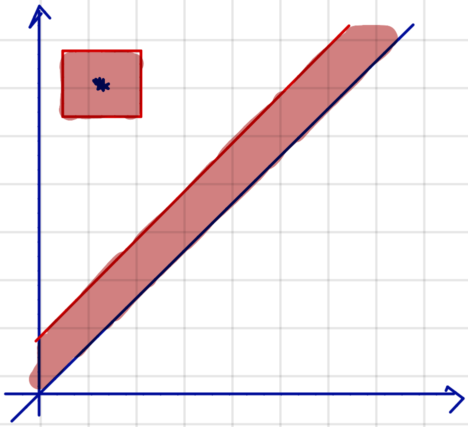
Topology of Random Spaces

What does stability tell us? There cannot be points outside of the shaded regions



What happens inside the shaded region?

Depends on the model

- Erdős-Rényi / Lineal-Meshulam
- Geometric - Poisson / Poisson-Boolean

Erdős-Renyi graph - given n points consider all possible $\frac{n(n-1)}{2}$ pairs/edges. Insert an edge with probability p. Each edge is independent of all others. To get higher dimensional simplices, make a clique complex (fill in all possible simplices that the edges allow)

k-simplices $\longleftrightarrow$ $(k+1)$-cliques (complete graph on $k+1$ vertices)

Linial - Meshulam - Again fix n vertices. Put in all possible $k-1$ simplices. (Complete $k-1$ skeleton). Put in each k simplex with prob p.

Geometric Models

Poisson points on manifold or $\mathbb{R}^d$

Three regimes
- subcritical $nr^d \to 0$ as $n \to \infty$
- critical - $nr^d \to c$ as $n \to \infty$
- supercritical $nr^d \to \infty$ as $n \to \infty$

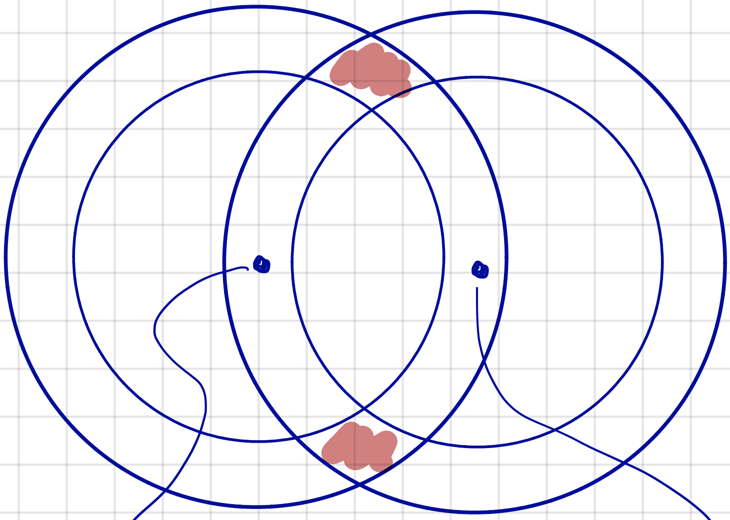
Notice that coverage occurs at $\left(\frac{\log n}{n}\right)^{1/d}$ so it is in the supercritical regime since

$$n \cdot \frac{\log n}{n} \to \infty \quad \text{as} \quad n \to \infty$$

In the sparse or subcritical regime

$\mathbb{P}(k\text{-cycle}) \approx$ probability of a particular configuration

Ex: 1-cycle



fix first point

second point needs to in annulus

third point in one of two regions marked by ●

In the subcritical regime cycles are far apart - so independent

Critical regime most difficult - well connected so no independence

Supercritical regime - uncovered region far apart, so those are indep.

