London Taught Course on Spectral Theory

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This lecture is based on Chapters 2 and 3 of STDO, which is 'Spectral Theory and Differential Operators', available in paperback from Amazon (for example) for about 28 pounds. A list of misprints, including the correction of a serious error in the proof of Theorem 1.2.10, may be downloaded from
`http://www.mth.kcl.ac.uk/staff/eb_davies/STDO.html`

The notions of self-adjointness and spectrum are defined for unbounded operators. Theorem 1.2.10 establishes that the spectrum of an unbounded SA operators is non-empty and real. Lemma 1.1.2 states that the spectrum is always closed. Four different forms of the spectral theorem are stated.

Theorem 2.3.1 is in terms of the existence of a suitable algebra homomorphism $T : C_0(\mathbf{R}) \to \mathcal{L}(\mathcal{H})$. This can be refined to the existence of such a map from $C_0(S)$ to $\mathcal{L}(\mathcal{H})$ where $S = \mathrm{Spec}(H)$. This leads to the notation $f(H) = T(f)$ where $H = H^*$ and $f \in C_0(S)$, called a functional calculus.

Theorem 2.5.3 requires an understanding of weak and strong convergence of operators. It asserts that Theorem 2.5.1 can be extended by replacing $C_0(\mathbf{R})$ by $B(\mathbf{R})$, the space of all bounded Borel measurable functions. $B(\mathbf{R})$ contains all constructively definable bounded functions on $\mathbf{R}$. This leads naturally to defining the orthogonal projections P_s for all $s \in \mathbf{R}$ as $T(\chi_{(-\infty,s]})$. It may be shown that $\mathcal{H}$ is the orthogonal direct sum of $\mathcal{L} = \mathrm{Ker}(P_s)$ and $\mathcal{M} = \mathrm{Ran}(P_s)$. Each of these subspaces is invariant under H and

$$\begin{aligned} \mathrm{Spec}(H|_{\mathcal{M}}) &= \mathrm{Spec}(H) \cap (-\infty, s], \\ \mathrm{Spec}(H|_{\mathcal{L}}) &= \mathrm{Spec}(H) \cap [s, \infty), \end{aligned}$$

except possibly at the point s.