Enumerative Combinatorics 10: Cayley's Theorem

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The course ends with four entirely different proofs of Cayley's theorem for the number of labelled trees on n vertices, some of which introduce new ideas. There is a direct bijective proof due to Prüfer; Joyal's proof using species; a proof using Kirchhoff's Matrix-Tree Theorem; and a proof using Lagrange inversion.

A *tree* is a connected graph without cycles. It is not hard to show by induction that a tree on n vertices has $n-1$ edges. There are 16 trees on the vertex set $\{1,2,3,4\}$: four of them are "stars" in which one vertex is joined to the other three, and the other twelve are "paths".

Theorem 10.1 *The number of labelled trees on the vertex set* $\{1,\ldots,n\}$ *is* n^{n-2}.

10.1 Prüfer codes

We construct a bijection between the set of all trees on the vertex set $\{1,\ldots,n\}$ and the set of all $(n-2)$-tuples of elements from this set. The tuple associated with a tree is called its *Prüfer code.*

First we describe the map from trees to Prüfer codes. Start with the empty code. Repeat the following procedure until only two vertices remain: select the leaf with smallest label; append the label of its unique neighbour to the code; and then remove the leaf and its incident edge.

Next, the construction of a tree from a Prüfer code P. We use an auxiliary list L of vertices added as leaves, which is initially empty. Now, while P is not empty, we join the first element of P to the smallest-numbered vertex

v which is not in either P or L, and then add v to L and remove the first element of P. When P is empty, two vertices have not been put into L; the final edge of the tree joins these two vertices.

I leave it as a (quite non-trivial) exercise to show that these maps are inverse bijections.

This proof gives extra information: the valency of vertex i of the tree is one more than the number of occurrences of i in its Prüfer code; so the number of trees with prescribed vertex valencies can be calculated.

10.2 A proof using species

Let **Lin** and **Perm** be the species of linear orders and permutations respectively. We have seen that these two species have the same counting function for labelled structures on n points (namely $n!$); so **Lin**[**F**] and **Perm**[**F**] will also have the same counting function for labelled structures, for any species **F**.

Joyal takes $\mathbf{F} = \mathbf{RTree}$, the species of rooted trees (trees with a distinguished vertex.

Now **Lin**[**RTree**] consists of a linear order on a set, say $\{1, 2, \ldots, k\}$ with the usual order, with a rooted tree at each point. We can identify the root of the tree at point i to be i itself. What we have constructed is a tree with a distinguished path $\{1, 2, \ldots, k\}$. Joyal calls such an object a *vertebrate*, since it has a "backbone" from the "head" 1 to the "tail" k. We get a vertebrate by taking a tree on n vertices and distinguishing two of them to be the head and the tail; in a tree there is a unique path between any two vertices. So the number of vertebrates is $n^2T(n)$, where $T(n)$ is the number of trees.

Also **Perm**[**RTree**] consists of a set of, say, k points carrying a permutation, with a rooted tree attached at each point. If we direct every edge of each tree towards the root, we have a picture representing what Joyal calls an *endofunction*, a function from $\{1, \ldots, n\}$ to itself. Such a function has a set of "periodic points" which return to their initial positions after finitely many steps; any other point is "transient", and the transient points feed into periodic points in a treelike fashion. The number of endofunctions is clearly n^n.

So $n^2T(n) = n^n$, giving the result.

10.3 The Matrix-Tree Theorem

This theorem, proved by Kirchhoff in the nineteenth century for analysis of electrical circuits, depends on the notion of the *Laplacian matrix* of a graph $G = (V, E)$. Assuming that $V = \{v_1, \ldots, v_n\}$, this is the $n \times n$ symmetric matrix whose (i, i) entry is the valency of vertex v_i, and whose (i, j) entry for $i \neq j$ is -1 if $\{v_i, v_j\}$ is an edge, and 0 otherwise. Note that the row sums of this matrix are all zero, so its determinant is zero.

Recall that the (i, j) *cofactor* of a square matrix A is the determinant of the matrix obtained from A by deleting the ith row and the jth column, multiplied by $(-1)^{i+j}$.

Theorem 10.2 *The cofactors of the Laplacian matrix of a graph are all equal to the number of spanning trees of the graph.*

A tree on the vertex set $\{1, \ldots, n\}$ is simply a spanning tree of the *complete graph*, the graph whose edges are all pairs of vertices. The Laplacian matrix of the complete graph is $nI_n - J_n$, where I_n and J_n denote the $n \times n$ identity and all-1 matrices. Deleting the last row and column gives $nI_{n-1} - J_{n-1}$.

We find the determinant of the last matrix by computing its eigenvalues. Every row and column sum is $n - (n - 1) = 1$, so the all-1 vector is an eigenvector with eigenvalue 1. If v is a vector orthogonal to the all-1 vector, then $J_{n-1}v = 0$, so v is an eigenvector with eigenvalue n. Thus $nI_{n-1} - J_{n-1}$ has eigenvalues 1 (multiplicity 1) and n (multiplicity $n-2$); so its determinant is n^{n-2}, which is thus the number of spanning trees.

