Modular Group and Modular Forms: Exercises 3, 5 Dec. 2008

These problems are on various aspects of lattices, automorphic forms and functions.

1. The *space of unmarked lattices* in $\mathbb{C}$ is denoted $\mathcal{L}$, topologised using the *geometric topology*: for a nbd of $\Lambda = \langle l_1, l_2 \rangle$, we use compact sets $K \subset \mathbb{C}$ and $\epsilon > 0$ to determine a subset of $\mathcal{L}$ as follows. For subsets S of $\mathbb{C}$, write $S^\epsilon = \{z : D(z, S) < \epsilon\}$, and then define

$$N(\Lambda) = \{L : L \cup K \subset \Lambda^\epsilon,\ \Lambda \subset L^\epsilon \|.$$

Verify that this provides a nbd base for a topology and that $\mathcal{L}$ is Hausdorff.

Define a map from $\mathcal{L}$ to $\mathbb{C} \setminus \mathbb{R}$ by the rule

$$\omega : \text{Ł} \mapsto \tau = l_1/l_2$$

where the generators of L are chosen using the rule given in lectures. Show that this mapping is continuous. What is the fibre over i of the map $\mathbf{p}$?

2. Prove that the Eisenstein series

$$E_k(\tau) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus(0,0)} (m + n\tau)^{-k}$$

converges absolutely and uniformly for τ in any compact set in $\mathcal{U}$ when $k \geq 3$. [Note: it is sufficient to consider compact subsets of $\mathcal{D}$: estimate the maximum value of $|m + n\tau|^2$ for $\tau \in \mathcal{D}$.]

Define the corresponding functions of lattices $\widetilde{E}_k$ by

$$\widetilde{E}_k(w_1, w_2) = w_2^{-k} E_k(w_1/w_2) = \sum_{w\in L(w_1,w_2)^*} w^{-k}.$$

Show that this function satisfies $\widetilde{E}_k(w_1, w_2) = E_k \circ \pi(L)$, and that $\widetilde{E}_k$ is invariant under the natural linear action of $SL(2, \mathbb{Z})$ on the lattice generators. Deduce that the function $E_k(\tau)$ is invariant under composition with the translation $T(\tau) = \tau + 1$, and of course with all powers of T in $SL(2, \mathbb{Z})$.

For τ restricted to the fundamental set $\mathcal{D}$, find the limit as $\Im(\tau) \to \infty$ of $E_k(\tau)$.

3. Write out the Laurent series expansion of the function $\wp(z, \Lambda)$ in (even) powers of z and use it to calculate the Laurent series of the Weierstrass function $\wp'(z)$ and check the validity of the differential equation

$$\wp'(z, \Lambda)^2 = 4\wp^3 - g_2(\Lambda)\wp - g_3(\Lambda).$$

Use the differential equation to show that each coefficient in the expansion of $\wp$ is a polynomial in the two coefficients c_2 and c_4.

4. Show that the lattice $L(\rho) = \mathbb{Z} + \rho\mathbb{Z}$, where $\rho = e^{\pi i/3} \in \partial\mathcal{D}$, is preserved by the plane rotation through angle $\pi/3$ and deduce that the torus $X(\rho) = \mathbb{C}/L(\rho)$ has an order 6 conformal automorphism. Verify that the Eisenstein series $E_6(\tau)$ vanishes at $\tau = \rho$.

Work out the corresponding facts for the square lattice $L(i)$.

5. Let f be an elliptic function for a lattice Λ. By integrating the 1-form $d \log f(z)$ around a fundamental polygon tile Π, show that f has an equal number of zeros and poles on the torus $X = \mathbb{C}/\Lambda$. Deduce that f takes on every value the same number of times on X, known as the *degree of* f. Use a similar argument to see that the degree of f cannot be 1.

Weekly course summary:

Modular group/ automorphic forms (Bill Harvey, KCL).

Week 1. The upper half plane, Moebius maps and hyperbolic plane geometry.

Week 2. Action of SL(2,Z) as hyperbolic isometries.

Summary notes for these weeks are posted on the LTCC website.

Week 3. Lattices, elliptic functions and Eisenstein series.

Notes on this are in preparation.

Week 4. More on Eisenstein series and the discriminant form. Automorphic forms in general; projective embedding of modular quotient spaces.

Week 5. Related topics, chosen from quadratic forms & theta functions, discussion of higher dimensional modular varieties and the Grothendieck-Belyi theory of arithmetically defined algebraic curves.