Measure Theory   First Week

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We follow the book by Donald Cohn, Measure Theory.

Algebras and Measures:

A collection $\mathcal{A}$ of subsets of X is called an *algebra* if

(a) all of X is a member of $\mathcal{A}$,

(b) for all $A \in \mathcal{A}$, the set $X \backslash A$ is in $\mathcal{A}$,

(c) for every $A_1, \ldots, A_n \in \mathcal{A}$, $\cup_{i=1}^n A_i \in \mathcal{A}$,

(d) for every $A_1, \ldots, A_n \in \mathcal{A}$, $\cap_{i=1}^n A_i \in \mathcal{A}$.

The condition (d) is superfluous,

as $\cap_{i=1}^n A_i = X \backslash (\cup_{i=1}^n (X \backslash A_i)$.

The empty set $\emptyset$ is always in the collection.

The collection $\mathcal{A}$ is called a sigma-algebra (or σ-algebra)

if (a) and (b) hold and additionally

for every infinite sequence $A_1, A_2, \ldots$ of sets in $\mathcal{A}$:

$\cup_{i=1}^{\infty} A_i \in \mathcal{A}$ and $\cap_{i=1}^{\infty} A_i \in \mathcal{A}$.

In general we work with sigma-algebras.

A set in the sigma-algebra $\mathcal{A}$ is called a $\mathcal{A}$-*measurable* set.

A space X with a sigma algebra $\mathcal{A}$ is expressed as $(X, \mathcal{A})$ and is called a measurable space.

Examples:

(a) $\mathcal{A}$ is the collection of all subsets of some set X. – sigma-algebra

(b) $\mathcal{A} = \{X, \emptyset\}$ – sigma-algebra

(c) X is infinite and $\mathcal{A}$ is the collection of sets A such that A is finite or $X \backslash A$ is finite.

– algebra, but not a sigma-algebra.

(d) X is infinite and $\mathcal{A}$ is the collection of sets A such that A is finite or countable or $X \backslash A$ is finite or countable.

– sigma-algebra.

(e) $X = \mathbf{R}$ and $\mathcal{A}$ is the collection of all intervals of $\mathbf{R}$.

– not an algebra

(f) $X = \mathbf{R}$ and $\mathcal{A}$ is the collection of all finite unions of intervals of $\mathcal{R}$.

– algebra, but not sigma-algebra.

If $\mathcal{A}$ were a sigma-algebra,

all points of $\mathbf{R}$ would be in $\mathcal{A}$,

and so the rational numbers $\mathbf{Q}$ should be in $\mathcal{A}$.

Given a collection of sets $\mathcal{A}$ that may not be a sigma-algebra,

we want to way to construct a collection $\sigma(\mathcal{A})$

that is a sigma-algebra

and is the sigma-algebra that is generated by $\mathcal{A}$–

meaning that all sets which must be in the sigma algebra are there and all that need not be there are not there.

We mean that it is the smallest sigma-algebra containing $\mathcal{A}$.

Lemma: Let X be a set and let $(\mathcal{A}_i \mid i \in I)$ be an arbitrary collection of sigma algebras.

The collection $\mathcal{A} := \{A \mid \forall i \in I \quad A \in \mathcal{A}_i\}$ is a sigma-algebra.

Proof:

(a) $X \in \mathcal{A}_i$ for every $i \in I \Rightarrow X \in \mathcal{A}$.

(b) $A \in \mathcal{A} \Rightarrow \forall\, i \in I \ \ A \in \mathcal{A}_i$

$\Rightarrow \forall\, i \in I \ \ X \setminus A \in \mathcal{A}_i \Rightarrow X \setminus A \in \mathcal{A}$.