The proof of the Matrix-Tree Theorem depends on the *Cauchy–Binet formula*, a nineteenth century determinant formula which asserts the following. et A be an $m \times n$ matrix, and B an $n \times m$ matrix, where $m < n$. Then

$$\det(AB) = \sum_X \det(A(X)) \det(B(X)),$$

where X ranges over all m-element subsets of $\{1, \ldots, n\}$. Here $A(X)$ is the $m \times m$ matrix whose columns are the columns of A with index in X, and $B(X)$ is the $m \times m$ matrix whose rows are the rows of B with index in X.

To prove the Matrix-Tree Theorem for the graph $G = (V, E)$ with Laplacian matrix $L(G)$, choose an arbitrary orientation of the edges of G, and let

M be the signed vertex-edge incidence matrix of G, with (v, e) entry $+1$ if v is the "head" of the arc e, -1 if v is the "tail" of e, and 0 otherwise. It is straightforward to show that $MM^\top = L(G)$. Let v be any vertex of G, and let $N = M_v$ be the matrix obtained by deleting the row of M indexed by e. It can be shown that, if X is a set of $n-1$ edges, then

$$\det(N(X)) = \begin{cases} \pm 1 & \text{if } X \text{ is the edge set of a spanning tree,} \\ 0 & \text{otherwise.} \end{cases}$$

By the Cauchy–Binet formula, $\det(NN^\top)$ is equal to the number of spanning trees. But $NN^\top$ is the principal cofactor of $L(G)$ obtained by deleting the row and column indexed by v.

The fact that all cofactors are equal is not really necessary for us, and can be proved by elementary linear algebra.

10.4 Lagrange inversion

Our final approach involves another general technique, Lagrange inversion.

Let G be the set of all formal power series (over the commutative ring R with identity) which have the form $x + \cdots$, that is, constant term is zero and coefficient of x is 1. Any of these series can be substituted into any other. We make a simple observation:

Proposition 10.3 *The set G, with the operation of substitution, is a group.*

This group is sometimes called the *Nottingham group*, for reasons that are a little obscure.

Proof Closure and the associative law are straightforward, and the formal power series x is the identity. Let $f(x) = x + a_2x^2 + a_3x^3 + \cdots$ be any element of G. We seek an inverse $g(x) = x + b_2x^2 + b_3x^3 + \cdots$ such that $f(g(x)) = x$. The coefficient of x^n in

$$f(g(x)) = g(x) + a_2g(x)^2 + a_3g(x)^3 + \cdots$$

is $b_n +$ stuff, where stuff involves the as and b_i for $i < n$. Equating it to zero gives b_n in terms of as and b_i for $i < n$; so the bs can be found recursively. In a similar way, we find a unique element $h(x) \in G$ for which $h(f(x)) = x$. Then

$$g(x) = h(f(g(x)) = h(x),$$

and the inverse is unique. □

The proof implicitly shows us how to find the inverse; Lagrange inversion gives a more direct approach.

Theorem 10.4 *The coefficient of x^n in $g(x)$ is*

$$\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(\frac{x}{f(x)}\right)^n\right]_{x=0}\Big/ n!.$$

I will not give the proof here; it involves working with Laurent series and extending the notion of poles and the calculus of residues to formal power series.

Now let **RTree** be the species of rooted trees, as before. We clearly have the equation

$$\mathbf{RTree} = \mathbf{E}\cdot\mathbf{Set}[\mathbf{RTree}],$$

where **E** is the species of 1-element sets; this is because a rooted tree is a (ppossibly empty) set of rooted trees all joined to a new root.

Thus the exponential generating function $T^*(x)$ for rooted trees satisfies

$$T^*(x) = x\exp(T^*(X)).$$

So the function $T^*(x)$ is the inverse (in the group G) of the function $x/\exp(x)$.

From Lagrange inversion, we find that the coefficient of $x^n/n!$ in $T^*(x)$ is

$$\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\exp(nx)\right]_{x=0} = n^{n-1}.$$

Since the number of rooted trees is n times the number of trees, we conclude that there are n^{n-2} trees on n vertices.

10.5 Stirling's formula

The most famous asymptotic formula in enumerative combinatorics is *Stirling's formula*, an estimate for the factorial function. We write $f \sim g$ to mean that $f(n)/g(n) \to 1$ as $n \to \infty$. Typically this is used with f a combinatorial counting function and g an analytic approximation to f. Stirling's formula is an example.

Theorem 10.5 $n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$.

It follows that, if $T(n)$ is the number of labelled trees on n vertices, then

$$\lim_{n\to\infty}\left(\frac{T(n)}{n!}\right)^{1/n} = \mathrm{e},$$

so the exponential generating function for $T(n)$ has radius of convergence $1/\mathrm{e}$.

Using more complicated methods, Otter showed that the number of unlabelled trees on n vertices is asymptotically $An^{-5/2}c^n$, where $A = 0.5349485\ldots$ and $c = 2.955765\ldots$.

Exercises

1 Calculate the chromatic polynomial of

(a) the path with n vertices,

(b) the cycle with n vertices.

2 A *forest* is a graph whose connected components are trees. Show that there is a bijection between labelled forests of rooted trees on n vertices, and labelled rooted trees on $n+1$ vertices with root $n+1$.

Use Stirling's formula to show that, if a forest of rooted trees on n vertices is chosen at random, then the probability that it is connected tends to the limit $1/\mathrm{e}$ as $n \to \infty$.

3 Count the labelled trees in which the vertex i has valency a_i for $1 \le i \le n$, where $a_1, \ldots, a_n$ are positive integers with sum $2n-2$.