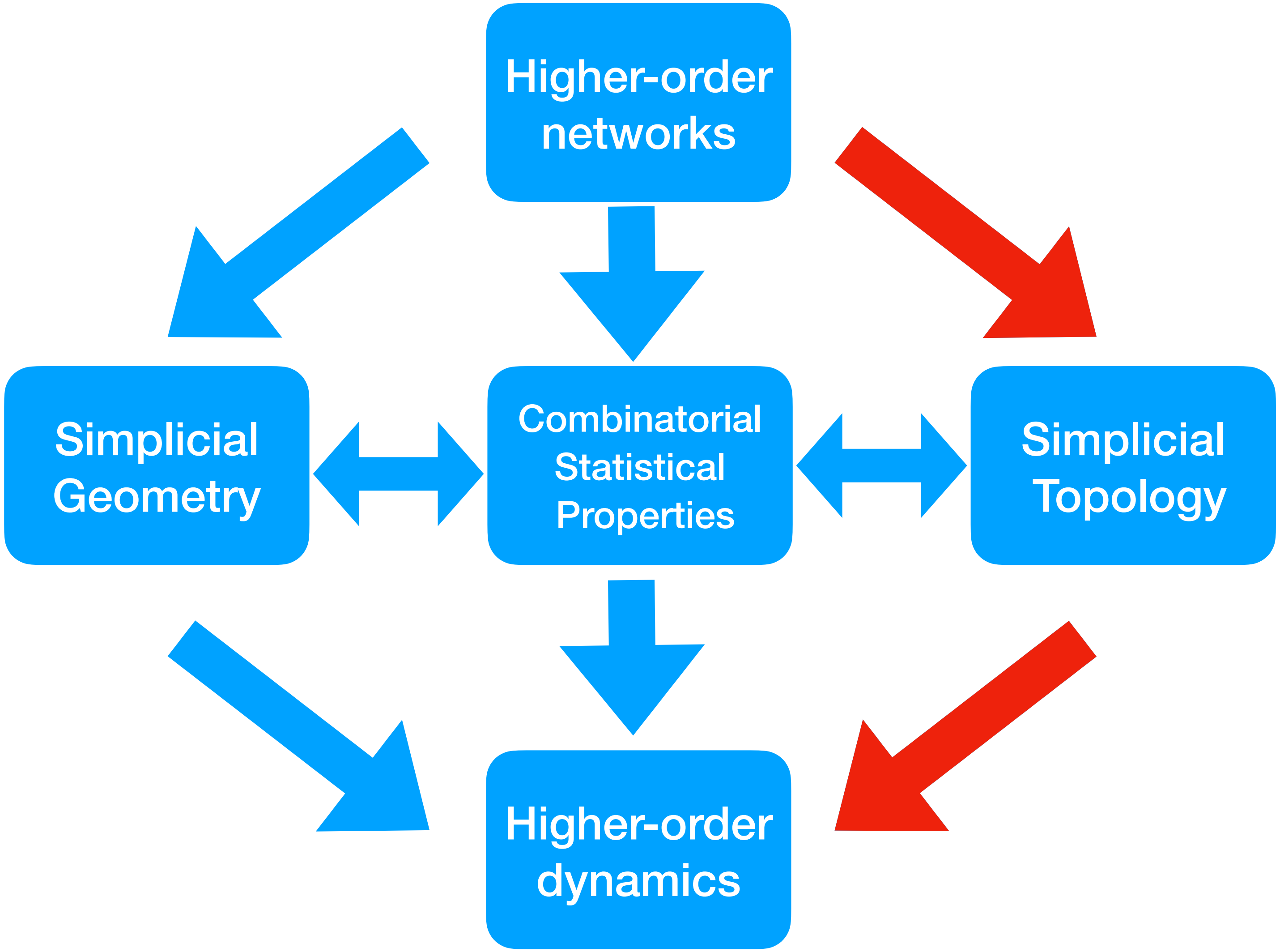
Lesson V:
The Dirac operator
on graphs and simplicial complexes

- **Dirac operator on graphs**
  - **Eigenvalues, Eigenvectors Chirality**
  - **Weighted and Normalised Dirac operator**
  - **Topological Dirac equation**
  - **Insights into the mathematical interpretation of gamma matrices**

