Enumerative Combinatorics 5: q-analogues

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In a sense, a q-analogue of a combinatorial formula is simply another formula involving a variable q which has the property that, as $q \to 1$, the second formula becomes the first. Of course there is more to it than that; some q-analogues are more important than others. What follows is nothing like a complete treatment; I will concentrate on a particularly important case, the *Gaussian* or *q-binomial coefficients*, which are, in the above sense, q-analogues of binomial coefficients.

5.1 Definition of Gaussian coefficients

The Gaussian (or q-binomial) coefficient is defined for non-negative integers n and k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

In other words, in the formula for the binomial coefficient, we replace each factor r by q^r-1. Note that this is zero if $k > n$; so we may assume that $k \leq n$.

Now observe that $\lim_{q\to 1} \frac{q^r-1}{q-1} = r$. This follows from l'Hôpital's rule: both numerator and denominator tend to 0, and their derivatives are rq^{r-1} and 1, whose ratio tends to r. Alternatively, use the fact that

$$\frac{q^r-1}{q-1} = 1 + q + \cdots + q^{r-1},$$

and we can now harmlessly substitute $q = 1$ in the right-hand side; each of the r terms becomes 1.

Hence if we replace each factor $(q^r - 1)$ in the definition of the Gaussian coefficient by $(q^r - 1)/(q - 1)$, then the factors $(q - 1)$ in numerator and denominator cancel, so the expression is unchanged; and now it is clear that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

5.2 Interpretations

Quantum calculus The letter q stands for "quantum", and the q-binomial coefficients do play a role in "quantum calculus" similar to that of the ordinary binomial coefficients in ordinary calculus. I will not discuss this further; see the book *Quantum Calculus*, by V. Kac and P. Cheung, Springer, 2002, for further details.

Vector spaces over finite fields The letter q is also routinely used for the number of elements in a finite field (which is necessarily a prime power, and indeed there is a unique finite field of any given prime power order q – a theorem of Galois).

Theorem 5.1 *Let V be an n-dimensional vector space over a field with q elements. Then the number of k-dimensional subspaces of v is* $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Proof The proof follows the standard proof for binomial coefficients counting subsets of a set.

A k-dimensional subspace of V is specified by choosing a basis, a sequence of k linearly independent vectors. Now the number of choices of the first vector is $q^n - 1$ (since every vector except the zero vector is eligible); the second can be chosen in $q^n - q$ ways (since the q multiples of the first vector are now ineligible); the third in $q^n - q^2$ ways (since the q^2 linear combinations of the first two are now ruled out); and so on. In total,

$$(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$$

choices.

We have to divide this by the number of k-tuples of vectors which form a basis for a given k-dimensional subspace. This number is obtained by replacing n by k in the above formula, that is,

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}).$$

Dividing, and cancelling the powers of q, gives the result.

Remark Let F denote a field of q elements. Then a set of k linearly independent vectors in F^n can be represented as a $k \times n$ matrix of rank k. We may put it into reduced echelon form by elementary row operations without changing the subspace it spans; and, indeed, any subspace has a unique basis in reduced echelon form. So as a corollary we obtain

Corollary 5.2 *The number of $k \times n$ matrices over a field of q elements which are in reduced echelon form is* $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

As a reminder, a matrix is in *reduced echelon form* if

(a) the first non-zero entry in any row is a 1 (a *leading* 1);

(b) the leading 1s occur further to the right in successive rows;

(c) all the other elements in the column of a leading 1 are 0.

This has two consequences. First, it gives us another way of calculating the Gaussian coefficients. For example, the 2×4 matrices in reduced echelon form are as follows, where $*$ denotes any element of the field:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So we have

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = q^4 + q^3 + q^2 + q^2 + q + 1 = (q^2+1)(q^2+q+1).$$

This expression, and the definition, are polynomials in q, which agree for every prime power q; so they coincide. In a similar way, any Gaussian coefficient can be written out as a polynomial.

The other consequence is that algebra is not required here. Over any alphabet of size q, containing two distinguished elements 0 and 1, the number of $k \times n$ matrices in "reduced echelon form" (satisfying (a)–(c) above) with no zero rows is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Lattice paths How many lattice paths are there from the origin to the point (m, n), where each step in the path moves one unit either north or east?

Clearly the number is $\binom{m+n}{m}$, since we must take $m+n$ steps of which m are north and n are east, and the northward steps may occur in any m of the $m+n$ positions.

Suppose we want to count the paths by the area under the path (that is, bounded by the X-axis, the line $x = m$, and the path). We use a generating function approach, so that a path enclosing an area of a units contributes q^a to the overall generating function. Here q is simply a formal variable; the answer is obviously a polynomial in q.

