Measure Theory Second Week

Outer Measures:

Let X be a set,

$\mathcal{P}(X)$ the collection of all subsets of X.

An outer measure $\mu : \mathcal{P}(X) \to [0, \infty]$ is a function such that

(a) $\mu(\emptyset) = 0$,

(b) if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,

(c) if $A_1, A_2, \ldots$ is a sequence of subsets then

$\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ (subadditive).

For outer measures μ that are not measures there is some sequence $A_1, A_2, \ldots$ of disjoint sets such that

$\sum_{i=1}^{\infty} \mu(A_i) > \mu(\cup_{i=1}^{\infty} A_i)$.

For finitely additive measures μ that are not measures there is some sequence $A_1, A_2, \ldots$ of disjoint sets such that

$\sum_{i=1}^{\infty} \mu(A_i) < \mu(\cup_{i=1}^{\infty} A_i)$.

In general, outer measures are not measures,

as they are defined on all subsets;

usually measures require some restriction to a collection of measurable subsets.

Examples:

(a) $\mu(A) = 0$ if $A = \emptyset$ and

$\mu(A) = 1$ if $A \neq \emptyset$.

(b) $\mu(A) = 0$ if A is countable and

$\mu(A) = 1$ if A is uncountable.

(c) Let $(X, \mathcal{A}, \mu)$ be a measurable space. Define $\mu^*(B) = \inf_{A \in \mathcal{A},\ A \supset B} \ \mu(A)$.

Lebesgue outer measure:

λ^* is defined on all subsets of $\mathbf{R}$.

$\lambda^*(A) =$

$\inf\{\sum_{i=1}^{\infty} b_i - a_i \mid \cup_{i=1}^{\infty} (a_i, b_i) \supset A\}$.

Lemma (1.3.2): Lebesgue outer measure is an outer measure and assigns to every interval its length.

Proof: The empty set is covered by any collection of open intervals, hence also of lengths $\epsilon/2, \epsilon/4, \ldots,$

therefore $\lambda^*(\emptyset) = 0$.

If $A \subseteq B$ then any collection of intervals covering B also covers A.

Hence the collection of coverings for A involves a larger collection than that for B,

and therefore $\lambda^*(A) \leq \lambda^*(B)$.

Let $\epsilon > 0$ be given. Any covering collection used to define $\mu(A_i)$ to within $\frac{\epsilon}{2^i}$ also is a covering collection for $\cup_i A_i$.

Hence after taking the infinum on all coverings of $\cup_i A_i$ and ignoring the ϵ

it follows that $\lambda^*(\cup_i A_i) \leq \sum_i \lambda^*(A_i)$.

Finally, letting I be any interval from a to b with $b > a$, be in closed, open, or open on one end and closed on the other,

the sequence $(a-\epsilon, b+\epsilon)$ covers the interval,

and so λ^* of the interval is no more than $b - a$.

One the other hand, it suffices to show that λ^* of the closed interval $[a, b]$ is at least $b-a$.

Because it is compact, any collection of covering open intervals can be reduced to a finite covering collection.

Now easy to show that if the lengths of this finite cover did add up to at least $b-a$ they could not reach from a to b.

Definition: Let μ be a outer measure on X. A subset B is μ-measurable if for every subset A of X it holds that

$\mu(A) = \mu(A \cap B) + \mu(A \backslash B)$.

Subadditivity of outer meaures implies already that $\mu(A) \leq \mu(A \cap B) + \mu(A \backslash B)$,

so only $\mu(A) < \infty$.

A *Lebesgue* measure set is one that is measurable with respect to Lebesgue outer measure,

and the measure λ is the the measurable λ^* restricted to the Lebesgue measurable sets.

Lemma: (1.3.5) Let μ be an outer measure on X. Every subset B such that $\mu(B) = 0$ or $\mu(X \backslash B) = 0$ is μ-measurable.

Proof: We need only show for every subset A that $\mu(A) \geq \mu(A \cap B) + \mu(A \cap (X \backslash B))$.