- **Dirac operator on simplicial complexes**

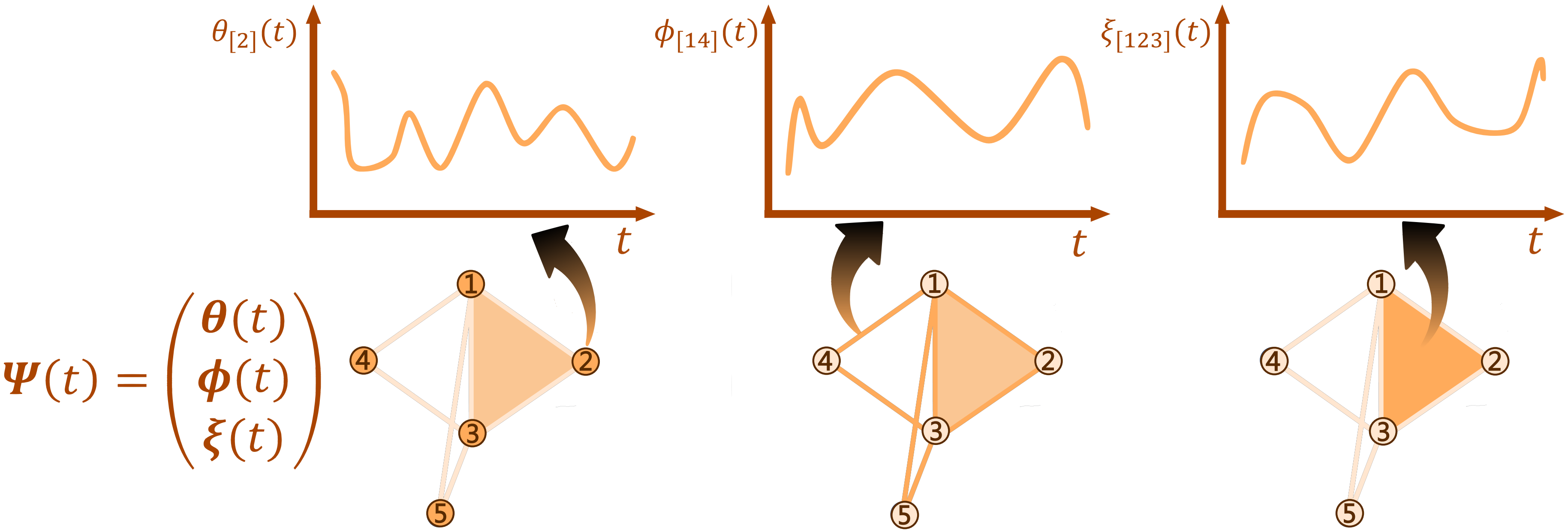
- **Dirac operator in dynamical systems and signal processing**
  - **Dirac Synchronisation and Global Dirac Synchronization**
  - **Dirac Turing patterns**
  - **Dirac signal processing**