Survey: Bobrowski-Kahle - Topology of Geometric Random Complexes

Sample of results

* When β_k is non-trivial (ie $\beta_k > 0$

* Central limit theorem

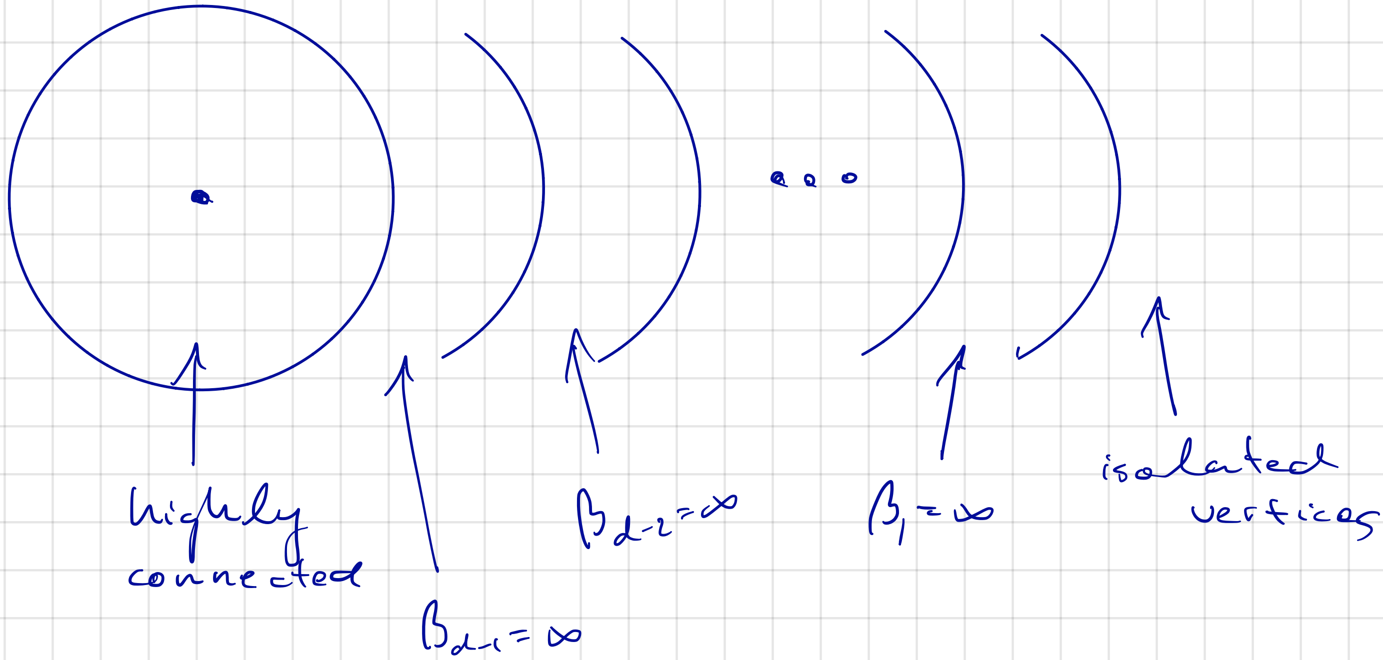
  Appropriately scaled $\beta_k \to N(0,1)$

* Crackle - Adler - Bobrowski : Crackle: the Topology of Noise

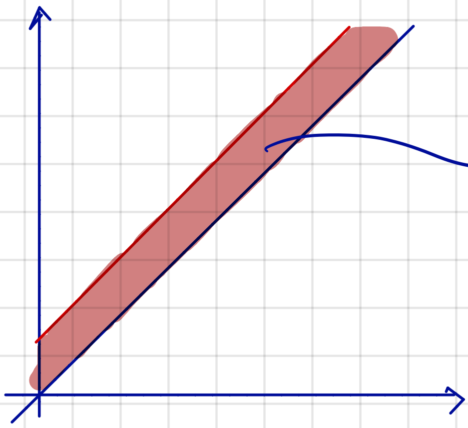
  - Consider a point at the origin and a distribution around the point

  - Gaussian - Niyogi - Smale - Weinberger
    only a few outliers - can be removed by considering connectivity (outliers are not highly connected)

  - Exponential / Power Law - all sorts of homology appears in shells



<u>CLT</u>:

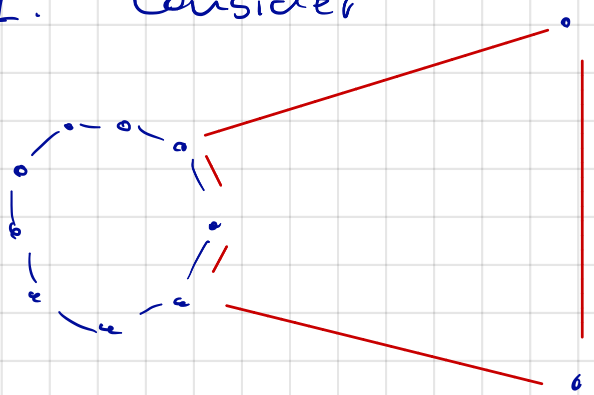


the central limit theorem holds near the diagonal.

<u>Maximal Persistence</u>: Bobrowski – Kahle – Skraba

Look at death/birth rather than death − birth

<u>Why</u>? Consider



death − birth: red cycle more persistent

$\frac{\text{death}}{\text{birth}}$: blue cycle more persistent.

Result:

$$\rho = \frac{\text{death}}{\text{birth}}$$

Poisson points in a convex region

$$\max \rho \asymp \left(\frac{\log n}{\log \log n} \right)^{1/i}$$

homological dimension

of the order of

Multiparameter Persistence

Say we have two functions $f, g : X \to \infty$

Define $X_{\alpha, \beta} = f^{-1}(-\infty, \alpha] \cap g^{-1}(-\infty, \beta]$

There is a bifiltration, let $\alpha < \alpha'$ & $\beta < \beta'$

$$\begin{tikzcd}
X_{\alpha,\beta'} \arrow[r] & X_{\alpha',\beta'} \\
X_{\alpha,\beta} \arrow[r] \arrow[u] & X_{\alpha',\beta} \arrow[u]
\end{tikzcd}$$

Interleaving as a concept works perfectly well, but there is no "barcode." Decompositions exist but are much more complicated.