With $\mu(B) = 0$ or $\mu(X \backslash B) = 0$ it follows by monotonicity.

If μ is an outer measure,

let $\mathcal{M}_\mu$ be the collection of μ measurable sets.

Theorem (1.3.6):

$\mathcal{M}_\mu$ is a sigma-algebra and

μ is a measure on $\mathcal{M}_\mu$.

Proof: From the previous lemma and the definintion of $\mathcal{M}_\mu$, X is in $\mathcal{M}_\mu$, $\mu(\emptyset) = 0$, and $A \in \mathcal{M}_\mu$ if and only if $X \backslash A \in \mathcal{M}_\mu$.

Next we show that $\mathcal{M}_\mu$ is an algebra and finitely additive.

Let $B_1, B_2 \in \mathcal{M}_\mu$; with closure by complementation already demonstrated, it suffices to show that $B_1 \cap B_2$ is also in $\mathcal{M}_\mu$.

Let A be any subset: as $A \cap B_1$ and $A \backslash B_2$ are also subsets

$\mu(A \cap B_1) =$

$\mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \backslash B_2)$ and

$\mu(A \backslash B_1) =$

$\mu((A \backslash B_1) \cap B_2) + \mu((A \backslash B_1) \backslash B_2)$.

With $\mu(A) = \mu(A \cap B_1) + \mu(A \backslash B_1)$

$\mu(A) = \mu(A \cap B_1 \cap B_2) + \mu((A \cap B_1) \backslash B_2) + \mu((A \backslash B_1) \cap B_2) + \mu((A \backslash B_1) \backslash B_2) \geq \mu(A \cap B_1 \cap B_2) + \mu(A \backslash (B_1 \cap B_2))$,

hence $B_1 \cap B_2$ is also in $\mathcal{M}_\mu$.

Furthermore, assuming $B_1, B_2 \in \mathcal{M}_\mu$ are disjoint,

and letting $A = B_1 \cup B_2$ be the set chosen,

we have $A \backslash B_1 = B_2$, $A \cap B_1 = B_1$

and $\mu(A) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$.

Therefore μ is finitely additive on $\mathcal{M}_\mu$.

It follows from finite additivity and induction (again chosing the sets) that for any infinite sequences $B_1, B_2, \ldots$ of mutually disjoint members of $\mathcal{M}_\mu$ and any subset A:

$\mu(A) = \sum_{i=1}^{n} \mu(A \cap B_i) + \mu(A \backslash (\cup_{i=1}^{n} B_i)$.

Letting n go to infinity,

$\mu(A \backslash (\cup_{i=1}^{\infty} B_i) \leq \lim_{n\to\infty} \mu(A \backslash (\cup_{i=1}^{n} B_i)$

but $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(A \cap B_i) = \sum_{i=1}^{\infty} \mu(A \cap B_i)$.

Therefore taking the limit of n to infinity, $\mu(A) \geq$

$\sum_{i=1}^{\infty} \mu(A \cap B_i) + \mu(A \backslash (\cup_{i=1}^{\infty} B_i)) \geq$

$\mu(A \cap (\cup_{i=1}^{\infty} B_i)) + \mu(A \backslash (\cup_{i=1}^{\infty} B_i) \geq \mu(A)$.

It follows that $\cup_{i=1} B_i$ is in $\mathcal{M}_\mu$ and it is a sigma algebra.

But now it is clear that μ is also sigma-additive on the disjoint sequence of the B_i,

as their finite additivity,

$\sum_{i=1}^{n} \mu(B_i) = \mu(\cup_{i=1}^{n} B_i)$ implies that

$\sum_{i=1}^{\infty} \mu(B_i) \leq \lim_{n\to\infty} \mu(\cup_{i=1}^{n} B_i) \leq \mu(\cup_{i=1}^{\infty} B_i)$

while the subadditivity implies that

$\sum_{i=1}^{n} \mu(B_i) \geq \mu(\cup_{i=1}^{\infty} B_i)$. $\square$

Lemma: Every Borel subset of $\mathbf{R}$ is Lebesgue measurable.