(c) $A_1, \cdots \in \mathcal{A} \Rightarrow \forall\, i \in I \ \ A_1, \cdots \in \mathcal{A}_i$

$\Rightarrow \forall\, i \in I \quad \cup_{j=1}^{\infty} A_j \in \mathcal{A}_i \Rightarrow \cup_{j=1}^{\infty} A_j \in \mathcal{A}$.
$\square$

Lemma: For any collection $\mathcal{A}$ of subsets of X there is a smallest sigma-algebra, called $\sigma(\mathcal{A})$, containing $\mathcal{A}$. This means that any other sigma algebra containing $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.

Proof: Let $(\mathcal{B}_i \mid i \in I)$ be all the sigma algebras of X that contain $\mathcal{A}$.

There is at least one member of the family, namely all the subsets of X.

By the above lemma, the collection $\mathcal{B} := \{B \mid \forall i \in I \quad B \in \mathcal{B}_i\}$ is a sigma-algebra.

Fix any sigma algebra $\mathcal{B}_i$ containing $\mathcal{A}$: any $B \in \mathcal{B}$ is also contained in $\mathcal{B}_i$. $\square$

Determining the smallest sigma algebra containing a collection of sets can be a difficult task.

Consider the collection $\mathcal{A}$ of subsets $A_1, A_2, \ldots$ of the integers such that $A_i = \{ni \,| n \text{ is an integer}\}$.

What is $\sigma(\mathcal{A})$?

Topology: A topology for a set X is a collection $\mathcal{A}$ of open subsets of X such that:

(a) $X \in \mathcal{A}$,

(b) $\forall A_1, A_2 \in \mathcal{A}\ A_1 \cap A_2 \in \mathcal{A}$,

(c) if $(A_i \mid i \in I)$ is any collection in $\mathcal{A}$ then $\cup_i A_i \in \mathcal{A}$.

As the empty intersection is included, the empty set is an open set.

In $\mathbf{R}^n$ a set A is open if for every $x \in A$ there is some δ such that the open ball $B_\delta(x) = \{y \mid ||y - x|| < \delta\}$ is contained in A.

It is easy to check that this definition satisfies the three conditions (a), (b), (c).

A *base* for a topology is a special collection $\mathcal{B}$ of open sets such that every open set A of the topology is a union of sets of the base.

Lemma: A base for the topology of $\mathbf{R}^n$ is the collection of sets $B_\delta(x)$ where δ is a rational number and each coordinate of x is rational.

Proof: Let x be a member of A, an open set.

With δ small enough so that $B_\delta(x) \subseteq A$,

let $\overline{x}$ be a point with rational coefficients within $\delta/3$ of x and let $\overline{\delta}$ be a rational number with $\delta/2 < \overline{\delta} < 2\delta/3$.

The ball $B_{\overline{\delta}}(\overline{x})$ both contains x and is contained within A.

Unioning such open balls for every $x \in A$ will recreate the set A. $\square$.

The above base of open sets of $\mathbf{R}^n$ is a countable collection,

meaning that every open set is the union of countably many open balls of this base.

For any space X with a topology

$\mathcal{B}(X)$ is defined to be the smallest sigma-algebra containing all the open sets of X.

This special sigma-algebra is called the collection of *Borel* sets.

$B(\mathbf{R}^n)$ include most of the sets one could imagine.

Let $\mathcal{F}$ be the collection of closed sets of $\mathbf{R}^n$ and

$\mathcal{G}$ the collection of open sets of $\mathbf{R}^n$.

Let $\mathcal{F}_\sigma$ be the collection of sets of the form $\cup_{i=1}^{\infty} A_i$ for some sequence $A_1, A_2, \ldots$ of closed sets (in $\mathcal{F}$).

Let $\mathcal{G}_\delta$ be the collection of sets of the form $\cap_{i=1}^{\infty} A_i$ for some sequence $A_1, A_2, \ldots$ of open sets (in $\mathcal{G}$).

Members of $\mathcal{G}_\delta$ are called G_δ sets and members of $\mathcal{F}_\sigma$ are called F_δ sets.

