

Chapter I. PROBABILITY BACKGROUND.**§1. MEASURE**

Measure-theoretic Probability Theory is based on *Measure Theory*. This originated with Henri LEBESGUE (1875-1941), in his 1902 thesis, '*Intégrale, longueur, aire*'. We begin with the simplest case.

Length.

The length $\mu(I)$ of an interval $I = (a, b), [a, b], [a, b)$ or $(a, b]$ should be $b - a$: $\mu(I) = b - a$. The length of the disjoint union $I = \bigcup_{r=1}^n I_r$ of intervals I_r should be the sum of their lengths:

$$\mu\left(\bigcup_{r=1}^n I_r\right) = \sum_{r=1}^n \mu(I_r) \quad (\text{finite additivity}).$$

Consider now an infinite sequence I_1, I_2, \dots (*ad infinitum*) of disjoint intervals. Letting $n \rightarrow \infty$ suggests that length should again be additive over disjoint intervals:

$$\mu\left(\bigcup_{r=1}^{\infty} I_r\right) = \sum_{r=1}^{\infty} \mu(I_r) \quad (\text{countable additivity}).$$

For I an interval, A a subset of length $\mu(A)$, the length of the complement $I \setminus A := I \cap A^c$ of A in I should be

$$\mu(I \setminus A) = \mu(I) - \mu(A) \quad (\text{complementation}).$$

If $A \subseteq B$ and B has length $\mu(B) = 0$, then A should have length 0 also:

$$A \subseteq B \ \& \ \mu(B) = 0 \ \Rightarrow \ \mu(A) = 0 \quad (\text{completeness}).$$

Let \mathcal{F} be the smallest class of sets $A \subset \mathbb{R}$ containing the intervals, closed under countable disjoint unions and complements, and complete (containing all subsets of sets of length 0 as sets of length 0). The above suggests – what Lebesgue showed – that length can be sensibly defined on the sets \mathcal{F} on the line, but on no others. There are others – but they are hard to construct (in technical language: the Axiom of Choice, or some variant of it such as Zorn's Lemma, is needed to demonstrate the existence of non-measurable sets – but all such proofs are highly non-constructive). So: some but not all subsets of

the line have a length. These are called the *Lebesgue-measurable sets*, and form the class \mathcal{F} described above; length, defined on \mathcal{F} is called *Lebesgue measure* μ (on the real line, \mathbb{R}).

Area.

The area of a rectangle $R = (a_1, b_1) \times (a_2, b_2)$ – with or without any of its perimeter included – should be $\mu(R) = (b_1 - a_1) \times (b_2 - a_2)$. The area of a finite or countably infinite union of disjoint rectangles should be the sum of their areas:

$$\mu \left(\bigcup_{n=1}^{\infty} R_n \right) = \sum_{n=1}^{\infty} \mu(R_n) \quad (\text{countable additivity}).$$

For R a rectangle and $A \subseteq R$ with area $\mu(A)$, its complement should have area

$$\mu(R \setminus A) = \mu(R) - \mu(A) \quad (\text{complementation}).$$

If $B \subseteq A$ and A has area 0, B should have area 0:

$$A \subseteq B \ \& \ \mu(B) = 0 \ \Rightarrow \ \mu(A) = 0 \quad (\text{completeness}).$$

Let \mathcal{F} be the smallest class of sets, containing the rectangles, closed under finite or countably infinite unions, closed under complements, and complete (containing all subsets of sets of area 0 as sets of area 0). Lebesgue showed that area can be sensibly defined on the sets in \mathcal{F} and no others. The sets $A \in \mathcal{F}$ are called the *Lebesgue-measurable sets* in the plane \mathbb{R}^2 ; area, defined on \mathcal{F} , is called *Lebesgue measure* in the plane. So: some but not all sets in the plane have an area.

Volume.

Similarly in 3-dimensional space \mathbb{R}^3 , starting with the volume of a cuboid $C = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ as

$$\mu(C) := (b_1 - a_1) \cdot (b_2 - a_2) \cdot (b_3 - a_3).$$

Euclidean Space.

Similarly in k -dimensional Euclidean space \mathbb{R}^k . We start with

$$\mu \left(\prod_{i=1}^k (a_i, b_i) \right) = \prod_{i=1}^k (b_i - a_i),$$

and obtain the class \mathcal{F} of *Lebesgue-measurable sets* in \mathbb{R}^k , and *Lebesgue measure* μ in \mathbb{R}^k .

