LTCC Examination 2023

Symmetry Methods for Differential Equations

1. Show that the groups

$$x^* = \frac{x}{1+\varepsilon x}, \qquad y^* = \frac{y}{1+\varepsilon y}, \tag{1a}$$

$$x^* = \frac{x(y+\varepsilon)}{y}, \qquad y^* = y+\varepsilon, \tag{1b}$$

are one-parameter groups of transformations.

2. Show that the Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x} + \frac{y^2}{x^3}, \tag{2}$$

is invariant under the projective group

$$x^* = \frac{x}{1-\varepsilon x}, \qquad y^* = \frac{y}{1-\varepsilon x}.$$

Find the associated invariant and hence solve equation (2).

3. The infinitesimals for the classical Boussinesq system

$$\begin{aligned} &u_t + uu_x + v_x = 0, \\ &v_t + (uv)_x + u_{xxx} = 0, \end{aligned} \tag{3}$$

are

$$\xi = \alpha x + \beta t + \gamma, \qquad \tau = 2\alpha t + \delta, \qquad \phi^{[u]} = \beta - \alpha u, \qquad \phi^{[v]} = -2\alpha v.$$

where α, β, γ and δ are arbitrary constants. Determine all the symmetry reductions and find the resulting ordinary differential equations. You do **not** have to solve these ordinary differential equations.

4. Consider the modified Boussinesq equation

$$u_{tt} + u_t u_{xx} - \tfrac{1}{2} u_x^2 u_{xx} + u_{xxxx} = 0. \tag{4}$$

Find the condition on the parameters μ and λ for

$$u(x,t) = w(z) + \mu x t, \qquad z = x + \lambda t^2,$$

to be a symmetry reduction of the modified Boussinesq equation (4) and find, but not solve, the ordinary differential equation which $w(z)$ satisfies. Determine whether this is a classical or nonclassical reduction of the modified Boussinesq equation (4).