Measure Theory:

1. Let $(X, \mathcal{B}, \mu)$ be a probability space and $A_1, A_2, \ldots$ be a sequence of measurable sets in $\mathcal{B}$ such that $\mu(A_i) \leq \frac{1}{i^2}$ for every i. Let A be the subset $\{x \mid x \in A_i \text{ for infinitely many } i\}$.

(a) Show that $\mu(A) = 0$.

(b) Given an example of where $\mu(A_i) \leq \frac{1}{i}$ however $\mu(A) > 0$ with A so defined as above.

2. For every $i = 1, 2, \ldots$ and positive integer $n < 2^i$ define $A_{n,i} \subseteq [0, 1]$ by $A_{n,i} = \{r \mid \frac{n}{2^i} \leq r < \frac{n+1}{2^i}\}$.

(a) What is the smallest algebra containing all the $A_{n,i}$?

(b) What is the smallest sigma algebra containing all the $A_{n,i}$?

(c) Given an example of two different algebras $\mathcal{A}$ and $\mathcal{B}$ such that the generated sigma algebras $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are the same.

3. Let A be a subset of the real numbers such that the Lebesgue measure $\lambda^*(A \cap (a, b)) = \frac{1}{2}(b - a)$ for all real numbers $b > a$. Show that A is not a Lebesgue measurable set.

4. Let f be a continuous and non-decreasing real valued function defined on the real numbers. Integration means Lebesgue integration.

(a) Show that $\lim_{n\to\infty} \inf n(f(x+\frac{1}{n}) - f(x-\frac{1}{n}))$ is a measurable function on x.

(b) Show that for any $a < b$ it follows that

$$\int_a^b \lim_{n\to\infty} \inf n\big(f(x+\frac{1}{n}) - f(x-\frac{1}{n})\big) \leq \lim_{n\to\infty} \inf \int_a^b n\big(f(x+\frac{1}{n}) - f(x-\frac{1}{n})\big).$$

(c) Given an example of such a function f such that

$$\int_a^b \lim_{n\to\infty} \inf n\big(f(x+\frac{1}{n}) - f(x-\frac{1}{n})\big) < \lim_{n\to\infty} \inf \int_a^b n\big(f(x+\frac{1}{n}) - f(x-\frac{1}{n})\big).$$

(d) Show that if f is non-decreasing, differentiable everywhere, and with some bound $M > 0$ such that $n\big(f(x + \frac{1}{n}) - f(x - \frac{1}{n})\big) \leq M$

for all x and n, then the derivative f' is a measurable function and the fundamental theorem of calculus holds even if f' is not necessarily continuous, namely that $\int_a^b f'(x) = f(b) - f(a)$.

5. Let f, g be Borel measurable real valued functions.

(a) Show that the composition $f \circ g$ is a Borel measurable function.

(b) Given an example of real valued functions f, g such that $f \circ g$ is a Borel measurable function however neither f nor g are Borel measurable.

6. Show that there is a closed subset C of $[0, 1]$ of positive Lebesgue measure that contains no open subset of $[0, 1]$.

7. Let X be a countable set and μ a finitely additive measure on an algebra $\mathcal{A}$ such that for every $i = 1, 2, \ldots$ there are two sets A_i and B_i in $\mathcal{A}$ such that $X = A_i \cup B_i$ and $A_i \cap B_i = \emptyset$, $\mu(A_i) = \mu(B_i) = \frac{1}{2}$, and for every disjoint finite sets U, V of the integers with $U \cap V = \emptyset$ the set $\cap_{i \in U} A_i \cap_{j \in V} B_j$ is not empty with $\mu(\cap_{i \in U} A_i \cap_{j \in V} B_j) = 2^{|U|+|V|}$.

(a) Show that μ cannot be extended to a sigma-additive measure on X.

(b) Show that such a set X, algebra $\mathcal{A}$ and finitely additive measure μ do exist.