Probability.

The unit cube $[0, 1]^k$ in \mathbb{R}^k has Lebesgue measure 1. It can be used to model the *uniform distribution* (density $f(x) = 1$ if $x \in [0, 1]^k$, 0 otherwise), with probability = length/area/volume if $k = 1/2/3$.

Note. If a property holds everywhere except on a set of measure zero, we say it holds *almost everywhere* (a.e.) [French: *presque partout*, p.p.; German: *fast überall*, f.u.]. If it holds everywhere except on a set of probability zero, we say it holds *almost surely* (a.s.) [or, with probability one].

§2 INTEGRAL.

1. Indicators.

We start in dimension $k = 1$ for simplicity, and consider the simplest calculus formula $\int_a^b 1 \, dx = b - a$. We rewrite this as

$$I(f) := \int_{-\infty}^{\infty} f(x) \, dx = b - a \quad \text{if } f(x) = I_{[a,b]}(x),$$

the *indicator* function of $[a, b]$ (1 in $[a, b]$, 0 outside it), and similarly for the other three choices about end-points.

2. Simple functions.

A function f is called *simple* if it is a finite linear combination of indicators: $f = \sum_{i=1}^n c_i f_i$ for constants c_i and indicator functions f_i of intervals I_i . One then extends the definition of the integral from indicator functions to simple functions by linearity:

$$I\left(\sum_{i=1}^n c_i f_i\right) := \sum_{i=1}^n c_i I(f_i)$$

for constants c_i and indicators f_i of intervals I_i .

3. Non-negative measurable functions.

Call f a (*Lebesgue-*) *measurable function* if, for all c , the sets $\{x : f(x) \leq c\}$ are Lebesgue-measurable (§1). If f is a non-negative measurable function, we quote that it is possible to construct f as the increasing limit of a sequence of simple functions f_n :

$$f_n(x) \uparrow f(x) \quad \text{for all } x \in \mathbb{R} \quad (n \rightarrow \infty), \quad f_n \text{ simple.}$$

We then define the integral of f as

$$I(f) := \lim_{n \rightarrow \infty} I(f_n) \quad (\leq \infty)$$

(we quote that this does indeed define $I(f)$: the value does not depend on *which* approximating sequence (f_n) we use). Since f_n increases in n , so does $I(f_n)$ (the integral is *order-preserving*), so either $I(f_n)$ increases to a finite limit, or diverges to ∞ . In the first case, we say f is (*Lebesgue-*) *integrable* with (*Lebesgue-*) *integral* $I(f) = \lim I(f_n)$, or $\int f(x) dx = \lim \int f_n(x) dx$, or simply $\int f = \lim \int f_n$.

4. Measurable functions.

If f is a measurable function that may change sign, we split it into its positive and negative parts, f_{\pm} :

$$f_+(x) := \max(f(x), 0), \quad f_-(x) := -\min(f(x), 0),$$

$$f(x) = f_+(x) - f_-(x), \quad |f(x)| = f_+(x) + f_-(x)$$

If both f_+ and f_- are integrable, we say that f is too, and define

$$\int f := \int f_+ - \int f_-.$$

Thus, in particular, f is also integrable, and

$$\int |f| = \int f_+ + \int f_-.$$

Note. The Lebesgue integral thus defined is, by construction, an *absolute integral*: f is integrable iff $|f|$ is integrable. Thus, for instance,

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

has no meaning for Lebesgue integrals, since $\int_1^{\infty} \frac{|\sin x|}{x} dx$ diverges to $+\infty$ like $\int_1^{\infty} \frac{1}{x} dx$. It has to be replaced by the limit relation

$$\int_0^X \frac{\sin x}{x} dx \rightarrow \frac{\pi}{2} \quad (X \rightarrow \infty).$$

The class of (*Lebesgue-*) integrable functions f on \mathbb{R} is written $L(\mathbb{R})$ or (for reasons explained below) $L_1(\mathbb{R})$ – abbreviated to L_1 or L .

Higher dimensions.

In \mathbb{R}^k , we start instead from the k -dimensional boxes in place of intervals. If f is the indicator of a box $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$,

$$\int f := \prod_{i=1}^k (b_i - a_i).$$

We then extend to simple functions (linear combinations of indicators of boxes) by linearity, to non-negative measurable functions by taking increasing limits, and to measurable functions by taking positive and negative parts.