Lemma: Every closed set of $\mathbf{R}^n$ is in $\mathcal{G}_\delta$ and every open set of $\mathbf{R}^n$ is in $\mathcal{F}_\sigma$.

Proof: Let C be a closed set of $\mathbf{R}^n$. For every $\epsilon > 0$ define $C_\epsilon := \{y \mid ||y - x|| < \epsilon$ for some $x \in C$.

C_ϵ is open: if $||y - x|| < \epsilon$ for some $x \in C$ then $B_{\frac{\epsilon - ||y-x||}{2}}(y)$ is also in C_ϵ.

We claim that $\cap_i C_{\frac{1}{i}}$ is equal to C.

Any points y of $\cap_i C_{\frac{1}{i}}$ has a sequence $x_1, x_2, \ldots$ of points in C such that $||y - x_i|| < \frac{1}{i}$.

Therefore the sequence x_i converges to y.

As C is closed, this means that y is in C.

Now take any open set A and consider its complement $B = \mathbf{R}^n \backslash A$.

As $B = \cap A_i$ for a sequence of open sets,

we can write $X \backslash B = A = \cup_{i=1}^{\infty} X \backslash A_i$

for a sequence $X \backslash A_1, X \backslash A_2, \ldots$ of closed sets. $\square$

Corollary: The Borel collection of $\mathbf{R}^n$ are the sigma-algebra generated by

(a) the open sets of $\mathbf{R}^n$,

(b) the closed sets of $\mathbf{R}^n$,

(c) a countable base for the topology of $\mathbf{R}^n$.

Any countable process of taking unions and intersections, starting with the open sets, will result in a Borel set.

Indeed, it is consistent logically to assume that all subsets of $\mathbf{R}^n$ are Borel sets,

and it requires some axiom of choice to prove that a non-Borel set exists.

Measure:

Let X be a set with an algebra $\mathcal{A}$. A function $\mu : \mathcal{A} \to [0, \infty]$ is called **finitely additive** if for every finite collection $A_1, A_2, \ldots, A_n$ of mutually disjoint sets in $\mathcal{A}$,

$\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.

It is called a *measure* if additionally $\mu(\emptyset) = 0$.

Let X be a set with a sigma algebra $\mathcal{A}$. A function $\mu : \mathcal{A} \to [0, \infty]$ is called **countably additive** if for every infinite collection $A_1, A_2, \ldots$ of mutually disjoint sets in $\mathcal{A}$,

$\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

It is called a *measure* if additionally $\mu(\emptyset) = 0$.

Countably additive implies finitely additive,

by equating $A_j = \emptyset$ for all $j > n$ such that n is large enough.

But does finite additivity on a sigma algebra imply countable additivity?

People thought so, until about 1900.

The theory of finite additive measures is deep, related to duality in functional analysis, and not handled in this course.

For us, measure means countably additive.

Examples:

(a) $\mu(A) = |A|$, the cardinality of A.

(b) $x \in X$ chosen, $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$.

(c) $\mathbf{Z}$ is the set of integers and $\mathcal{A}$ is the collection of subsets such that A is in $\mathcal{A}$ if and only if A is finite or $\mathbf{Z} \backslash A$ is finite.

$\mu(A) = 1$ if $\mathbf{Z} \backslash A$ is finite and $\mu(A) = 0$ if A is finite.

Now try to extend μ to the sigma algebra $\sigma(\mathcal{A})$, which are all the subsets of $\mathbf{Z}$.

Problem: what should be the weight given to the set of even numbers and the set of odd numbers? And what of the other ways to partition the integers into finitely many disjoint infinite subsets?

It can be done, but never in a sigma additive way. If so, $\mu(\{i\}) = 0$ for all $i \in \mathbf{Z}$, but then $\mu(\mathbf{Z}) = 0$, a contradiction.

Measures are always monotonic,

meaning that A, B measurable and $A \subseteq B$ implies that $\mu(A) \leq \mu(B)$.

