$$\underset{\substack{\uparrow \\ \text{mean} \\ \text{energy} \\ \text{(internal} \\ \text{energy)}}}{F = U} - T\underset{\substack{\uparrow \\ \text{entropy}}}{S} \qquad \underset{\substack{\uparrow \\ \text{temperature}}}{T} = \frac{1}{\beta}$$

<u>Proof</u>

$$l = F = -\frac{1}{\beta} \log Z(\beta) = -\frac{1}{\beta} \log \sum_i e^{-\beta E_i}$$

$r = U - TS$ to show $l = r$

$$p_i = \frac{1}{Z} e^{-\beta E_i}$$

$$\log p_i = \log \frac{1}{Z} + \log \left(e^{-\beta E_i}\right)$$

$$= -\log Z - \beta E_i$$

$$\Rightarrow \quad S = -\sum_i p_i \log p_i$$

$$= +\sum p_i \left(\log Z + \beta E_i\right)$$

$$= \log Z \cdot 1 + \beta \underbrace{\sum_i p_i E_i}_{U}$$

$$r = U - TS = U - \frac{1}{\beta} S$$

$$= \cancel{\sum p_i E_i} - \frac{1}{\beta} \log Z - \cancel{\sum p_i E_i}$$

$$= -\frac{1}{\beta}\log Z \qquad \text{q.e.d.}$$
$$= F$$

Note that for the canonical ensemble

$$\Psi[p] = -S + \alpha + \beta U$$

$$\frac{1}{\beta}\Psi[p] = -TS + \frac{\alpha}{\beta} + U = F + \frac{\alpha}{\beta}$$

Ψ has minimum

$\Leftrightarrow$ F has a minimum

$\Leftrightarrow$ S has a maximum

Generalized statistical mechanics

Start from more general information measures

$$I[p] = -S[p] = \sum p_i h(p_i)$$

some function

Example: Tsallis entropy

$$S_q = \frac{1}{q-1}\left(1 - \sum p_i^{\,q}\right)$$

$$q \to 1 \implies S_q \to S = -\sum p_i \log p_i$$

✓

For this example

$$h(p_i) = \frac{p_i^{\,q-1} - 1}{q-1}$$

One defines a q- logarithm as

$$\log_q(x) = \frac{x^{1-q} - 1}{1-q}$$

[exercise : Show that

$$\lim_{q\to 1} \log_q(x) = \log x$$]

The inverse function of the q- logarithm is the q- exponential

$$\exp_q(x) := \left(1 + (1-q)x\right)^{\frac{1}{1-q}}$$

$$q \to 1 \implies \exp_q(x) \to e^x$$

Now let's do generalized stat. mech.

$$\Psi[p] = \sum p_i h(p_i) + \alpha \sum p_i + \beta \sum p_i E_i$$

$$\frac{\partial}{\partial p_i} \Psi[p] = 0$$

$\Rightarrow$ we obtain

$$\underbrace{h(p_i) + p_i h'(p_i)}_{g(p_i)} + \alpha + \beta E_i = 0$$

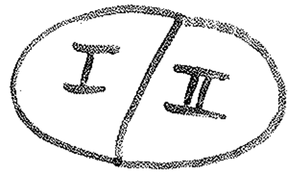
$$g(p_i) = -\alpha - \beta E_i$$

$$p_i = g^{-1}(-\alpha - \beta E_i)$$

Nonextensive stat. mech.
= generalized stat. mech. based on maximisation of Tsallis entropies

Why that name?
Take two independent systems I and II. Then the Tsallis entropy of the

joint system I, II is non-additive:



<u>Theorem</u>

$$S_q^{I,II} = S_q^{I} + S_q^{II} - (q-1)\, S_q^{I} \cdot S_q^{II}$$

<u>Proof</u>

$$\sum_i (p_i^{I})^q = 1 - (q-1)\, S_q^{I} \qquad (1)$$

$$\sum_j (p_j^{II})^q = 1 - (q-1)\, S_q^{II} \qquad (2)$$

$$\sum_{i,j} p_{ij}^{\,q} \underset{\substack{\uparrow \\ \text{independence}}}{=} \sum_i (p_i^{I})^q \sum_j (p_j^{II})^q$$

$$= 1 - (q-1)\, S_q^{I,II}$$

eq (1) × eq. (2)

$$\Rightarrow \sum_i (p_i^{I})^q \cdot \sum_j (p_j^{II})^q$$

$$= 1 - (q-1)\, S_q^{I} - (q-1)\, S_q^{II}$$

$$+ (q-1)^2\, S_q^{I}\, S_q^{II}$$

$$S_q^{I,II} = S_q^{I} + S_q^{II} - (q-1)\, S_q^{I} S_q^{II}$$

q.e.d.

A few further (nice) properties of the Tsallis entropy

— convexity $\quad S_q = \frac{1}{q-1}\left(1-\sum_i p_i^{\,q}\right)$

$$\frac{\partial}{\partial p_i} S_q = -\frac{q}{q-1}\, p_i^{\,q-1}$$

$$\frac{\partial^2}{\partial p_i \partial p_j} S_q = -q\, p_i^{\,q-2}\, \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

— stability

Tsallis entropies are Lesche-stable, whereas for example Renyi entropies are not.

Conclusion

$\Rightarrow$ After the Shannon entropy, the next-best entropy measures are the Tsallis entropies.

$$\frac{\partial}{\partial p_i} \underset{\uparrow}{I[p]} + \sum_\sigma \beta_\sigma M_i^\sigma = 0$$

take now Tsallis information

For the canonical ensemble

$$\frac{\partial}{\partial p_i} I_q^T [p] = \frac{q}{q-1} p_i^{q-1}$$

$$\Rightarrow \frac{q}{q-1} p_i^{q-1} + \alpha + \beta E_i = 0$$

$$\Rightarrow p_i = \frac{1}{Z_q} \left(1 - \beta(q-1) E_i\right)^{\frac{1}{q-1}}$$

Today's notation uses $q' = 2-q$
and then renames $q' \to q$

$$\Rightarrow p_i = \frac{1}{Z_q} \left(1 + \beta(q-1) E_i\right)^{-\frac{1}{q-1}}$$

Further important entropy measures:

<u>Kaniadakis entropy</u>

$$S_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}$$

$\kappa \to 0$ this again reduces to the Shannon entropy.

<u>Sharma - Mittal entropies</u>

$$S_{\kappa, r} = -\sum_i p_i^r \left(\frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa} \right)$$

reduces to – Tsallis entropy for $r = \kappa$, $q = 1 - 2\kappa$

– Kaniadakis entropy for $r = 0$

– Abe entropy for $\kappa = \frac{1}{2}(q - q^{-1})$

$r = \frac{1}{2}(q + q^{-1}) - 1$

[illegible]