L_p spaces.

For $p \geq 1$, the L_p spaces $L_p(\mathbb{R}^k)$ on \mathbb{R}^k are the spaces of measurable functions f with L_p -norm

$$\|f\|_p := \left(\int |f|^p \right)^{\frac{1}{p}} < \infty.$$

Riemann integrals.

We first meet the ‘Sixth-Form integral’, taught non-rigorously at school. Mathematics undergraduates are taught a rigorous integral (in their 1st or 2nd years), the *Riemann integral* [G.B. RIEMANN (1826-1866)] – essentially just a rigourization of the school integral. It is much easier to set up than the Lebesgue integral, but much harder to handle.

For finite intervals $[a, b]$, we quote:

- (i) for any function f Riemann-integrable on $[a, b]$, it is Lebesgue-integrable to the same value (but many more functions are Lebesgue integrable),
- (ii) f is Riemann-integrable on $[a, b]$ iff it is continuous a.e. on $[a, b]$. Thus the question, “Which functions are Riemann-integrable?” cannot be answered without the language of measure theory – which then gives one the technically superior Lebesgue integral anyway.

Note. Integration is like summation (which is why Leibniz gave us the integral sign \int , as an elongated S). Lebesgue was a very practical man – his father was a tradesman – and used to think about integration in the following way. Think of a shopkeeper totalling up his day’s takings. The Riemann integral is like adding up the takings – notes and coins – *in the order in which they arrived*. By contrast, the Lebesgue integral is like totalling up the takings *in order of size* – from the smallest coins up to the largest notes. This is obviously better! In mathematical effect, it exchanges ‘integrating by x -values’ (abscissae) with ‘integrating by y -values (ordinates).

Lebesgue-Stieltjes integral.

Suppose that $F(x)$ is a *non-decreasing* function on \mathbb{R} :

$$F(x) \leq F(y) \quad \text{if } x \leq y$$

(prime example: F a probability distribution function). Such functions can have at most countably many discontinuities, which are at worst jumps. We may without loss re-define F at jumps so as to be *right-continuous*.

We now generalize the starting points above:

(i) *Measure*. We take $\mu((a, b]) := F(b) - F(a)$.

(ii) *Integral*. We take $\int_a^b 1 := F(b) - F(a)$.

Now follow through the successive extension procedures above. We obtain:

(i) *Lebesgue-Stieltjes measure* μ , or μ_F ,

(ii) *Lebesgue-Stieltjes integral* $\int f d\mu$, or $\int f d\mu_F$, or even $\int f dF$.

Similarly in higher dimensions; we omit further details.

Finite variation (FV).

If instead of being monotone non-decreasing, F is the *difference* of two such functions, $F = F_1 - F_2$, define $\int f dF_1$, $\int f dF_2$ as above, and then

$$\int f dF = \int f d(F_1 - F_2) := \int f dF_1 - \int f dF_2.$$

If $[a, b]$ is a finite interval and F is defined on $[a, b]$, a finite collection of points, x_0, x_1, \dots, x_n with $a = x_0 < x_1 < \dots < x_n = b$, is called a *partition* of $[a, b]$, \mathcal{P} say. The sum $\sum_{i=1}^n |F(x_i) - F(x_{i-1})|$ is called the *variation* of F over the partition. The least upper bound of this over all partitions \mathcal{P} is called the *variation* of F over the interval $[a, b]$, $V_a^b(F)$:

$$V_a^b(F) := \sup_{\mathcal{P}} \sum |F(x_i) - F(x_{i-1})|.$$

This may be $+\infty$; but if $V_a^b(F) < \infty$, F is said to be of *finite variation* on $[a, b]$, $F \in FV_a^b$. If F is of bounded variation on all finite intervals, F is said to be *locally of finite variation*, $F \in FV_{loc}$; if F is of finite variation on the real line \mathbb{R} , F is of *finite variation*, $F \in FV$.

We quote (*Jordan's theorem*) that the following are equivalent:

(i) F is locally of finite variation,

(ii) F is the difference $F = F_1 - F_2$ of two monotone functions.

So the above procedure defines the integral $\int f dF$ when the *integrator* F is of *finite variation*.

§3 PROBABILITY. Probability spaces.