This follows because $B = A \cup (B \backslash A)$,

so that $\mu(B) = \mu(A) + \mu(B \backslash A)$ and $\mu(B \backslash A) \geq 0$.

Likewise $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} A_i$,

as we can always write, for every $i \geq 2$,

$A_i = (A_i \cap (\cup_{j=1}^{i-1} A_j)) \cup (A_i \backslash (\cup_{j=1}^{i-1} A_j))$

and recognise $\cup_{i=1}^{\infty} A_i$ as a disjoint union of A_1 with the $(A_i \backslash (\cup_{j=1}^{i-1} A_j))$,

and use that $\mu((A_i \backslash (\cup_{j=1}^{i-1} A_j)) \leq \mu(A_i)$.

By a measure space $(X, \mathcal{A}, \mu)$ we mean a set X, a sigma algebra $\mathcal{A}$ of subsets of X, and a sigma additive measure defined on all sets in $\mathcal{A}$.

A finite measure μ on X is one where $\mu(X) < \infty$.

μ is σ-finite if X is the union of countably many subsets $A_1, A_2, \ldots$ such that $\mu(A_i) < \infty$ for every $i = 1, 2, \ldots$.

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space.

If $A_1, A_2, \ldots$ is an increasing sequence of sets belonging to $\mathcal{A}$ (meaning $A_1 \subseteq A_2 \subseteq \cdots$ then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

If $A_1, A_2, \ldots$ is a decreasing sequence of sets belonging to $\mathcal{A}$ (meaning $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_k) < \infty$ for some k, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

Proof:

Let $A = \cup_{i=1}^{\infty} A_i$ and define $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_i = A_i \backslash A_{i-1}$.

As the union of the B_i is equal to A the B_i are disjoint,

we have $\sum_{i=1}^{\infty} \mu(B_i) = \mu(A)$.

But also $\cup_{k=1}^{i} B_k$ is a disjoint union equal to A_i.

So $\mu(A_i) = \sum_{k=1}^{i} \mu(B_k)$.

The conclusion follows from $\lim_{i\to\infty} \mu(A_i) = \lim_{i\to\infty} \sum_{k=1}^{i} \mu(B_k) = \sum_{k=1}^{\infty} \mu(B_k) = \mu(A)$

If $A = \cap_{i=1}^{\infty}$ and $\mu(A_n) < \infty$ for some n,

we can start at A_n and look at the sequence $C_k = A_n \backslash A_k$ for all $k > n$.

This is an increasing sequence, with $\lim_{k\to\infty} \mu(C_k) = \mu(\cup_{k>n} C_k) < \infty$.

It follows that $A = A_n \backslash \cup_{k>n} C_k$ and $A_k = A_n \backslash C_k$ for all $k > n$,

and the equality follows by subtracting from $\mu(A_n)$.

Lemma: Let $(X, \mathcal{A})$ be a measurable space (with $\mathcal{A}$ a sigma algebra) and μ a finitely additive measure defined on $\mathcal{A}$.

For μ to be a (sigma additive) measure, it suffices that for every increasing sequence $A_1, A_2, \ldots$ of sets in $\mathcal{A}$

it follows that $\lim_{i\to\infty} \mu(A_i) = \mu(\cup_{i=1}^{\infty} A_i)$.

Proof: Let $B_1, B_2, \ldots$ be an infinite sequence of mutually disjoint sets of $\mathcal{A}$.

We need to know that $\mu(\cup_i B_i) = \sum_i \mu(B_i)$.

Define $A_i = \cup_{k=1}^{i} B_i$, and let $B := \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$.

The A_i are an increasing sequence of sets,

and so $\lim_{i\to\infty} \mu(A_i) = \mu(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} B_i) = \mu(B)$.

Now notice that

$\mu(A_i) = \sum_{k=1}^{i} \mu(B_k)$ from finite additivity,

so $\lim_{i\to\infty} \mu(A_i)$ is also equal to $\sum_{k=1}^{\infty} \mu(B_k)$.