Proof: Given that the Lebesgue measure is an outer measure, hence the measurable sets are a sigma algebra,

it suffices to show for every interval of the form $I = (-\infty, c]$ and subset A that

$\lambda^*(A) = \lambda^*(A \cap I) + \lambda^*(A \backslash I)$.

We can break the ith open interval (a_i, b_i) covering A into two inteverals, $(a_i, c + \frac{\epsilon}{2^i})$ and (c, b_i) whenever $a_i < c < b_i$.

In this way we cover both $A \cap I$ and $A \backslash I$ and show that $\lambda^*(A \cap I) + \lambda^*(A \backslash I) \leq \lambda^*(A) + \epsilon$ for every $\epsilon > 0$;

together with subadditivity, the equality follows.

More on Lebesgue measure:

Lemma (regularity: Let B be a Lebesgue measurable subset of finite measure.

For every $\epsilon > 0$ there is an open set A and a compact set C such that $C \subseteq B \subseteq A$

and $\lambda(A \backslash C) < \epsilon$.

Proof:

As the measure $\lambda(B)$ is approximated by open covers,

there is an open cover of B whose union A has measure less than $\lambda(B) + \epsilon/3$

By sigma additivity,

there is an n large enough so that

$\lambda(B \cap [-n.n]) > \lambda(B) - \epsilon/3.$

Cover $[-n, n] \backslash B$ with an open set G so that $\lambda(G) > \lambda([-n, n] \backslash B) + \epsilon/3.$

$C = [-n, n] \backslash G$ is a closed set contained in B whose measure is more than $\lambda(B) - 2\epsilon/3.$

Lemma: Lebesgue measure is translation invariant,

meaing that for any given $r \in R$,

a set A is Lebesgue measurable

if and only if $A + r := \{a + r \mid a \in A\}$ is Lebesgue measurable

and $\lambda^*(A) = \lambda^*(A + r)$.

Proof: Let $(I_i \mid i = 1, 2, \dots)$ be a collection of open intervals covering A.

The intervals $(I_i + r)$ cover $A + r$ and each interval has the same length.

This shows that $\lambda^*(A + r) \leq \lambda^*(A)$,

and the same arguement shifting by $-r$ shows the opposite inequality.

Likewise the intersection property with any subset of $\mathbf{R}$ that confirms that A and $X \setminus A$ are Lebesgue measurable

shows the same for $A + r$ and $(X \setminus A) + r$ after all sets are shifted by r.

Theorem: Given the axiom of choice,

there is a subset of $[0, 1)$ that is not Lebesgue measurable.

Proof: Define an equivalence relation on $r, s \in [0, 1)$

by $r \sim s \Leftrightarrow r - s$ is rational.

Define addition modulo 1,

so that $b + c$ is $b + c - 1$ if $b + c \geq 1$.

List the rational numbers $a_1, a_2, \ldots$ in $[0, 1)$.

Let B be a set of representatives for the equivalence relation (Axiom of Choice)

meaning that B intersections every equivalence class one and only once,

or that for every $r \in [0, 1)$ there is one and only one i with $r + a_i \in B$.

This means that $\cup_{i=1}^{\infty}(B - a_i)$ partitions $[0, 1)$:

for every r there is some $b \in B$ and a_i such that $r = b - a_i$

and if $r \in B - a_i \cap B - a_j \neq \emptyset$ for distinct $a_i \neq a_j$

then $r = b_i - a_i = b_j - a_j$ for some $b_i, b_j \in B$ and the equivalence relation sharing both $r + a_i$ and $r + a_j$ have two representatives in B, a contradiction.

Assume that B is Lesbegue measurable.

Notice that translation invariance holds also in the modulo arithmetic,

due to a secondary shift of the measurable subset that went over the value of 1.

So every $B + a_i$ must be Lebesgue measurable and have the same measure.

This measure can neither be 0 or anything positive,

as that would imply that the whole set $[0, 1)$ is either infinite in measure or zero in measure,

when it is really of measure one.