The mathematical theory of probability can be traced to 1654, to correspondence between PASCAL (1623-1662) and FERMAT (1601-1665). However, the theory remained both incomplete and non-rigorous till the 20th century. It turns out that the Lebesgue theory of measure and integral sketched above is exactly the machinery needed to construct a rigorous theory of probability adequate for modelling reality (option pricing, etc.) for us. This was realised by the great Russian mathematician and probabilist A.N.KOLMOGOROV (1903-1987), whose classic book of 1933, *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Foundations of probability theory) [Kol] inaugurated the modern era in probability.

Recall from your first course on probability that, to describe a random experiment mathematically, we begin with the *sample space* Ω , the set of all possible outcomes. Each point ω of Ω , or *sample point*, represents a possible – random – outcome of performing the random experiment. For a set $A \subseteq \Omega$ of points ω we want to know the probability $P(A)$ (or $\text{pr}(A)$). We need

1. $P(\emptyset) = 0$, $P(\Omega) = 1$,
2. $P(A) \geq 0$ for all A ,
3. If A_1, A_2, \dots, A_n are disjoint, $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$ (finite additivity), which, as above we will strengthen to
- 3*. If $A_1, A_2 \dots$ (*ad inf.*) are disjoint,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad (\text{countable additivity}).$$

4. If $B \subseteq A$ and $P(A) = 0$, then $P(B) = 0$ (completeness).

Then by 1 and 3 (with $A = A_1$, $\Omega \setminus A = A_2$),

$$P(A^c) = P(\Omega \setminus A) = 1 - P(A).$$

So the class \mathcal{F} of subsets of Ω whose probabilities $P(A)$ are defined should be closed under countable, disjoint unions and complements, and contain the empty set \emptyset and the whole space Ω . Such a class is called a σ -*field* of subsets of Ω [or sometimes a σ -algebra, which one would write \mathcal{A}]. For each $A \in \mathcal{F}$, $P(A)$ should be defined (and satisfy 1, 2, 3*, 4 above). So, $P : \mathcal{F} \rightarrow [0, 1]$ is a set-function,

$$P : A \mapsto P(A) \in [0, 1] \quad (A \in \mathcal{F}).$$

The sets $A \in \mathcal{F}$ are called *events*. Finally, 4 says that all subsets of null-sets (events) with probability zero (we will call the empty set \emptyset empty, not null)

should be null-sets (completeness). A *probability space*, or *Kolmogorov triple*, is a triple (Ω, \mathcal{F}, P) satisfying these *Kolmogorov axioms* 1,2,3*,4 above. A probability space is a mathematical model of a random experiment.

Random variables.

Next, recall random variables X from your first probability course. Given a random outcome ω , you can calculate the value $X(\omega)$ of X (a scalar – a real number, say; similarly for vector-valued random variables, or random vectors). So, X is a function from Ω to \mathbb{R} , $X : \omega \rightarrow X(\omega)$ ($\omega \in \Omega$). Recall also that the *distribution function* of X is defined by

$$F(x), \quad \text{or} \quad F_X(x), \quad := P\left(\{\omega : X(\omega) \leq x\}\right), \quad \text{or} \quad P(X \leq x), \quad (x \in \mathbb{R}).$$

We can only deal with functions X for which all these probabilities are defined. So, for each x , we need $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$. We summarize this by saying that X is *measurable* with respect to the σ -field \mathcal{F} (of events), briefly, X is \mathcal{F} -*measurable*. Then, X is called a *random variable* [non- \mathcal{F} -measurable X cannot be handled, and so are left out]. So,

- (i) a random variable X is an \mathcal{F} -measurable function on Ω ,
- (ii) a function on Ω is a random variable (is measurable) iff its distribution function is defined.

Generated σ -fields.

The smallest σ -field containing all the sets $\{\omega : X(\omega) \leq x\}$ for all real x [equivalently, $\{X < x\}$, $\{X \geq x\}$, $\{X > X\}$] is called the σ -field *generated* by X , written $\sigma(X)$. Thus,

$$X \text{ is } \mathcal{F}\text{-measurable [is a random variable] iff } \sigma(X) \subseteq \mathcal{F}.$$

When the (random) value $X(\omega)$ is *known*, we know *which* of the events in the σ -field generated by X have happened: these are the events $\{\omega : X(\omega) \in B\}$, where B runs through the Borel σ -field [generated by the intervals] on \mathbb{R} .