Theorem 5.3 *The generating function for lattice paths from* $(0, 0)$ *to* (m, n) *by area under the path is* $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$.

We will see why in the next section. Note that, as $q \to 1$, we expect the formula to tend to $\binom{m+n}{m}$.

A non-commutative interpretation Let x and y be elements of a (non-commutative) algebra which satisfy $yx = qxy$, where the coefficient q is a "scalar" and commutes with x and y. Then we have the following analogue of the Binomial Theorem (see Exercises):

Theorem 5.4

$$(x+y)^n = \sum_{k=0} n \begin{bmatrix} n \\ k \end{bmatrix}_q x^(n-k) y^k.$$

For example,

$$(x+y)^3 = xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy.$$

We can use the relation to move the y's to the end in each term; each time we jump a y over an x we pick up a factor q. So

$$(x+y)^3 = x^3 + (1+q+q^2)x^2y + (1+q+q^2)xy^2 + y^3,$$

in agreement with the theorem.

5.3 Combinatorial properties

These properties can be proved in two ways: by using the counting interpretation involving subspaces of a vector space, or directly from the formula (usually easiest). The proofs are all relatively straightforward; I will just give outlines where appropriate.

Proposition 5.5 $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$

This is straightforward from the formula. Alternatively we can invoke vector space duality: there is a bijection between subspaces of dimension k of a vector space and their annihilators (subspaces of codimension k of the dual space).

Proposition 5.6 $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1,$ *and* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ *for* $0 < k < n$.

Again, straightforward from the formula. Alternatively, consider $k \times n$ matrices in reduced echelon. If the leading 1 in the last row is in the last column, then the other entries in the last row and column are zero, and deleting them gives a $(k-1) \times (n-1)$ matrix in reduced echelon. Otherwise, the last column is arbitrary (so there are q^k possibilities for it; deleting it leaves a $k \times (n-1)$ matrix in reduced echelon.

Remark From this we can prove Theorem 5.3, as follows. Let $Q(n,k)$ be the sum of the weights of lattice paths from $(0,0)$ to $(n-k,k)$, where the weight of a path is q^a if the area below it is a. Clearly $Q(n,0) = Q(n,n) = 1$.

Consider all the lattice paths from $(0,0)$ to $(n-k,k)$, and divide them into two classes: those in which the last step is vertical, and those in which it is horizontal. In the first case, the last step is from the end of a path counted by $Q(n-1,k-1)$ (ending at $(n-k,k-1)$), and adds no area to the path. In the second step, it is from the end of a path counted by $Q(n-1,k)$ (ending at $(n-k-1,k)$), and increases the area by k, adding $q^kQ(n-1,k)$ to the sum. So the numbers $Q(n,k)$ have the same recurrence and boundary conditions as the Gaussian coefficients, and must be equal to them.

From the last two results, we can deduce an alternative recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

5.4 The q-binomial theorem

The q-analogue of the Binomial Theorem states:

Theorem 5.7 *For any positive integer n,*

$$\prod_{i=1}^{n}(1+q^{i-1}z) = \sum_{k=0}^{n} q^{k(k-1)/2}z^k\begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The proof is by induction on n; starting the induction at $n=1$ is trivial. Suppose that the result is true for $n-1$. For the inductive step, we must compute

$$\left(\sum_{k=0}^{n-1} q^{k(k-1)/2}z^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q\right)\left(1+q^{n-1}z\right).$$

The coefficient of z^k in this expression is

$$\begin{aligned} & q^{k(k-1)/2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{(k-1)(k-2)/2+n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\ = \;& q^{k(k-1)/2}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right) \\ = \;& q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q \end{aligned}$$

by the alternative recurrence relation.

I state without proof here *Heine's formula*, the q-analogue of the negative binomial theorem:

$$\prod_{i=1}^{n}(1-q^{i-1}z)^{-1} = \sum_{j=0}^{\infty}\begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q z^j.$$

5.5 Jacobi's Triple Product Identity

This is only loosely connected with the topics of this chapter, but is interesting in its own right.

Theorem 5.8 (Jacobi's Triple Product Identity)

$$\prod_{n>0}(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})(1-q^{2n}) = \sum_{l\in\mathbb{Z}} q^{l^2}z^l.$$

The sharp-eyed will notice that the series on the right breaks my rules that formal Laurent series should have only finitely many negative terms. Well, this just shows that formal power series are more flexible than might first appear! You can check that the three infinite products on the left contribute only finitely many terms to each power, positive or negative, of z.

By replacing q by $q^{1/2}$ and moving the third term in the product to the right-hand side, the identity takes the form

$$\prod_{n>0}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}) = \left(\sum_{l\in\mathbb{Z}} q^{l^2/2}z^l\right)\left(\prod_{n>0}(1-q^n)^{-1}\right),$$

in which form we will prove it. The proof here is a remarkable argument by Richard Borcherds, and this write-up from my Combinatorics textbook.

