LTCC Examination 2023

Symmetry Methods for Differential Equations

1. *Show that the groups*

$$x^* = \frac{x}{1+\varepsilon x}, \qquad y^* = \frac{y}{1+\varepsilon y}, \tag{1a}$$

$$x^* = \frac{x(y+\varepsilon)}{y}, \qquad y^* = y+\varepsilon, \tag{1b}$$

are one-parameter groups of transformations.

Answer: For these transformations one has to show that

(i) If $\varepsilon = 0$ then $x^* = x$ and $t^* = t$, so $\varepsilon = 0$ is the identity transformation.

(ii) Show that $-\varepsilon$ is the inverse transformation.

(iii) Show that the product (composition) of two transformations is a transformation with parameter $\varepsilon + \delta$.

2. *Show that the Riccati equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x} + \frac{y^2}{x^3}, \tag{2}$$

is invariant under the projective group

$$x^* = \frac{x}{1-\varepsilon x}, \qquad y^* = \frac{y}{1-\varepsilon x}.$$

Find the associated invariant and hence solve equation (2)*.*

Answer:If $y(x)$ satisfies the Riccati equation (2) then $y^*(x^*)$ satisfies

$$\frac{\mathrm{d}y^*}{\mathrm{d}x^*} - \frac{y^*+1}{x^*} - \frac{(y^*)^2}{(x^*)^3}.$$

The invariant is y/x so letting $y(x) = xv(x)$ in the Riccati equation (2) gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v^2}{x} \qquad \Rightarrow \qquad v(x) = \tan\left(C - \frac{1}{x}\right).$$

3. *The infinitesimals for the classical Boussinesq system*

$$\begin{aligned} &u_t + uu_x + v_x = 0,\\ &v_t + (uv)_x + u_{xxx} = 0, \end{aligned} \tag{3}$$

are

$$\xi = \alpha x + \beta t + \gamma, \qquad \tau = 2\alpha t + \delta, \qquad \phi^{[u]} = \beta - \alpha u, \qquad \phi^{[v]} = -2\alpha v.$$

where α, β, γ and δ are arbitrary constants. Determine all the symmetry reductions and find the resulting ordinary differential equations. You do ***not*** *have to solve these ordinary differential equations.*

Answer: To determine the symmetry reductions, it is necessary to solve the equations

$$\frac{\mathrm{d}x}{\alpha x + \beta t + \gamma} = \frac{\mathrm{d}t}{2\alpha t + \delta} = \frac{\mathrm{d}u}{\beta - \alpha u} = \frac{\mathrm{d}v}{-2\alpha v}.$$

There are three cases to consider: (i), $\alpha \neq 0$; (ii), $\alpha = 0$ and $\beta \neq 0$; and (iii), $\alpha = \beta = 0$.

(i) If $\alpha \neq 0$ then set $\alpha = 1$ and $\gamma = \delta = 0$, without loss of generality. The symmetry reduction is

$$u(x,t) = \frac{U(z)}{\sqrt{t}} + \beta, \qquad v(z,t) = \frac{V(z)}{t}, \qquad z = \frac{x-\beta t}{\sqrt{t}},$$

where $U(z)$ and $V(z)$ satisfy the ODEs

$$\frac{\mathrm{d}^3U}{\mathrm{d}z^3} + \frac{\mathrm{d}U}{\mathrm{d}z}V + U\frac{\mathrm{d}V}{\mathrm{d}z} = \tfrac{1}{2}z\frac{\mathrm{d}V}{\mathrm{d}z} + V, \qquad \frac{\mathrm{d}V}{\mathrm{d}z} + U\frac{\mathrm{d}U}{\mathrm{d}z} = \tfrac{1}{2}z\frac{\mathrm{d}U}{\mathrm{d}z} + U.$$

(ii) If $\alpha = 0$ and $\beta \neq 0$ then set $\beta = 1$ and $\gamma = 0$, without loss of generality. The symmetry reduction is

$$u(x,t) = U(z) + \mu t, \qquad v(x,t) = V(z), \qquad z = x - \tfrac{1}{2}\mu t^2,$$

where $U(z)$ and $V(z)$ satisfy the ODEs

$$\frac{\mathrm{d}^3 U}{\mathrm{d}z^3} + \frac{\mathrm{d}U}{\mathrm{d}z}V + U\frac{\mathrm{d}V}{\mathrm{d}z} = 0, \qquad \frac{\mathrm{d}V}{\mathrm{d}z} + U\frac{\mathrm{d}U}{\mathrm{d}z} + \mu = 0.$$

(iii) If $\alpha = \beta = 0$ then we obtain the travelling wave reduction

$$u(x,t) = U(z), \qquad v(x,t) = V(z), \qquad z = x - ct,$$

where $U(z)$ and $V(z)$ satisfy the ODEs

$$\frac{\mathrm{d}^3 U}{\mathrm{d}z^3} + \frac{\mathrm{d}U}{\mathrm{d}z}V + U\frac{\mathrm{d}V}{\mathrm{d}z} = c\frac{\mathrm{d}V}{\mathrm{d}z}, \qquad \frac{\mathrm{d}V}{\mathrm{d}z} + U\frac{\mathrm{d}U}{\mathrm{d}z} = c\frac{\mathrm{d}U}{\mathrm{d}z}.$$

4. *Consider the modified Boussinesq equation*

$$u_{tt} + u_t u_{xx} - \tfrac{1}{2}u_x^2 u_{xx} + u_{xxxx} = 0. \tag{4}$$

Find the condition on the parameters μ and λ for

$$u(x,t) = w(z) + \mu x t, \qquad z = x + \lambda t^2,$$

to be a symmetry reduction of the modified Boussinesq equation (4) *and find, but not solve, the ordinary differential equation which $w(z)$ satisfies. Determine whether this is a classical or nonclassical reduction of the modified Boussinesq equation* (4).

Answer: If

$$u(x,t) = w(z) + \mu x t, \qquad z = x + \lambda t^2, \tag{5}$$

then to getan ordinary differential equation necessarily $\mu = 2\lambda$, which gives

$$\left\{2\lambda z - \tfrac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\right\}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2\lambda\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\mathrm{d}^4 w}{\mathrm{d}z^4} = 0.$$

The infinitesimals for the modified Boussinesq equation are

$$\xi = \alpha x + \beta, \qquad \tau = 2\alpha t + \gamma, \qquad \phi = \delta.$$

where α, β, γ and δ are arbitrary constants. These yield a scaling reduction (if $\alpha \neq 0$) and a travelling wave reduction (if $\alpha = 0$). Hence the reduction is a nonclassical reduction.