Interpretation. Think of $\sigma(X)$ as representing *what we know when we know X* , or in other words *the information contained in X* (or in knowledge of X). This is reflected in the following result, due to J. L. DOOB (1910-2004), which we quote:

$$\sigma(X) \subseteq \sigma(Y) \quad \text{iff} \quad X = g(Y)$$

for some measurable function g . For, knowing Y means we know $X := g(Y)$ – but not vice-versa, unless the function g is one-to-one [injective], when the inverse function g^{-1} exists, and we can go back via $Y = g^{-1}(X)$.

Expectation.

A measure (§1) determines an integral (§2). A probability measure P , being a special kind of measure [a measure of total mass one] determines a special kind of integral, called an *expectation*.

Definition.

The *expectation* E of a random variable X on (Ω, \mathcal{F}, P) is defined by

$$EX := \int_{\Omega} X dP, \text{ or } \int_{\Omega} X(\omega) dP(\omega).$$

If X is real-valued, say, with distribution function F , recall that EX is defined in your first course on probability by

$$EX := \int x f(x) dx \text{ if } X \text{ has a density } f$$

or if X is discrete, taking values X_n , ($n = 1, 2, \dots$) with probability function $f(x_n) (\geq 0)$, ($\sum x_n f(x_n) = 1$),

$$EX := \sum x_n f(x_n).$$

These two formulae are the special cases (for the density and discrete cases) of the general formula

$$EX := \int_{-\infty}^{\infty} x dF(x)$$

where the integral on the right is a Lebesgue-Stieltjes integral. This in turn agrees with the definition above, since if F is the distribution function of X ,

$$\int_{\Omega} X dP = \int_{-\infty}^{\infty} x dF(x)$$

follows by the *change of variable formula* for the measure-theoretic integral, on applying the map $X : \Omega \rightarrow \mathbb{R}$ (we quote this).

Glossary.

We now have two parallel languages, measure-theoretic and probabilistic:

Measure	Probability
Integral	Expectation
Measurable set	Event
Measurable function	Random variable
almost-everywhere (a.e.)	almost-surely (a.s.).

§4. Equivalent measures and Radon-Nikodym derivatives.

Given two measures P and Q defined on the same σ -field \mathbf{F} , we say that P is *absolutely continuous* with respect to Q , written

$$P \ll Q,$$

if $P(A) = 0$ whenever $Q(A) = 0$, $A \in \mathcal{F}$. We quote from measure theory the vitally important *Radon-Nikodym theorem*: $P \ll Q$ iff there exists a (\mathcal{F} -) measurable function f such that

$$P(A) = \int_A f dQ \quad \forall A \in \mathcal{F}$$

(note that since the integral of anything over a null set is zero, any P so representable is certainly absolutely continuous with respect to Q – the point is that the converse holds). Since $P(A) = \int_A dP$, this says that $\int_A dP = \int_A f dQ$ for all $A \in \mathcal{F}$. By analogy with the chain rule of ordinary calculus, we write dP/dQ for f ; then

$$\int_A dP = \int_A \frac{dP}{dQ} dQ \quad \forall A \in \mathcal{F} : \quad \text{if } P \ll Q, \quad dP = \frac{dP}{dQ} dQ.$$

The measurable function [random variable] dP/dQ is called the *Radon-Nikodym derivative* [RN-derivative] of P with respect to Q .

If $P \ll Q$ and also $Q \ll P$, we call P and Q *equivalent* measures, written $P \sim Q$. Then dP/dQ and dQ/dP both exist, and $dP/dQ = 1/(dQ/dP)$.

For $P \sim Q$, $P(A) = 0$ iff $Q(A) = 0$: P and Q *have the same null sets*. Taking negations: $P \sim Q$ iff P, Q have the same sets of positive measure. Taking complements: $P \sim Q$ iff P, Q have the same sets of probability one [the same a.s. sets]. Thus the following are equivalent: $P \sim Q$ iff P, Q have the same null sets/the same a.s. sets/the same sets of positive measure.

Note. Far from being an abstract theoretical result, the Radon-Nikodym theorem is of key practical importance, in two ways:

- (a) It is the key to *conditioning*, which is of central importance.
- (b) The concept of equivalent measures is central to the key idea of mathematical finance, *risk-neutrality*, and hence to its main results, the *Black-Scholes formula*, the *Fundamental Theorem of Asset Pricing*, etc. The key to all this is that prices should be the *discounted expected values under the equivalent martingale measure*. Thus equivalent measures, and the operation of *change of measure*, are of central economic and financial importance. Mathematically, change of measure is effected by *Girsanov's theorem*.