A *level* is a number of the form $n+\frac{1}{2}$, where n is an integer. A *state* is a set of levels which contains all but finitely many negative levels and only finitely many positive levels. The state consisting of all the negative levels and no positive ones is called the *vacuum*. Given a state S, we define the *energy* of S to be

$$\sum\{l : l>0, l\in S\} - \sum\{l : l<0, l\notin S\},$$

while the *particle number* of S is

$$|\{l : l>0, l\in S\}| - |\{l : l<0, l\notin S\}|.$$

Although it is not necessary for the proof, a word about the background is in order!

Dirac showed that relativistic electrons could have negative as well as positive energy. Since they jump to a level of lower energy if possible, Dirac hypothesised that, in a vacuum, all the negative energy levels are occupied.

Since electrons obey the exclusion principle, this prevents further electrons from occupying these states. Electrons in negative levels are not detectable. If an electron gains enough energy to jump to a positive level, then it becomes 'visible'; and the 'hole' it leaves behind behaves like a particle with the same mass but opposite charge to an electron. (A few years later, positrons were discovered filling these specifications.) If the vacuum has no net particles and zero energy, then the energy and particle number of any state should be relative to the vacuum, giving rise to the definitions given.

We show that the coefficient of $q^m z^l$ on either side of the equation is equal to the number of states with energy m and particle number l. This will prove the identity.

For the left-hand side this is straightforward. A term in the expansion of the product is obtained by selecting $q^{n-\frac{1}{2}}z$ or $q^{n-\frac{1}{2}}z^{-1}$ from finitely many factors. These correspond to the presence of an electron in positive level $n - \frac{1}{2}$ (contributing $n - \frac{1}{2}$ to the energy and 1 to the particle number), or a hole in negative level $-(n - \frac{1}{2})$ (contributing $n - \frac{1}{2}$ to the energy and -1 to the particle number). So the coefficient of $q^m z^l$ is as claimed.

The right-hand side is a little harder. Consider first the states with particle number 0. Any such state can be obtained in a unique way from the vacuum by moving the electrons in the top k negative levels up by $n_1, n_2, \ldots, n_k$, say, where $n_1 \geq n_2 \geq \ldots \geq n_k$. (The monotonicity is equivalent to the requirement that no electron jumps over another. The jumping process allows the possibility that some electrons jump from negative levels to higher but still negative levels, so k is not the number of occupied positive levels.) The energy of the state is thus $m = n_1 + \ldots + n_k$. Thus, the number of states with energy m and particle number 0 is equal to the number $p(m)$ of partitions of m, which is the coefficient of q^m in $P(q) = \prod_{n>0}(1 - q^n)^{-1}$, as we saw in lecture 1.

Now consider states with positive particle number l. There is a unique *ground state*, in which all negative levels and the first l positive levels are filled; its energy is

$$\frac{1}{2} + \frac{3}{2} + \ldots + \frac{2l-1}{2} = \frac{1}{2}l^2,$$

and its particle number is l. Any other state with particle number l is obtained from this one by 'jumping' electrons up as before; so the number of such states with energy m is $p(m - \frac{1}{2}l^2)$, which is the coefficient of $q^m z^l$ in $q^{l^2/2} z^l P(q)$, as required.

The argument for negative particle number is similar.

Exercises

1 Prove that, for fixed n, the Gaussian coefficients are unimodal.

2 For fixed n and k, the Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q of degree $k(n-k)$, whose coefficients $a_0, \ldots, a_{k(n-k)}$ are non-negative integers. Prove that the coefficients are symmetric: that is, $a_i = a_{k(n-k)-i}$.

Remark It is also true that the coefficients are unimodal, but this is not so easy to prove. The polynomial does not have all its roots real and negative!

3 Show that, for a, b equal to 0 or 1,

$$\begin{bmatrix} 2m+a \\ 2l+b \end{bmatrix}_{-1} = \begin{cases} 0 & \text{if } a = 0 \text{ and } b = 1, \\ \binom{m}{l} & \text{otherwise.} \end{cases}$$

Remark For a more challenging exercise, find a formula for $\begin{bmatrix} n \\ k \end{bmatrix}_\omega$, where ω is a primitive dth root of unity.

4 Deduce Euler's Pentagonal Numbers Theorem from Jacobi's Triple Product Identity. (*Hint*: put $q = t^{3/2}$, $z = -t^{-1/2}$.)

5 Consider the algebra generated by two non-commuting variables x and y satisfying the relation $yx = qxy$. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k.$$