Higher-order structure and dynamics

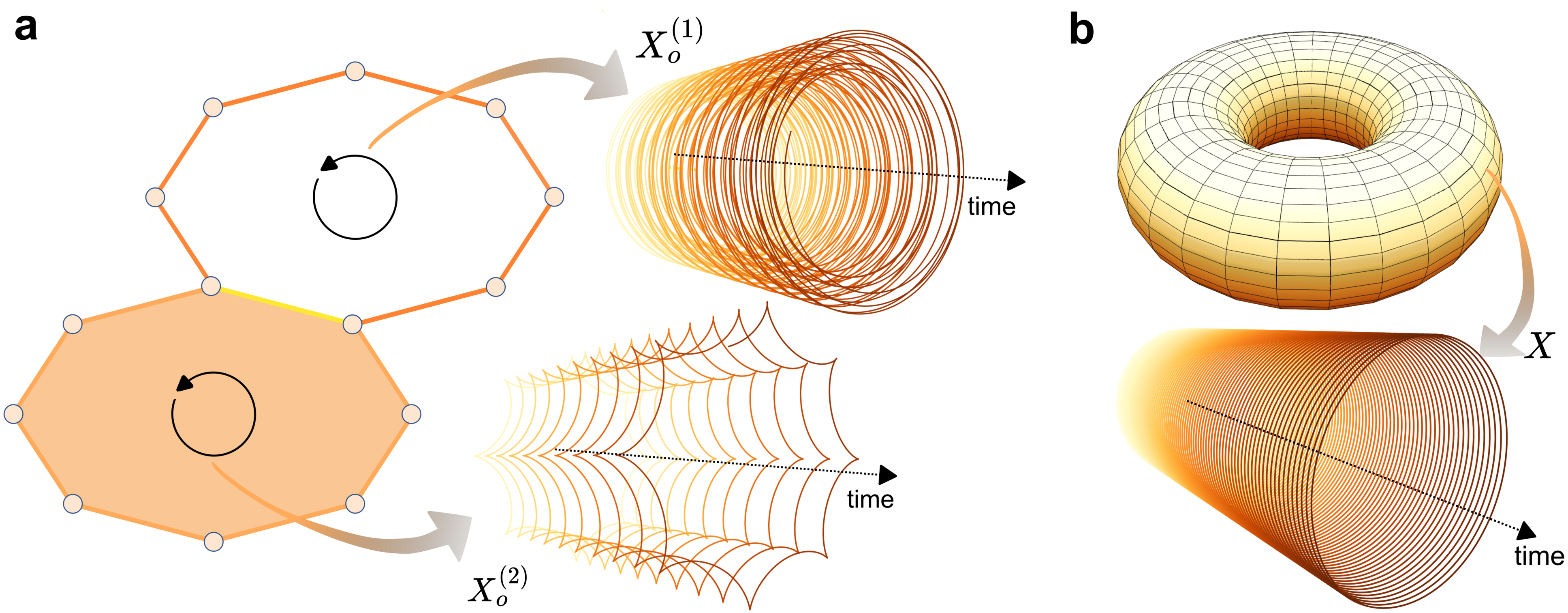


Topological signals

Beyond the node centered description of network dynamics
The dynamical state of a simplicial complex includes
node, edge, and higher-order topological signals

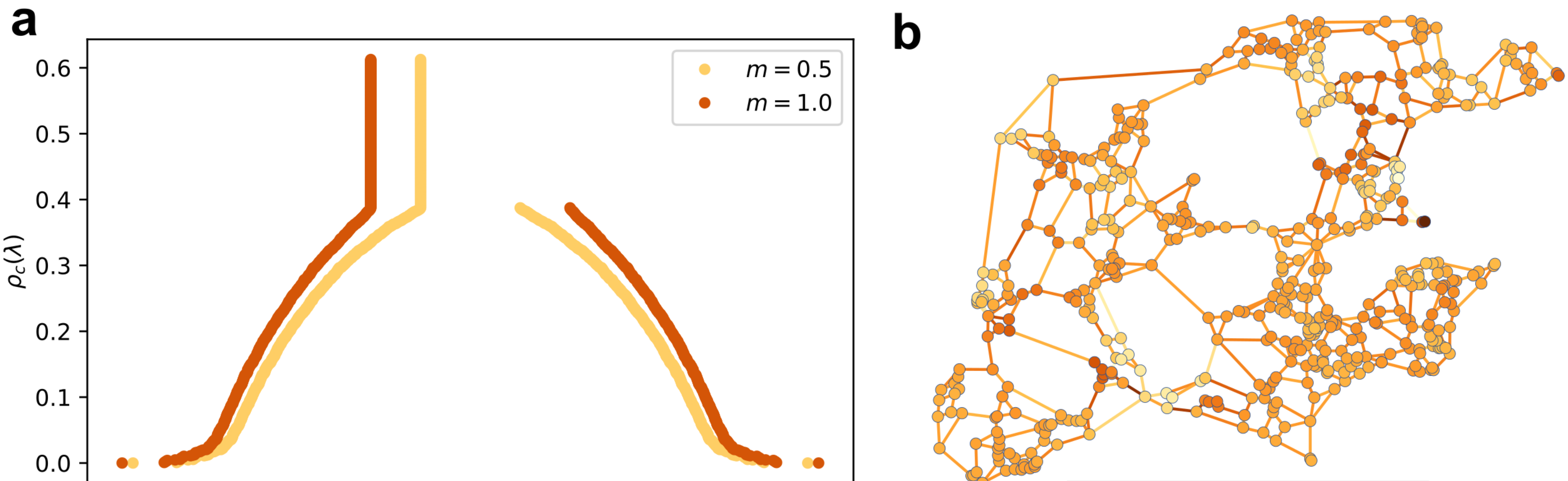


Topological synchronisation



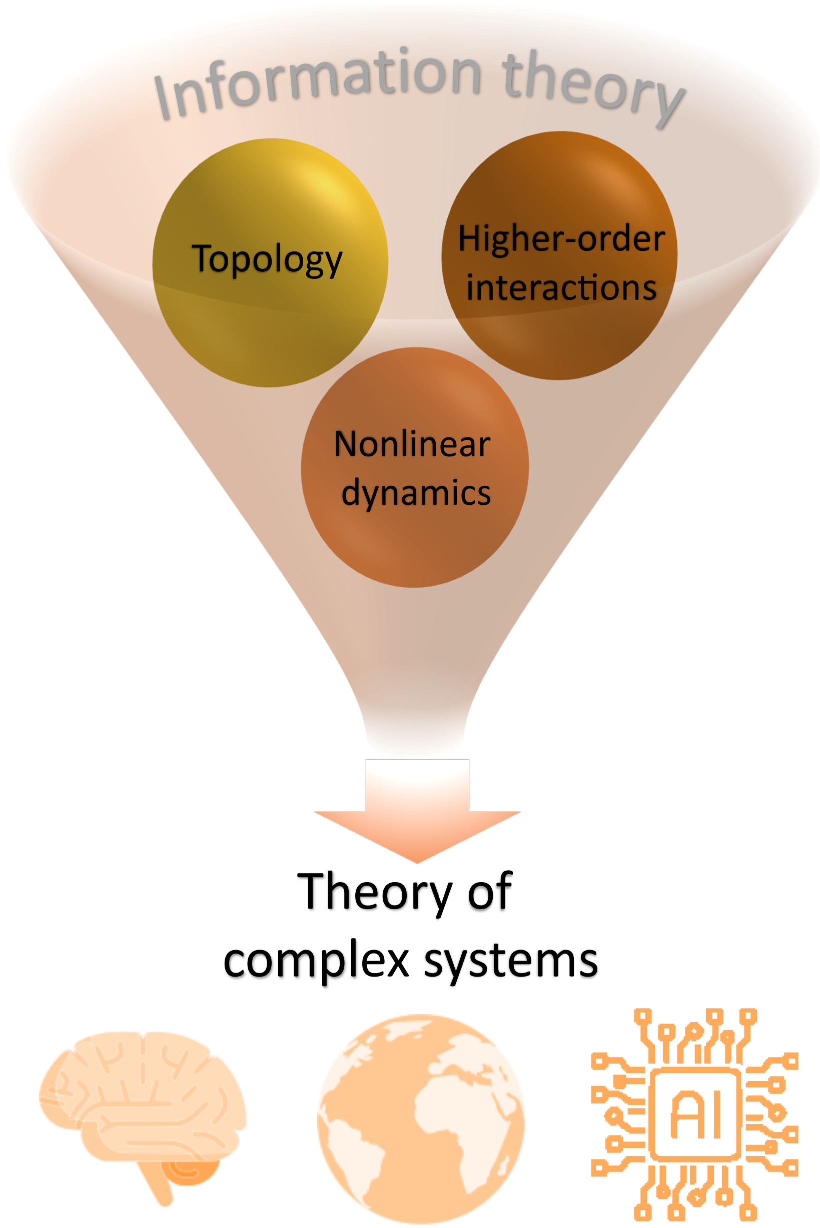
A. Millan, J. Torres and GB PRL 2020 T. Carletti, L. Giambagli and GB PRL 2023

Dirac equation spectrum and eigenstates



$$\mathrm{i}\partial_t \mathbf{\Psi} = (\mathbf{D} + \boldsymbol{\gamma}_0 m)\mathbf{\Psi}$$

G. Bianconi JPhys Complexity 2021



Complexity challenge